7 Vectors

Exercises 7.1

1. **(a)** $6\mathbf{i}+12\mathbf{j}$ **(b)** $\mathbf{i}+8\mathbf{j}$ **(c)** $3\mathbf{i}$ **(d)** $\sqrt{65}$ **(e)** 3

2. **(a)** $\langle 3,3\rangle$ **(b)** $\langle 3,4\rangle$ **(c)** $\langle -1,-2\rangle$ **(d)** 5 **(e)** $\sqrt{5}$

3. **(a)** $\langle 12,0\rangle$ **(b)** $\langle 4,-5\rangle$ **(c)** $\langle 4,5\rangle$ **(d)** $\sqrt{41}$ **(e)** $\sqrt{41}$

4. **(a)** $\frac{1}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}$ **(b)** $\frac{2}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}$ **(c)** $-\frac{1}{3}\mathbf{i}-\mathbf{j}$ **(d)** $2\sqrt{2}/3$ **(e)** $\sqrt{10}/3$

5. **(a)** $-9\mathbf{i}+6\mathbf{j}$ **(b)** $-3\mathbf{i}+9\mathbf{j}$ **(c)** $-3\mathbf{i}-5\mathbf{j}$ **(d)** $3\sqrt{10}$ **(e)** $\sqrt{34}$

6. **(a)** $\langle 3,9\rangle$ **(b)** $\langle -4,-12\rangle$ **(c)** $\langle 6,18\rangle$ **(d)** $4\sqrt{10}$ **(e)** $6\sqrt{10}$

7. **(a)** $-6\mathbf{i}+27\mathbf{j}$ **(b)** $\mathbf{0}$ **(c)** $-4\mathbf{i}+18\mathbf{j}$ **(d)** 0 **(e)** $2\sqrt{85}$

8. **(a)** $\langle 21,30\rangle$ **(b)** $\langle 8,12\rangle$ **(c)** $\langle 6,8\rangle$ **(d)** $4\sqrt{13}$ **(e)** 10

9. **(a)** $\langle 4,-12\rangle-\langle -2,2\rangle=\langle 6,-14\rangle$ **(b)** $\langle -3,9\rangle-\langle -5,5\rangle=\langle 2,4\rangle$

10. **(a)** $(4\mathbf{i}+4\mathbf{j})-(6\mathbf{i}-4\mathbf{j})=-2\mathbf{i}+8\mathbf{j}$ **(b)** $(-3\mathbf{i}-3\mathbf{j})-(15\mathbf{i}-10\mathbf{j})=-18\mathbf{i}+7\mathbf{j}$

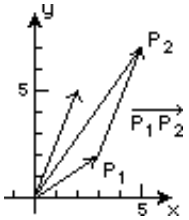
11. **(a)** $(4\mathbf{i}-4\mathbf{j})-(-6\mathbf{i}+8\mathbf{j})=10\mathbf{i}-12\mathbf{j}$ **(b)** $(-3\mathbf{i}+3\mathbf{j})-(-15\mathbf{i}+20\mathbf{j})=12\mathbf{i}-17\mathbf{j}$

12. **(a)** $\langle 8,0\rangle-\langle 0,-6\rangle=\langle 8,6\rangle$ **(b)** $\langle -6,0\rangle-\langle 0,-15\rangle=\langle -6,15\rangle$

13. **(a)** $\langle 16,40\rangle-\langle -4,-12\rangle=\langle 20,52\rangle$ **(b)** $\langle -12,-30\rangle-\langle -10,-30\rangle=\langle -2,0\rangle$

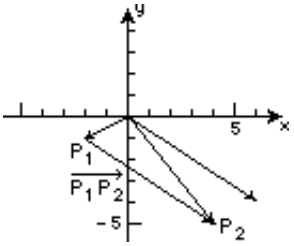
14. **(a)** $\langle 8,12\rangle-\langle 10,6\rangle=\langle -2,6\rangle$ **(b)** $\langle -6,-9\rangle-\langle 25,15\rangle=\langle -31,-24\rangle$

15.



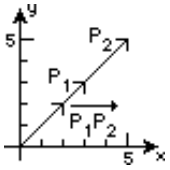
$\overrightarrow{P_1P_2}=\langle 2,5\rangle$

16.



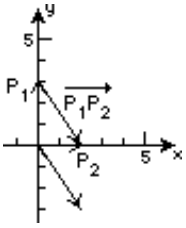
$\overrightarrow{P_1P_2}=\langle 6,-4\rangle$

17.



$\overrightarrow{P_1P_2}=\langle 2,2\rangle$

18.



$\overrightarrow{P_1P_2}=\langle 2,-3\rangle$

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19. Since $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$, $\overrightarrow{OP_2} = \overrightarrow{P_1P_2} + \overrightarrow{OP_1} = (4\mathbf{i} + 8\mathbf{j}) + (-3\mathbf{i} + 10\mathbf{j}) = \mathbf{i} + 18\mathbf{j}$, and the terminal point is $(1, 18)$.

20. Since $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$, $\overrightarrow{OP_1} = \overrightarrow{OP_2} - \overrightarrow{P_1P_2} = \langle 4, 7\rangle - \langle -5, -1\rangle = \langle 9, 8\rangle$, and the initial point is $(9, 8)$.

21. $a(= -\mathbf{a})$, $b(= -\frac{1}{4}\mathbf{a})$, $c(= \frac{5}{2}\mathbf{a})$, $e(= 2\mathbf{a})$, and $f(= -\frac{1}{2}\mathbf{a})$ are parallel to $\mathbf{a}$.

22. We want $-3\mathbf{b} = \mathbf{a}$, so $c = -3(9) = -27$.

23. $\langle 6, 15\rangle$

24. $\langle 5, 2\rangle$

25. $\|\mathbf{a}\| = \sqrt{4+4} = 2\sqrt{2}$; **(a)** $\mathbf{u} = \frac{1}{2\sqrt{2}}\langle 2, 2\rangle = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$; **(b)** $-\mathbf{u} = \langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle$

26. $\|\mathbf{a}\| = \sqrt{9+16} = 5$; **(a)** $\mathbf{u} = \frac{1}{5}\langle -3, 4\rangle = \langle -\frac{3}{5}, \frac{4}{5}\rangle$; **(b)** $-\mathbf{u} = \langle \frac{3}{5}, -\frac{4}{5}\rangle$

27. $\|\mathbf{a}\| = 5$; **(a)** $\mathbf{u} = \frac{1}{5}\langle 0, -5\rangle = \langle 0, -1\rangle$; **(b)** $-\mathbf{u} = \langle 0, 1\rangle$

28. $\|\mathbf{a}\| = \sqrt{1+3} = 2$; **(a)** $\mathbf{u} = \frac{1}{2}\langle 1, -\sqrt{3}\rangle = \langle \frac{1}{2}, -\frac{\sqrt{3}}{2}\rangle$; **(b)** $-\mathbf{u} = \langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$

29. $\|\mathbf{a}+\mathbf{b}\| = \|\langle 5, 12\rangle\| = \sqrt{25+144} = 13$; $\mathbf{u} = \frac{1}{13}\langle 5, 12\rangle = \langle \frac{5}{13}, \frac{12}{13}\rangle$

30. $\|\mathbf{a}+\mathbf{b}\| = \|\langle -5, 4\rangle\| = \sqrt{25+16} = \sqrt{41}$; $\mathbf{u} = \frac{1}{\sqrt{41}}\langle -5, 4\rangle = \langle -\frac{5}{\sqrt{41}}, \frac{4}{\sqrt{41}}\rangle$

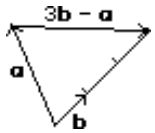
31. $\|\mathbf{a}\| = \sqrt{9+49} = \sqrt{58}$; $\mathbf{b} = 2(\frac{1}{\sqrt{58}})(3\mathbf{i} + 7\mathbf{j}) = \frac{6}{\sqrt{58}}\mathbf{i} + \frac{14}{\sqrt{58}}\mathbf{j}$

32. $\|\mathbf{a}\| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}$; $\mathbf{b} = 3(\frac{1}{1/\sqrt{2}})(\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}) = \frac{3\sqrt{2}}{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$

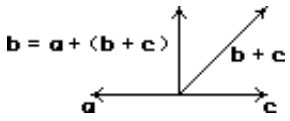
33. $-\frac{3}{4}\mathbf{a} = \langle -3, -15/2\rangle$

34. $5(\mathbf{a}+\mathbf{b}) = 5\langle 0, 1\rangle = \langle 0, 5\rangle$

35.



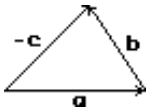
36.



37. $\mathbf{x} = -(\mathbf{a}+\mathbf{b}) = -\mathbf{a} - \mathbf{b}$

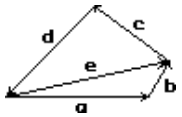
38. $\mathbf{x} = 2(\mathbf{a}-\mathbf{b}) = 2\mathbf{a} - 2\mathbf{b}$

39.



$\mathbf{b} = (-\mathbf{c}) - \mathbf{a}$; $(\mathbf{b}+\mathbf{c}) + \mathbf{a} = \mathbf{0}$; $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$

40.



From Problem 39, $\mathbf{e}+\mathbf{c}+\mathbf{d} = \mathbf{0}$. But $\mathbf{b} = \mathbf{e} - \mathbf{a}$ and $\mathbf{e} = \mathbf{a}+\mathbf{b}$, so $(\mathbf{a}+\mathbf{b}) + \mathbf{c} + \mathbf{d} = \mathbf{0}$.

41. From $2\mathbf{i} + 3\mathbf{j} = k_1\mathbf{b} + k_2\mathbf{c} = k_1(\mathbf{i}+\mathbf{j}) + k_2(\mathbf{i}-\mathbf{j}) = (k_1+k_2)\mathbf{i} + (k_1-k_2)\mathbf{j}$ we obtain the system of equations $k_1 + k_2 = 2$, $k_1 - k_2 = 3$. Solving, we find $k_1 = \frac{5}{2}$ and $k_2 = -\frac{1}{2}$. Then $\mathbf{a} = \frac{5}{2}\mathbf{b} - \frac{1}{2}\mathbf{c}$.

42. From $2\mathbf{i} + 3\mathbf{j} = k_1\mathbf{b} + k_2\mathbf{c} = k_1(-2\mathbf{i} + 4\mathbf{j}) + k_2(5\mathbf{i} + 7\mathbf{j}) = (-2k_1 + 5k_2)\mathbf{i} + (4k_1 + 7k_2)\mathbf{j}$ we obtain the system of equations $-2k_1 + 5k_2 = 2$, $4k_1 + 7k_2 = 3$. Solving, we find $k_1 = \frac{1}{34}$ and $k_2 = \frac{7}{17}$.

43. From $y' = \frac{1}{2}x$ we see that the slope of the tangent line at $(2, 2)$ is 1. A vector with slope 1 is $\mathbf{i}+\mathbf{j}$. A unit vector is $(\mathbf{i}+\mathbf{j})/\|\mathbf{i}+\mathbf{j}\| = (\mathbf{i}+\mathbf{j})/\sqrt{2} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. Another unit vector tangent to the curve is $-\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$.

44. From $y' = -2x + 3$ we see that the slope of the tangent line at $(0, 0)$ is 3. A vector with slope 3 is $\mathbf{i} + 3\mathbf{j}$. A unit vector is $(\mathbf{i}+3\mathbf{j})/\|\mathbf{i}+3\mathbf{j}\| = (\mathbf{i}+3\mathbf{j})/\sqrt{10} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j}$. Another unit vector is $-\frac{1}{\sqrt{10}}\mathbf{i} - \frac{1}{\sqrt{10}}\mathbf{j}$.

45. **(a)** Since $\mathbf{F}_f = -\mathbf{F}_g$, $\|\mathbf{F}_g\| = \|\mathbf{F}_f\| = \mu\|\mathbf{F}_n\|$ and $\tan\theta = \|\mathbf{F}_g\|/\|\mathbf{F}_n\| = \mu\|\mathbf{F}_n\|/\|\mathbf{F}_n\| = \mu$.

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(b) $\theta = \tan^{-1} 0.6 \approx 31°$.

46. Since $\mathbf{w} + \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0}$,

$$-200\mathbf{j} + \|\mathbf{F}_1\| \cos 20°\mathbf{i} + \|\mathbf{F}_1\| \sin 20°\mathbf{j} - \|\mathbf{F}_2\| \cos 15°\mathbf{i} + \|\mathbf{F}_2\| \sin 15°\mathbf{j} = \mathbf{0}$$

or $$(\|\mathbf{F}_1\| \cos 20° - \|\mathbf{F}_2\| \cos 15°)\mathbf{i} + (\|\mathbf{F}_1\| \sin 20° + \|\mathbf{F}_2\| \sin 15° - 200)\mathbf{j} = \mathbf{0}.$$

Thus, $\|\mathbf{F}_1\| \cos 20° - \|\mathbf{F}_2\| \cos 15° = 0$; $\|\mathbf{F}_1\| \sin 20° + \|\mathbf{F}_2\| \sin 15° - 200 = 0$. Solving this system for $\|\mathbf{F}_1\|$ and $\|\mathbf{F}_2\|$, we obtain

$$\|\mathbf{F}_1\| = \frac{200 \cos 15°}{\sin 15° \cos 20° + \cos 15° \sin 20°} = \frac{200 \cos 15°}{\sin(15° + 20°)} = \frac{200 \cos 15°}{\sin 35°} \approx 336.8 \text{ lb}$$

and

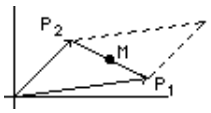
$$\|\mathbf{F}_2\| = \frac{200 \cos 20°}{\sin 15° \cos 20° + \cos 15° \sin 20°} = \frac{200 \cos 20°}{\sin 35°} \approx 327.7 \text{ lb}.$$

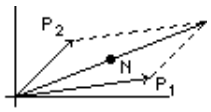
47. Since $y/2a(L^2 + y^2)^{3/2}$ is an odd function on $[-a, a]$, $F_y = 0$. Now, using the fact that $L/(L^2 + y^2)^{3/2}$ is an even function, we have

$$\begin{aligned}
\int_{-a}^{a} \frac{L\,dy}{2a(L^2 + y^2)^{3/2}} &= \frac{L}{a} \int_0^a \frac{dy}{(L^2 + y^2)^{3/2}} \qquad \boxed{y = L\tan\theta,\ dy = L\sec^2\theta\,d\theta} \\
&= \frac{L}{a} \int_0^{\tan^{-1} a/L} \frac{L\sec^2\theta\,d\theta}{L^3(1 + \tan^2\theta)^{3/2}} = \frac{1}{La} \int_0^{\tan^{-1} a/L} \frac{\sec^2\theta\,d\theta}{\sec^3\theta} \\
&= \frac{1}{La} \int_0^{\tan^{-1} a/L} \cos\theta\,d\theta = \frac{1}{La} \sin\theta \Big|_0^{\tan^{-1} a/L} \\
&= \frac{1}{La} \frac{a}{\sqrt{L^2 + a^2}} = \frac{1}{L\sqrt{L^2 + a^2}}.
\end{aligned}$$

Then $F_x = qQ/4\pi\epsilon_0 L\sqrt{L^2 + a^2}$ and $\mathbf{F} = (qQ/4\pi\epsilon_0 L\sqrt{L^2 + a^2}\,)\mathbf{i}$.

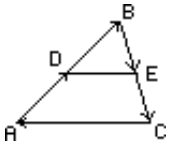
48. Place one corner of the parallelogram at the origin and let two adjacent sides be $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$. Let M be the midpoint of the diagonal connecting P_1 and P_2 and N be the midpoint of the other diagonal.





By Problem 37, $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2})$. Since $\overrightarrow{OP_1} + \overrightarrow{OP_2}$ is the main diagonal of the parallelogram and N is its midpoint, $\overrightarrow{ON} = \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2})$. Thus, $\overrightarrow{OM} = \overrightarrow{ON}$ and the diagonals bisect each other.

49. By Problem 39, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$ and $\overrightarrow{AD} + \overrightarrow{DE} + \overrightarrow{EC} + \overrightarrow{CA} = \mathbf{0}$. From the first equation, $\overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{CA}$. Since D and E are midpoints, $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB}$ and $\overrightarrow{EC} = \frac{1}{2}\overrightarrow{BC}$. Then, $\frac{1}{2}\overrightarrow{AB} + \overrightarrow{DE} + \frac{1}{2}\overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$ and



$$\overrightarrow{DE} = -\overrightarrow{CA} - \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = -\overrightarrow{CA} - \frac{1}{2}(-\overrightarrow{CA}) = -\frac{1}{2}\overrightarrow{CA}.$$

Thus, the line segment joining the midpoints D and E is parallel to the side AC and half its length.

50. We have $\overrightarrow{OA} = 150\cos 20°\mathbf{i} + 150\sin 20°\mathbf{j}$, $\overrightarrow{AB} = 200\cos 113°\mathbf{i} + 200\sin 113°\mathbf{j}$, $\overrightarrow{BC} = 240\cos 190°\mathbf{i} + 240\sin 190°\mathbf{j}$. Then

$$\begin{aligned}
\mathbf{r} &= (150\cos 20° + 200\cos 113° + 240\cos 190°)\mathbf{i} + (150\sin 20° + 200\sin 113° + 240\sin 190°)\mathbf{j} \\
&\approx -173.55\mathbf{i} + 193.73\mathbf{j}
\end{aligned}$$

and $\|\mathbf{r}\| \approx 260.09$ miles.

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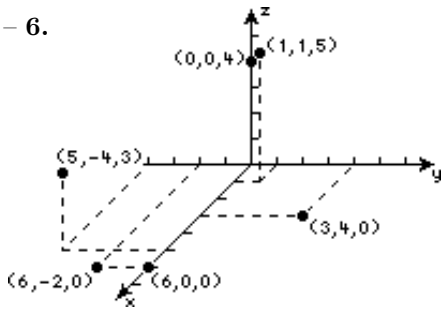
Exercises 7.2

1. – **6.**



7. A plane perpendicular to the z-axis, 5 units above the xy-plane

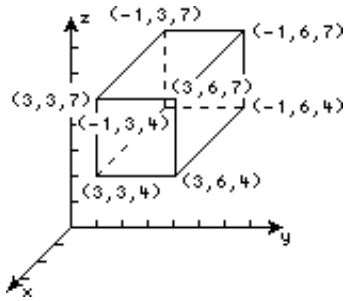
8. A plane perpendicular to the x-axis, 1 unit in front of the yz-plane

9. A line perpendicular to the xy-plane at $(2, 3, 0)$

10. A single point located at $(4, -1, 7)$

11. $(2, 0, 0)$, $(2, 5, 0)$, $(2, 0, 8)$, $(2, 5, 8)$, $(0, 5, 0)$, $(0, 5, 8)$, $(0, 0, 8)$, $(0, 0, 0)$

12.



13. **(a)** xy-plane: $(-2, 5, 0)$, xz-plane: $(-2, 0, 4)$, yz-plane: $(0, 5, 4)$; **(b)** $(-2, 5, -2)$

(c) Since the shortest distance between a point and a plane is a perpendicular line, the point in the plane $x = 3$ is $(3, 5, 4)$.

14. We find planes that are parallel to coordinate planes: **(a)** $z = -5$; **(b)** $x = 1$ and $y = -1$; **(c)** $z = 2$

15. The union of the planes $x = 0$, $y = 0$, and $z = 0$

16. The origin $(0, 0, 0)$

17. The point $(-1, 2, -3)$

18. The union of the planes $x = 2$ and $z = 8$

19. The union of the planes $z = 5$ and $z = -5$

20. The line through the points $(1, 1, 1)$, $(-1, -1, -1)$, and the origin

21. $d = \sqrt{(3-6)^2 + (-1-4)^2 + (2-8)^2} = \sqrt{70}$

22. $d = \sqrt{(-1-0)^2 + (-3-4)^2 + (5-3)^2} = 3\sqrt{6}$

23. **(a)** 7; **(b)** $d = \sqrt{(-3)^2 + (-4)^2} = 5$

24. **(a)** 2; **(b)** $d = \sqrt{(-6)^2 + 2^2 + (-3)^2} = 7$

25. $d(P_1, P_2) = \sqrt{3^2 + 6^2 + (-6)^2} = 9$; $d(P_1, P_3) = \sqrt{2^2 + 1^2 + 2^2} = 3$

$d(P_2, P_3) = \sqrt{(2-3)^2 + (1-6)^2 + (2-(-6))^2} = \sqrt{90}$; The triangle is a right triangle.

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26. $d(P_1, P_2) = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}\,; \quad d(P_1, P_3) = \sqrt{3^2 + 2^2 + (2\sqrt{2})^2} = \sqrt{21}$

$d(P_2, P_3) = \sqrt{(3-1)^2 + (2-2)^2 + (2\sqrt{2}-4)^2} = \sqrt{28 - 16\sqrt{2}}$

The triangle is an isosceles triangle.

27. $d(P_1, P_2) = \sqrt{(4-1)^2 + (1-2)^2 + (3-3)^2} = \sqrt{10}$

$d(P_1, P_3) = \sqrt{(4-1)^2 + (6-2)^2 + (4-3)^2} = \sqrt{26}$

$d(P_2, P_3) = \sqrt{(4-4)^2 + (6-1)^2 + (4-3)^2} = \sqrt{26}\,;$ The triangle is an isosceles triangle.

28. $d(P_1, P_2) = \sqrt{(1-1)^2 + (1-1)^2 + (1-(-1))^2} = 2$

$d(P_1, P_3) = \sqrt{(0-1)^2 + (-1-1)^2 + (1-(-1))^2} = 3$

$d(P_2, P_3) = \sqrt{(0-1)^2 + (-1-1)^2 + (1-1)^2} = \sqrt{5}\,;$ The triangle is a right triangle.

29. $d(P_1, P_2) = \sqrt{(-2-1)^2 + (-2-2)^2 + (-3-0)^2} = \sqrt{34}$

$d(P_1, P_3) = \sqrt{(7-1)^2 + (10-2)^2 + (6-0)^2} = 2\sqrt{34}$

$d(P_2, P_3) = \sqrt{(7-(-2))^2 + (10-(-2))^2 + (6-(-3))^2} = 3\sqrt{34}$

Since $d(P_1, P_2) + d(P_1, P_3) = d(P_2, P_3)$, the points P_1, P_2, and P_3 are collinear.

30. $d(P_1, P_2) = \sqrt{(1-2)^2 + (4-3)^2 + (4-2)^2} = \sqrt{6}$

$d(P_1, P_3) = \sqrt{(5-2)^2 + (0-3)^2 + (-4-2)^2} = 3\sqrt{6}$

$d(P_2, P_3) = \sqrt{(5-1)^2 + (0-4)^2 + (-4-4)^2} = 4\sqrt{6}$

Since $d(P_1, P_2) + d(P_1, P_3) = d(P_2, P_3)$, the points P_1, P_2, and P_3 are collinear.

31. $\sqrt{(2-x)^2 + (1-2)^2 + (1-3)^2} = \sqrt{21} \implies x^2 - 4x + 9 = 21 \implies x^2 - 4x + 4 = 16$

$\implies (x-2)^2 = 16 \implies x = 2 \pm 4$ or $x = 6,\ -2$

32. $\sqrt{(0-x)^2 + (3-x)^2 + (5-1)^2} = 5 \implies 2x^2 - 6x + 25 = 25 \implies x^2 - 3x = 0 \implies x = 0,\ 3$

33. $\left(\dfrac{1+7}{2}, \dfrac{3+(-2)}{2}, \dfrac{1/2+5/2}{2}\right) = (4, 1/2, 3/2)$

34. $\left(\dfrac{0+4}{2}, \dfrac{5+1}{2}, \dfrac{-8+(-6)}{2}\right) = (2, 3, -7)$

35. $(x_1 + 2)/2 = -1,\ \ x_1 = -4; \quad (y_1 + 3)/2 = -4,\ \ y_1 = -11; \quad (z_1 + 6)/2 = 8, \quad z_1 = 10$

The coordinates of P_1 are $(-4, -11, 10)$.

36. $(-3 + (-5))/2 = x_3 = -4; \quad (4 + 8)/2 = y_3 = 6; \quad (1 + 3)/2 = z_3 = 2.$

The coordinates of P_3 are $(-4, 6, 2)$.

(a) $\left(\dfrac{-3+(-4)}{2}, \dfrac{4+6}{2}, \dfrac{1+2}{2}\right) = (-7/2, 5, 3/2)$

(b) $\left(\dfrac{-4+(-5)}{2}, \dfrac{6+8}{2}, \dfrac{2+3}{2}\right) = (-9/2, 7, 5/2)$

37. $\overrightarrow{P_1P_2} = \langle -3, -6, 1\rangle$

38. $\overrightarrow{P_1P_2} = \langle 8, -5/2, 8\rangle$

39. $\overrightarrow{P_1P_2} = \langle 2, 1, 1\rangle$

40. $\overrightarrow{P_1P_2}\langle -3, -3, 7\rangle$

41. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = \langle 2, 4, 12\rangle$

42. $2\mathbf{a} - (\mathbf{b} - \mathbf{c}) = \langle 2, -6, 4\rangle - \langle -3, -5, -8\rangle = \langle 5, -1, 12\rangle$

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43. $\mathbf{b} + 2(\mathbf{a} - 3\mathbf{c}) = \langle -1, 1, 1\rangle + 2\langle -5, -21, -25\rangle = \langle -11, -41, -49\rangle$

44. $4(\mathbf{a} + 2\mathbf{c}) - 6\mathbf{b} = 4\langle 5, 9, 20\rangle - \langle -6, 6, 6\rangle = \langle 26, 30, 74\rangle$

45. $\|\mathbf{a} + \mathbf{c}\| = \|\langle 3, 3, 11\rangle\| = \sqrt{9 + 9 + 121} = \sqrt{139}$

46. $\|\mathbf{c}\|\|2\mathbf{b}\| = (\sqrt{4 + 36 + 81}\,)(2)(\sqrt{1 + 1 + 1}\,) = 22\sqrt{3}$

47. $\left\| \dfrac{\mathbf{a}}{\|\mathbf{a}\|} \right\| + 5 \left\| \dfrac{\mathbf{b}}{|\mathbf{b}\|} \right\| = \dfrac{1}{\|\mathbf{a}\|}\|\mathbf{a}\| + 5\dfrac{1}{\|\mathbf{b}\|}\|\mathbf{b}\| = 1 + 5 = 6$

48. $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b} = \sqrt{1 + 1 + 1}\,\langle 1, -3, 2\rangle + \sqrt{1 + 9 + 4}\,\langle -1, 1, 1\rangle = \langle \sqrt{3}\,, -3\sqrt{3}\,, 2\sqrt{3}\,\rangle + \langle -\sqrt{14}\,, \sqrt{14}\,, \sqrt{14}\,\rangle$
$= \langle \sqrt{3} - \sqrt{14}\,, -3\sqrt{3} + \sqrt{14}\,, 2\sqrt{3} + \sqrt{14}\,\rangle$

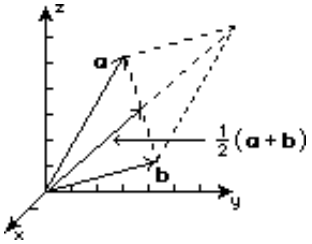
49. $\|\mathbf{a}\| = \sqrt{100 + 25 + 100} = 15; \quad \mathbf{u} = -\dfrac{1}{15}\langle 10, -5, 10\rangle = \langle -2/3, 1/3, -2/3\rangle$

50. $\|\mathbf{a}\| = \sqrt{1 + 9 + 4} = \sqrt{14}; \quad \mathbf{u} = \dfrac{1}{\sqrt{14}}(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) = \dfrac{1}{\sqrt{14}}\mathbf{i} - \dfrac{3}{\sqrt{14}}\mathbf{j} + \dfrac{2}{\sqrt{14}}\mathbf{k}$

51. $\mathbf{b} = 4\mathbf{a} = 4\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$

52. $\|\mathbf{a}\| = \sqrt{36 + 9 + 4} = 7; \quad \mathbf{b} = -\dfrac{1}{2}\left(\dfrac{1}{7}\right)\langle -6, 3, -2\rangle = \langle \dfrac{3}{7}\,, -\dfrac{3}{14}\,, \dfrac{1}{7}\rangle$

53.



Exercises 7.3

1. $\mathbf{a} \cdot \mathbf{b} = 10(5)\cos(\pi/4) = 25\sqrt{2}$

2. $\mathbf{a} \cdot \mathbf{b} = 6(12)\cos(\pi/6) = 36\sqrt{3}$

3. $\mathbf{a} \cdot \mathbf{b} = 2(-1) + (-3)2 + 4(5) = 12$

4. $\mathbf{b} \cdot \mathbf{c} = (-1)3 + 2(6) + 5(-1) = 4$

5. $\mathbf{a} \cdot \mathbf{c} = 2(3) + (-3)6 + 4(-1) = -16$

6. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = 2(2) + (-3)8 + 4(4) = -4$

7. $\mathbf{a} \cdot (4b) = 2(-4) + (-3)8 + 4(20) = 48$

8. $\mathbf{b} \cdot (\mathbf{a} - \mathbf{c}) = (-1)(-1) + 2(-9) + 5(5) = 8$

9. $\mathbf{a} \cdot \mathbf{a} = 2^2 + (-3)^2 + 4^2 = 29$

10. $(2\mathbf{b}) \cdot (3\mathbf{c}) = (-2)9 + 4(18) + 10(-3) = 24$

11. $\mathbf{a} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) = 2(4) + (-3)5 + 4(8) = 25$

12. $(2\mathbf{a}) \cdot (\mathbf{a} - 2\mathbf{b}) = 4(4) + (-6)(-7) + 8(-6) = 10$

13. $\left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right)\mathbf{b} = \left[\dfrac{2(-1) + (-3)2 + 4(5)}{(-1)^2 + 2^2 + 5^2}\right]\langle -1, 2, 5\rangle = \dfrac{12}{30}\langle -1, 2, 5\rangle = \langle -2/5, 4/5, 2\rangle$

14. $(\mathbf{c} \cdot \mathbf{b})\mathbf{a} = [3(-1) + 6(2) + (-1)5]\langle 2, -3, 4\rangle = 4\langle 2, -3, 4\rangle = \langle 8, -12, 16\rangle$

15. a and f, b and e, c and d

16. **(a)** $\mathbf{a} \cdot \mathbf{b} = 2 \cdot 3 + (-c)2 + 3(4) = 0 \implies c = 9$

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(b) $\mathbf{a}\cdot\mathbf{b} = c(-3) + \frac{1}{2}(4) + c^2 = c^2 - 3c + 2 = (c-2)(c-1) = 0 \implies c = 1,\ 2$

17. Solving the system of equations $3x_1 + y_1 - 1 = 0$, $-3x_1 + 2y_1 + 2 = 0$ gives $x_1 = 4/9$ and $y_1 = -1/3$. Thus, $\mathbf{v} = \langle 4/9, -1/3, 1\rangle$.

18. If $\mathbf{a}$ and $\mathbf{b}$ represent adjacent sides of the rhombus, then $\|\mathbf{a}\| = \|\mathbf{b}\|$, the diagonals of the rhombus are $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, and

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b} = \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 = 0.$$

Thus, the diagonals are perpendicular.

19. Since

$$\mathbf{c}\cdot\mathbf{a} = \left(\mathbf{b} - \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}\right)\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}(\mathbf{a}\cdot\mathbf{a}) = \mathbf{b}\cdot\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\|\mathbf{a}\|^2 = \mathbf{b}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} = 0,$$

the vectors $\mathbf{c}$ and $\mathbf{a}$ are orthogonal.

20. $\mathbf{a}\cdot\mathbf{b} = 1(1) + c(1) = c+1$; $\|\mathbf{a}\| = \sqrt{1+c^2}$, $\|\mathbf{b}\| = \sqrt{2}$

$\cos 45^\circ = \dfrac{1}{\sqrt{2}} = \dfrac{c+1}{\sqrt{1+c^2}\sqrt{2}} \implies \sqrt{1+c^2} = c+1 \implies 1 + c^2 = c^2 + 2c + 1 \implies c = 0$

21. $\mathbf{a}\cdot\mathbf{b} = 3(2) + (-1)2 = 4$; $\|\mathbf{a}\| = \sqrt{10}$, $\|\mathbf{b}\| = 2\sqrt{2}$

$\cos\theta = \dfrac{4}{(\sqrt{10})(2\sqrt{2})} = \dfrac{1}{\sqrt{5}} \implies \theta = \cos^{-1}\dfrac{1}{\sqrt{5}} \approx 1.11 \text{ rad} \approx 63.43^\circ$

22. $\mathbf{a}\cdot\mathbf{b} = 2(-3) + 1(-4) = -10$; $\|\mathbf{a}\| = \sqrt{5}$, $\|\mathbf{b}\| = 5$

$\cos\theta = \dfrac{-10}{(\sqrt{5})5} = -\dfrac{2}{\sqrt{5}} \implies \theta = \cos^{-1}(-2/\sqrt{5}) \approx 2.68 \text{ rad} \approx 153.43^\circ$

23. $\mathbf{a}\cdot\mathbf{b} = 2(-1) + 4(-1) + 0(4) = -6$; $\|\mathbf{a}\| = 2\sqrt{5}$, $\|\mathbf{b}\| = 3\sqrt{2}$

$\cos\theta = \dfrac{-6}{(2\sqrt{5})(3\sqrt{2})} = -\dfrac{1}{\sqrt{10}} \implies \theta = \cos^{-1}(-1/\sqrt{10}) \approx 1.89 \text{ rad} \approx 108.43^\circ$

24. $\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(2) + \frac{1}{2}(-4) + \frac{3}{2}(6) = 8$; $\|\mathbf{a}\| = \sqrt{11}/2$, $\|\mathbf{b}\| = 2\sqrt{14}$

$\cos\theta = \dfrac{8}{(\sqrt{11}/2)(2\sqrt{14})} = \dfrac{8}{\sqrt{154}} \implies \theta = \cos^{-1}(8/\sqrt{154}) \approx 0.87 \text{ rad} \approx 49.86^\circ$

25. $\|\mathbf{a}\| = \sqrt{14}$; $\cos\alpha = 1/\sqrt{14}$, $\alpha \approx 74.50^\circ$; $\cos\beta = 2/\sqrt{14}$, $\beta \approx 57.69^\circ$; $\cos\gamma = 3/\sqrt{14}$, $\gamma \approx 36.70^\circ$

26. $\|\mathbf{a}\| = 9$; $\cos\alpha = 2/3$, $\alpha \approx 48.19^\circ$; $\cos\beta = 2/3$, $\beta \approx 48.19^\circ$; $\cos\gamma = -1/3$, $\gamma \approx 109.47^\circ$

27. $\|\mathbf{a}\| = 2$; $\cos\alpha = 1/2$, $\alpha = 60^\circ$; $\cos\beta = 0$, $\beta = 90^\circ$; $\cos\gamma = -\sqrt{3}/2$, $\gamma = 150^\circ$

28. $\|\mathbf{a}\| = \sqrt{78}$; $\cos\alpha = 5/\sqrt{78}$, $\alpha \approx 55.52^\circ$; $\cos\beta = 7/\sqrt{78}$, $\beta \approx 37.57^\circ$; $\cos\gamma = 2/\sqrt{78}$, $\gamma \approx 76.91^\circ$

29. Let θ be the angle between $\overrightarrow{AD}$ and $\overrightarrow{AB}$ and a be the length of an edge of the cube. Then $\overrightarrow{AD} = a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$, $\overrightarrow{AB} = a\mathbf{i}$ and

$$\cos\theta = \frac{\overrightarrow{AD}\cdot\overrightarrow{AB}}{\|\overrightarrow{AD}\|\,\|\overrightarrow{AB}\|} = \frac{a^2}{\sqrt{3a^2}\sqrt{a^2}} = \frac{1}{\sqrt{3}}$$

so $\theta \approx 0.955317$ radian or 54.7356°. Letting ϕ be the angle between $\overrightarrow{AD}$ and $\overrightarrow{AC}$ and noting that $\overrightarrow{AC} = a\mathbf{i} + a\mathbf{j}$ we have

$$\cos\phi = \frac{\overrightarrow{AD}\cdot\overrightarrow{AC}}{\|\overrightarrow{AD}\|\,\|\overrightarrow{AC}\|} = \frac{a^2 + a^2}{\sqrt{3a^2}\sqrt{2a^2}} = \sqrt{\frac{2}{3}}$$

so $\phi \approx 0.61548$ radian or 35.2644°.

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30. If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, then $\mathbf{a}\cdot\mathbf{b}=0$ and

$$\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2=\frac{a_1}{\|\mathbf{a}\|}\frac{b_1}{\|\mathbf{b}\|}+\frac{a_2}{\|\mathbf{a}\|}\frac{b_2}{\|\mathbf{b}\|}+\frac{a_3}{\|\mathbf{a}\|}\frac{b_3}{\|\mathbf{b}\|}$$
$$=\frac{1}{\|\mathbf{a}\|\,\|\mathbf{b}\|}(a_1b_1+a_2b_2+a_3b_3)=\frac{1}{\|\mathbf{a}\|\,\|\mathbf{b}\|}(\mathbf{a}\cdot\mathbf{b})=0.$$

31. $\mathbf{a}=\langle 5,7,4\rangle$; $\|\mathbf{a}\|=3\sqrt{10}$; $\cos\alpha=5/3\sqrt{10}$, $\alpha\approx 58.19°$; $\cos\beta=7/3\sqrt{10}$, $\beta\approx 42.45°$; $\cos\gamma=4/3\sqrt{10}$, $\gamma\approx 65.06°$

32. We want $\cos\alpha=\cos\beta=\cos\gamma$ or $a_1=a_2=a_3$. Letting $a_1=a_2=a_3=1$ we obtain the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$. A unit vector in the same direction is $\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}$.

33. $\text{comp}_{\mathbf{b}}\mathbf{a}=\mathbf{a}\cdot\mathbf{b}/\|\mathbf{b}\|=\langle 1,-1,3\rangle\cdot\langle 2,6,3\rangle/7=5/7$

34. $\text{comp}_{\mathbf{a}}\mathbf{b}=\mathbf{b}\cdot\mathbf{a}/\|\mathbf{a}\|=\langle 2,6,3\rangle\cdot\langle 1,-1,3\rangle/\sqrt{11}=5/\sqrt{11}$

35. $\mathbf{b}-\mathbf{a}=\langle 1,7,0\rangle$; $\text{comp}_{\mathbf{a}}(\mathbf{b}-\mathbf{a})=(\mathbf{b}-\mathbf{a})\cdot\mathbf{a}/\|\mathbf{a}\|=\langle 1,7,0\rangle\cdot\langle 1,-1,3\rangle/\sqrt{11}=-6/\sqrt{11}$

36. $\mathbf{a}+\mathbf{b}=\langle 3,5,6\rangle$; $2\mathbf{b}=\langle 4,12,6\rangle$; $\text{comp}_{2\mathbf{b}}(\mathbf{a}+\mathbf{b})\cdot 2\mathbf{b}/|2\mathbf{b}|=\langle 3,5,6\rangle\cdot\langle 4,12,6\rangle/14=54/7$

37. $\overrightarrow{OP}=3\mathbf{i}+10\mathbf{j}$; $\|\overrightarrow{OP}\|=\sqrt{109}$; $\text{comp}_{\overrightarrow{OP}}\mathbf{a}=\mathbf{a}\cdot\overrightarrow{OP}/\|\overrightarrow{OP}\|=(4\mathbf{i}+6\mathbf{j})\cdot(3\mathbf{i}+10\mathbf{j})/\sqrt{109}=72/\sqrt{109}$

38. $\overrightarrow{OP}=\langle 1,-1,1\rangle$; $\|\overrightarrow{OP}\|=\sqrt{3}$; $\text{comp}_{\overrightarrow{OP}}\mathbf{a}=\mathbf{a}\cdot\overrightarrow{OP}/\|\overrightarrow{OP}\|=\langle 2,1,-1\rangle\cdot\langle 1,-1,1\rangle/\sqrt{3}=0$

39. **(a)** $\text{comp}_{\mathbf{b}}\mathbf{a}=\mathbf{a}\cdot\mathbf{b}/\|\mathbf{b}\|=(-5\mathbf{i}+5\mathbf{j})\cdot(-3\mathbf{i}+4\mathbf{j})/5=7$
$\text{proj}_{\mathbf{b}}\mathbf{a}=(\text{comp}_{\mathbf{b}}\mathbf{a})\mathbf{b}/\|\mathbf{b}\|=7(-3\mathbf{i}+4\mathbf{j})/5=-\frac{21}{5}\mathbf{i}+\frac{28}{5}\mathbf{j}$

(b) $\text{proj}_{\mathbf{b}^\perp}\mathbf{a}=\mathbf{a}-\text{proj}_{\mathbf{b}}\mathbf{a}=(-5\mathbf{i}+5\mathbf{j})-(-\frac{21}{5}\mathbf{i}+\frac{28}{5}\mathbf{j})=-\frac{4}{5}\mathbf{i}-\frac{3}{5}\mathbf{j}$

40. **(a)** $\text{comp}_{\mathbf{b}}\mathbf{a}=\mathbf{a}\cdot\mathbf{b}/\|\mathbf{b}\|=(4\mathbf{i}+2\mathbf{j})\cdot(-3\mathbf{i}+\mathbf{j})/\sqrt{10}=-\sqrt{10}$
$\text{proj}_{\mathbf{b}}\mathbf{a}=(\text{comp}_{\mathbf{b}}\mathbf{a})\mathbf{b}/\|\mathbf{b}\|=-\sqrt{10}(-3\mathbf{i}+\mathbf{j})/\sqrt{10}=3\mathbf{i}-\mathbf{j}$

(b) $\text{proj}_{\mathbf{b}^\perp}\mathbf{a}=\mathbf{a}-\text{proj}_{\mathbf{b}}\mathbf{a}=(4\mathbf{i}+2\mathbf{j})-(3\mathbf{i}-\mathbf{j})=\mathbf{i}+3\mathbf{j}$

41. **(a)** $\text{comp}_{\mathbf{b}}\mathbf{a}=\mathbf{a}\cdot\mathbf{b}/\|\mathbf{b}\|=(-\mathbf{i}-2\mathbf{j}+7\mathbf{k})\cdot(6\mathbf{i}-3\mathbf{j}-2\mathbf{k})/7=-2$
$\text{proj}_{\mathbf{b}}\mathbf{a}=(\text{comp}_{\mathbf{b}}\mathbf{a})\mathbf{b}/\|\mathbf{b}\|=-2(6\mathbf{i}-3\mathbf{j}-2\mathbf{k})/7=-\frac{12}{7}\mathbf{i}+\frac{6}{7}\mathbf{j}+\frac{4}{7}\mathbf{k}$

(b) $\text{proj}_{\mathbf{b}^\perp}\mathbf{a}=\mathbf{a}-\text{proj}_{\mathbf{b}}\mathbf{a}=(-\mathbf{i}-2\mathbf{j}+7\mathbf{k})-(-\frac{12}{7}\mathbf{i}+\frac{6}{7}\mathbf{j}+\frac{4}{7}\mathbf{k})=\frac{5}{7}\mathbf{i}-\frac{20}{7}\mathbf{j}+\frac{45}{7}\mathbf{k}$

42. **(a)** $\text{comp}_{\mathbf{b}}\mathbf{a}=\mathbf{a}\cdot\mathbf{b}/\|\mathbf{b}\|=\langle 1,1,1\rangle\cdot\langle -2,2,-1\rangle/3=-1/3$
$\text{proj}_{\mathbf{b}}\mathbf{a}=(\text{comp}_{\mathbf{b}}\mathbf{a})\mathbf{b}/\|\mathbf{b}\|=-\frac{1}{3}\langle -2,2,-1\rangle/3=\langle 2/9,-2/9,1/9\rangle$

(b) $\text{proj}_{\mathbf{b}^\perp}\mathbf{a}=\mathbf{a}-\text{proj}_{\mathbf{b}}\mathbf{a}=\langle 1,1,1\rangle-\langle 2/9,-2/9,1/9\rangle=\langle 7/9,11/9,8/9\rangle$

43. $\mathbf{a}+\mathbf{b}=3\mathbf{i}+4\mathbf{j}$; $\|\mathbf{a}+\mathbf{b}\|=5$; $\text{comp}_{(\mathbf{a}+\mathbf{b})}\mathbf{a}=\mathbf{a}\cdot(\mathbf{a}+\mathbf{b})/\|\mathbf{a}+\mathbf{b}\|=(4\mathbf{i}+3\mathbf{j})\cdot(3\mathbf{i}+4\mathbf{j})/5=24/5$
$\text{proj}_{(\mathbf{a}+\mathbf{b})}\mathbf{a}=(\text{comp}_{(\mathbf{a}+\mathbf{b})}\mathbf{a})(\mathbf{a}+\mathbf{b})/\|\mathbf{a}+\mathbf{b}\|=\frac{24}{5}(3\mathbf{i}+4\mathbf{j})/5=\frac{72}{25}\mathbf{i}+\frac{96}{25}\mathbf{j}$

44. $\mathbf{a}-\mathbf{b}=5\mathbf{i}+2\mathbf{j}$; $\|\mathbf{a}-\mathbf{b}\|=\sqrt{29}$; $\text{comp}_{(\mathbf{a}-\mathbf{b})}\mathbf{b}=\mathbf{b}\cdot(\mathbf{a}-\mathbf{b})/\|\mathbf{a}-\mathbf{b}\|=(-\mathbf{i}+\mathbf{j})\cdot(5\mathbf{i}+2\mathbf{j})/\sqrt{29}=-3/\sqrt{29}$
$\text{proj}_{(\mathbf{a}-\mathbf{b})}\mathbf{b}=(\text{comp}_{(\mathbf{a}-\mathbf{b})}\mathbf{b})(\mathbf{a}-\mathbf{b})/\|\mathbf{a}-\mathbf{b}\|=-\frac{3}{\sqrt{29}}(5\mathbf{i}+2\mathbf{j})/\sqrt{29}=-\frac{15}{29}\mathbf{i}-\frac{6}{29}\mathbf{j}$
$\text{proj}_{(\mathbf{a}-\mathbf{b})^\perp}\mathbf{b}=\mathbf{b}-\text{proj}_{(\mathbf{a}-\mathbf{b})}\mathbf{b}=(-\mathbf{i}+\mathbf{j})-(-\frac{15}{29}\mathbf{i}-\frac{6}{29}\mathbf{j})=-\frac{14}{29}\mathbf{i}+\frac{35}{29}\mathbf{j}$

45. We identify $\|\mathbf{F}\|=20$, $\theta=60°$ and $\|\mathbf{d}\|=100$. Then $W=\|\mathbf{F}\|\,\|\mathbf{d}\|\cos\theta=20(100)(\frac{1}{2})=1000$ ft-lb.

46. We identify $\mathbf{d}=-\mathbf{i}+3\mathbf{j}+8\mathbf{k}$. Then $W=\mathbf{F}\cdot\mathbf{d}=\langle 4,3,5\rangle\cdot\langle -1,3,8\rangle=45$ N-m.

47. **(a)** Since $\mathbf{w}$ and $\mathbf{d}$ are orthogonal, $W=\mathbf{w}\cdot\mathbf{d}=0$.

(b) We identify $\theta=0°$. Then $W=\|\mathbf{F}\|\,\|\mathbf{d}\|\cos\theta=30(\sqrt{4^2+3^2})=150$ N-m.

48. Using $\mathbf{d}=6\mathbf{i}+2\mathbf{j}$ and $\mathbf{F}=3(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j})$, $W=\mathbf{F}\cdot\mathbf{d}=\langle\frac{9}{5},\frac{12}{5}\rangle\cdot\langle 6,2\rangle=\frac{78}{5}$ ft-lb.

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49. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors from the center of the carbon atom to the centers of two distinct hydrogen atoms. The distance between two hydrogen atoms is then

$$\begin{aligned}\|\mathbf{b}-\mathbf{a}\| &= \sqrt{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})} = \sqrt{\mathbf{b}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{a}} \\ &= \sqrt{\|\mathbf{b}\|^2+\|\mathbf{a}\|^2-2\|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta} = \sqrt{(1.1)^2+(1.1)^2-2(1.1)(1.1)\cos 109.5^\circ} \\ &= \sqrt{1.21+1.21-2.42(-0.333807)} \approx 1.80 \text{ angstroms.}\end{aligned}$$

50. Using the fact that $|\cos\theta| \le 1$, we have $|\mathbf{a}\cdot\mathbf{b}| = \|\mathbf{a}\|\,\|\mathbf{b}\||\cos\theta| = \|\mathbf{a}\|\,\|\mathbf{b}\||\cos\theta| \le \|\mathbf{a}\|\,\|\mathbf{b}\|$.

51. $\|\mathbf{a}+\mathbf{b}\|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}+2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b} = \|\mathbf{a}\|^2+2\mathbf{a}\cdot\mathbf{b}+\|\mathbf{b}\|^2$

$\le \|\mathbf{a}\|^2+2|\mathbf{a}\cdot\mathbf{b}|+\|\mathbf{b}\|^2$ $\boxed{\text{since } x \le |x|}$

$\le \|\mathbf{a}\|^2+2\|\mathbf{a}\|\,\|\mathbf{b}\|+\|\mathbf{b}\|^2 = (\|\mathbf{a}\|+\|\mathbf{b}\|)^2$ $\boxed{\text{by Problem 50}}$

Thus, since $\|\mathbf{a}+\mathbf{b}\|$ and $\|\mathbf{a}\|+\|\mathbf{b}\|$ are positive, $\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\|+\|\mathbf{b}\|$.

52. Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ be distinct points on the line $ax+by=-c$. Then

$$\begin{aligned}\mathbf{n}\cdot\overrightarrow{P_1P_2} &= \langle a,b\rangle\cdot\langle x_2-x_1,y_2-y_1\rangle = ax_2-ax_1+by_2-by_1 \\ &= (ax_2+by_2)-(ax_1+by_1) = -c-(-c) = 0,\end{aligned}$$

and the vectors are perpendicular. Thus, $\mathbf{n}$ is perpendicular to the line.

53. Let θ be the angle between $\mathbf{n}$ and $\overrightarrow{P_2P_1}$. Then

$$\begin{aligned}d &= \|\overrightarrow{P_1P_2}\|\,|\cos\theta| = \frac{|\mathbf{n}\cdot\overrightarrow{P_2P_1}|}{\|\mathbf{n}\|} = \frac{|\langle a,b\rangle\cdot\langle x_1-x_2,y_1-y_2\rangle|}{\sqrt{a^2+b^2}} = \frac{|ax_1-ax_2+by_1-by_2|}{\sqrt{a^2+b^2}} \\ &= \frac{|ax_1+by_1-(ax_2+by_2)|}{\sqrt{a^2+b^2}} = \frac{|ax_1+by_1-(-c)|}{\sqrt{a^2+b^2}} = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}\,.\end{aligned}$$

Exercises 7.4

1. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&0\\0&3&5\end{vmatrix} = \begin{vmatrix}-1&0\\3&5\end{vmatrix}\mathbf{i} - \begin{vmatrix}1&0\\0&5\end{vmatrix}\mathbf{j} + \begin{vmatrix}1&-1\\0&3\end{vmatrix}\mathbf{k} = -5\mathbf{i}-5\mathbf{j}+3\mathbf{k}$

2. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&0\\4&0&-1\end{vmatrix} = \begin{vmatrix}1&0\\0&-1\end{vmatrix}\mathbf{i} - \begin{vmatrix}2&0\\4&-1\end{vmatrix}\mathbf{j} + \begin{vmatrix}2&1\\4&0\end{vmatrix}\mathbf{k} = -\mathbf{i}+2\mathbf{j}-4\mathbf{k}$

3. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-3&1\\2&0&4\end{vmatrix} = \begin{vmatrix}-3&1\\0&4\end{vmatrix}\mathbf{i} - \begin{vmatrix}1&1\\2&4\end{vmatrix}\mathbf{j} + \begin{vmatrix}1&-3\\2&0\end{vmatrix}\mathbf{k} = \langle -12,-2,6\rangle$

4. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&1\\-5&2&3\end{vmatrix} = \begin{vmatrix}1&1\\2&3\end{vmatrix}\mathbf{i} - \begin{vmatrix}1&1\\-5&3\end{vmatrix}\mathbf{j} + \begin{vmatrix}1&1\\-5&2\end{vmatrix}\mathbf{k} = \langle 1,-8,7\rangle$

5. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&2\\-1&3&-1\end{vmatrix} = \begin{vmatrix}-1&2\\3&-1\end{vmatrix}\mathbf{i} - \begin{vmatrix}2&2\\-1&-1\end{vmatrix}\mathbf{j} + \begin{vmatrix}2&-1\\-1&3\end{vmatrix}\mathbf{k} = -5\mathbf{i}+5\mathbf{k}$

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6. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&1&-5\\2&3&-1\end{vmatrix}=\begin{vmatrix}1&-5\\3&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}4&-5\\2&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}4&1\\2&3\end{vmatrix}\mathbf{k}=14\mathbf{i}-6\mathbf{j}+10\mathbf{k}$

7. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1/2&0&1/2\\4&6&0\end{vmatrix}=\begin{vmatrix}0&1/2\\6&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1/2&1/2\\4&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1/2&0\\4&6\end{vmatrix}\mathbf{k}=\langle-3,2,3\rangle$

8. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&5&0\\2&-3&4\end{vmatrix}=\begin{vmatrix}5&0\\-3&4\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&0\\2&4\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&5\\2&-3\end{vmatrix}\mathbf{k}=\langle20,0,-10\rangle$

9. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&2&-4\\-3&-3&6\end{vmatrix}=\begin{vmatrix}2&-4\\-3&6\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&-4\\-3&6\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&2\\-3&-3\end{vmatrix}\mathbf{k}=\langle0,0,0\rangle$

10. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\8&1&-6\\1&-2&10\end{vmatrix}=\begin{vmatrix}1&-6\\-2&10\end{vmatrix}\mathbf{i}-\begin{vmatrix}8&-6\\1&10\end{vmatrix}\mathbf{j}+\begin{vmatrix}8&1\\1&-2\end{vmatrix}\mathbf{k}=\langle-2,-86,-17\rangle$

11. $\overrightarrow{P_1P_2}=(-2,2,-4)$; $\overrightarrow{P_1P_3}=(-3,1,1)$

$\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2&2&-4\\-3&1&1\end{vmatrix}=\begin{vmatrix}2&-4\\1&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}-2&-4\\-3&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}-2&2\\-3&1\end{vmatrix}\mathbf{k}=6\mathbf{i}+14\mathbf{j}+4\mathbf{k}$

12. $\overrightarrow{P_1P_2}=(0,1,1)$; $\overrightarrow{P_1P_3}=(1,2,2)$; $\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&1\\1&2&2\end{vmatrix}=\begin{vmatrix}1&1\\2&2\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&1\\1&2\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&1\\1&2\end{vmatrix}\mathbf{k}=\mathbf{j}-\mathbf{k}$

13. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&7&-4\\1&1&-1\end{vmatrix}=\begin{vmatrix}7&-4\\1&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&-4\\1&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&7\\1&1\end{vmatrix}\mathbf{k}=-3\mathbf{i}-2\mathbf{j}-5\mathbf{k}$

is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

14. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&-2&4\\4&-1&0\end{vmatrix}=\begin{vmatrix}-2&4\\-1&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1&4\\4&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1&-2\\4&-1\end{vmatrix}\mathbf{k}=\langle4,16,9\rangle$

is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

15. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\5&-2&1\\2&0&-7\end{vmatrix}=\begin{vmatrix}-2&1\\0&-7\end{vmatrix}\mathbf{i}-\begin{vmatrix}5&1\\2&-7\end{vmatrix}\mathbf{j}+\begin{vmatrix}5&-2\\2&0\end{vmatrix}\mathbf{k}=\langle14,37,4\rangle$

$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=\langle5,-2,-1\rangle\cdot\langle14,37,4\rangle=70-74+4=0$; $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})=\langle2,0,-7\rangle\cdot\langle14,37,4\rangle=28+0-28=0$

16. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1/2&-1/4&0\\2&-2&6\end{vmatrix}=\begin{vmatrix}-1/4&0\\-2&6\end{vmatrix}\mathbf{i}-\begin{vmatrix}1/2&0\\2&6\end{vmatrix}\mathbf{j}+\begin{vmatrix}1/2&-1/4\\2&-2\end{vmatrix}\mathbf{k}=-\frac{3}{2}\mathbf{i}-3\mathbf{j}-\frac{1}{2}\mathbf{k}$

$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=(\frac{1}{2}\mathbf{i}-\frac{1}{4}\mathbf{j})\cdot(-\frac{3}{2}\mathbf{i}-3\mathbf{j}-\frac{1}{2}\mathbf{k})=-\frac{3}{4}+\frac{3}{4}+0=0$

$\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})=(2\mathbf{i}-2\mathbf{j}+6\mathbf{k})\cdot(-\frac{3}{2}\mathbf{i}-3\mathbf{j}-\frac{1}{2}\mathbf{k})=-3+6-3=0$

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17. **(a)** $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&1\\3&1&1\end{vmatrix}=\begin{vmatrix}1&1\\1&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&1\\3&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&1\\3&1\end{vmatrix}\mathbf{k}=\mathbf{j}-\mathbf{k}$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&2\\0&1&-1\end{vmatrix}=\begin{vmatrix}-1&2\\1&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&2\\0&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&-1\\0&1\end{vmatrix}\mathbf{k}=-\mathbf{i}+\mathbf{j}+\mathbf{k}$

(b) $\mathbf{a}\cdot\mathbf{c}=(\mathbf{i}-\mathbf{j}+2\mathbf{k})\cdot(3\mathbf{i}+\mathbf{j}+\mathbf{k})=4;\ \ (\mathbf{a}\cdot\mathbf{c})\mathbf{b}=4(2\mathbf{i}+\mathbf{j}+\mathbf{k})=8\mathbf{i}+4\mathbf{j}+4\mathbf{k}$

$\mathbf{a}\cdot\mathbf{b}=(\mathbf{i}-\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}+\mathbf{k})=3;\ \ (\mathbf{a}\cdot\mathbf{b})\mathbf{c}=3(3\mathbf{i}+\mathbf{j}+\mathbf{k})=9\mathbf{i}+3\mathbf{j}+3\mathbf{k}$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}=(8\mathbf{i}+4\mathbf{j}+4\mathbf{k})-(9\mathbf{i}+3\mathbf{j}+3\mathbf{k})=-\mathbf{i}+\mathbf{j}+\mathbf{k}$

18. **(a)** $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-1\\-1&5&8\end{vmatrix}=\begin{vmatrix}2&-1\\5&8\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&-1\\-1&8\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&2\\-1&5\end{vmatrix}\mathbf{k}=21\mathbf{i}-7\mathbf{j}+7\mathbf{k}$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&0&-4\\21&-7&7\end{vmatrix}=\begin{vmatrix}0&-4\\-7&7\end{vmatrix}\mathbf{i}-\begin{vmatrix}3&-4\\21&7\end{vmatrix}\mathbf{j}+\begin{vmatrix}3&0\\21&-7\end{vmatrix}\mathbf{k}=-28\mathbf{i}-105\mathbf{j}-21\mathbf{k}$

(b) $\mathbf{a}\cdot\mathbf{c}=(3\mathbf{i}-4\mathbf{k})\cdot(-\mathbf{i}+5\mathbf{j}+8\mathbf{k})=-35;\ \ (\mathbf{a}\cdot\mathbf{c})\mathbf{b}=-35(\mathbf{i}+2\mathbf{j}-\mathbf{k})=-35\mathbf{i}-70\mathbf{j}+35\mathbf{k}$

$\mathbf{a}\cdot\mathbf{b}=(3\mathbf{i}-4\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-\mathbf{k})=7;\ \ (\mathbf{a}\cdot\mathbf{b})\mathbf{c}=7(-\mathbf{i}+5\mathbf{j}+8\mathbf{k})=-7\mathbf{i}+35\mathbf{j}+56\mathbf{k}$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}=(-35\mathbf{i}-70\mathbf{j}+35\mathbf{k})-(-7\mathbf{i}+35\mathbf{j}+56\mathbf{k})=-28\mathbf{i}-105\mathbf{j}-21\mathbf{k}$

19. $(2\mathbf{i})\times\mathbf{j}=2(\mathbf{i}\times\mathbf{j})=2\mathbf{k}$

20. $\mathbf{i}\times(-3\mathbf{k})=-3(\mathbf{i}\times\mathbf{k})=-3(-\mathbf{j})=3\mathbf{j}$

21. $\mathbf{k}\times(2\mathbf{i}-\mathbf{j})=\mathbf{k}\times(2\mathbf{i})+\mathbf{k}\times(-\mathbf{j})=2(\mathbf{k}\times\mathbf{i})-(\mathbf{k}\times\mathbf{j})=2\mathbf{j}-(-\mathbf{i})=\mathbf{i}+2\mathbf{j}$

22. $\mathbf{i}\times(\mathbf{j}\times\mathbf{k})=\mathbf{i}\times\mathbf{i}=\mathbf{0}$

23. $[(2\mathbf{k})\times(3\mathbf{j})]\times(4\mathbf{j})=[2\cdot3(\mathbf{k}\times\mathbf{j})\times(4\mathbf{j})]=6(-\mathbf{i})\times4\mathbf{j}=(-6)(4)(\mathbf{i}\times\mathbf{j})=-24\mathbf{k}$

24. $(2\mathbf{i}-\mathbf{j}+5\mathbf{k})\times\mathbf{i}=(2\mathbf{i}\times\mathbf{i})+(-\mathbf{j}\times\mathbf{i})+(5\mathbf{k}\times\mathbf{i})=2(\mathbf{i}\times\mathbf{i})+(\mathbf{i}\times\mathbf{j})+5(\mathbf{k}\times\mathbf{i})=5\mathbf{j}+\mathbf{k}$

25. $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}+5\mathbf{k})=[(\mathbf{i}+\mathbf{j})\times\mathbf{i}]+[(\mathbf{i}+\mathbf{j})\times5\mathbf{k}]=(\mathbf{i}\times\mathbf{i})+(\mathbf{j}\times\mathbf{i})+(\mathbf{i}\times5\mathbf{k})+(\mathbf{j}\times5\mathbf{k})$

$=-\mathbf{k}+5(-\mathbf{j})+5\mathbf{i}=5\mathbf{i}-5\mathbf{j}-\mathbf{k}$

26. $\mathbf{i}\times\mathbf{k}-2(\mathbf{j}\times\mathbf{i})=-\mathbf{j}-2(-\mathbf{k})=-\mathbf{j}+2\mathbf{k}$

27. $\mathbf{k}\cdot(\mathbf{j}\times\mathbf{k})=\mathbf{k}\cdot\mathbf{i}=0$

28. $\mathbf{i}\cdot[\mathbf{j}\times(-\mathbf{k})]=\mathbf{i}\cdot[-(\mathbf{j}\times\mathbf{k})]=\mathbf{i}\cdot(-\mathbf{i})=-(\mathbf{i}\cdot\mathbf{i})=-1$

29. $\|4\mathbf{j}-5(\mathbf{i}\times\mathbf{j})\|=\|4\mathbf{j}-5\mathbf{k}\|=\sqrt{41}$

30. $(\mathbf{i}\times\mathbf{j})\cdot(3\mathbf{j}\times\mathbf{i})=\mathbf{k}\cdot(-3\mathbf{k})=-3(\mathbf{k}\cdot\mathbf{k})=-3$

31. $\mathbf{i}\times(\mathbf{i}\times\mathbf{j})=\mathbf{i}\times\mathbf{k}=-\mathbf{j}$

32. $(\mathbf{i}\times\mathbf{j})\times\mathbf{i}=\mathbf{k}\times\mathbf{i}=\mathbf{j}$

33. $(\mathbf{i}\times\mathbf{i})\times\mathbf{j}=\mathbf{0}\times\mathbf{j}=\mathbf{0}$

34. $(\mathbf{i}\cdot\mathbf{i})(\mathbf{i}\times\mathbf{j})=1(\mathbf{k})=\mathbf{k}$

35. $2\mathbf{j}\cdot[\mathbf{i}\times(\mathbf{j}-3\mathbf{k})]=2\mathbf{j}\cdot[(\mathbf{i}\times\mathbf{j})+(\mathbf{i}\times(-3\mathbf{k})]=2\mathbf{j}\cdot[\mathbf{k}+3(\mathbf{k}\times\mathbf{i})]=2\mathbf{j}\cdot(\mathbf{k}+3\mathbf{j})=2\mathbf{j}\cdot\mathbf{k}+2\mathbf{j}\cdot3\mathbf{j}$

$=2(\mathbf{j}\cdot\mathbf{k})+6(\mathbf{j}\cdot\mathbf{j})=2(0)+6(1)=6$

36. $(\mathbf{i}\times\mathbf{k})\times(\mathbf{j}\times\mathbf{i})=(-\mathbf{j})\times(-\mathbf{k})=(-1)(-1)(\mathbf{j}\times\mathbf{k})=\mathbf{j}\times\mathbf{k}=\mathbf{i}$

37. $\mathbf{a}\times(3\mathbf{b})=3(\mathbf{a}\times\mathbf{b})=3(4\mathbf{i}-3\mathbf{j}+6\mathbf{k})=12\mathbf{i}-9\mathbf{j}+18\mathbf{k}$

38. $\mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}=-(\mathbf{a}\times\mathbf{b})=-4\mathbf{i}+3\mathbf{j}-6\mathbf{k}$

39. $(-\mathbf{a})\times\mathbf{b}=-(\mathbf{a}\times\mathbf{b})=-4\mathbf{i}+3\mathbf{j}-6\mathbf{k}$

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40. $|\mathbf{a}\times\mathbf{b}| = \sqrt{4^2+(-3)^2+6^2} = \sqrt{61}$

41. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & 6 \\ 2 & 4 & -1 \end{vmatrix} = \begin{vmatrix} -3 & 6 \\ 4 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & 6 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & -3 \\ 2 & -4 \end{vmatrix}\mathbf{k} = -21\mathbf{i}+16\mathbf{j}+22\mathbf{k}$

42. $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = 4(2)+(-3)4+6(-1) = -10$

43. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = 4(2)+(-3)4+6(-1) = -10$

44. $(4\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c}) = (4\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = 4(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = 16(2)+(-12)4+24(-1) = -40$

45. **(a)** Let $A=(1,3,0)$, $B=(2,0,0)$, $C=(0,0,4)$, and $D=(1,-3,4)$. Then $\overrightarrow{AB} = \mathbf{i}-3\mathbf{j}$, $\overrightarrow{AC} = -\mathbf{i}-3\mathbf{j}+4\mathbf{k}$, $\overrightarrow{CD} = \mathbf{i}-3\mathbf{j}$, and $\overrightarrow{BD} = -\mathbf{i}-3\mathbf{j}+4\mathbf{k}$. Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the quadrilateral is a parallelogram.

(b) Computing

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 0 \\ -1 & -3 & 4 \end{vmatrix} = -12\mathbf{i}-4\mathbf{j}-6\mathbf{k}$$

we find that the area is $\|-12\mathbf{i}-4\mathbf{j}-6\mathbf{k}\| = \sqrt{144+16+36} = 14$.

46. **(a)** Let $A=(3,4,1)$, $B=(-1,4,2)$, $C=(2,0,2)$ and $D=(-2,0,3)$. Then $\overrightarrow{AB} = -4\mathbf{i}+\mathbf{k}$, $\overrightarrow{AC} = -\mathbf{i}-4\mathbf{j}+\mathbf{k}$, $\overrightarrow{CD} = -4\mathbf{i}+\mathbf{k}$, and $\overrightarrow{BD} = -\mathbf{i}-4\mathbf{j}+\mathbf{k}$. Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the quadrilateral is a parallelogram.

(b) Computing

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 0 & 1 \\ -1 & -4 & 1 \end{vmatrix} = 4\mathbf{i}+3\mathbf{j}+16\mathbf{k}$$

we find that the area is $\|4\mathbf{i}+3\mathbf{j}+16\mathbf{k}\| = \sqrt{16+9+256} = \sqrt{281} \approx 16.76$.

47. $\overrightarrow{P_1P_2} = \mathbf{j}$; $\overrightarrow{P_2P_3} = -\mathbf{j}+\mathbf{k}$

$$\overrightarrow{P_1P_2}\times\overrightarrow{P_2P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix}\mathbf{k} = \mathbf{i}; \quad A = \tfrac{1}{2}\|\mathbf{i}\| = \tfrac{1}{2} \text{ sq. unit}$$

48. $\overrightarrow{P_1P_2} = \mathbf{j}+2\mathbf{k}$; $\overrightarrow{P_2P_3} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$

$$\overrightarrow{P_1P_2}\times\overrightarrow{P_2P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\mathbf{k} = -4\mathbf{i}+4\mathbf{j}-2\mathbf{k}$$

$A = \frac{1}{2}\|-4\mathbf{i}+4\mathbf{j}-2\mathbf{k}\| = 3$ sq. units

49. $\overrightarrow{P_1P_2} = -3\mathbf{j}-\mathbf{k}$; $\overrightarrow{P_2P_3} = -2\mathbf{i}-\mathbf{k}$

$$\overrightarrow{P_1P_2}\times\overrightarrow{P_2P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & -1 \\ -2 & 0 & -1 \end{vmatrix} = \begin{vmatrix} -3 & -1 \\ 0 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -1 \\ -2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & -3 \\ -2 & 0 \end{vmatrix}\mathbf{k} = 3\mathbf{i}+2\mathbf{j}-6\mathbf{k}$$

$A = \frac{1}{2}\|3\mathbf{i}+2\mathbf{j}-6\mathbf{k}\| = \frac{7}{2}$ sq. units

50. $\overrightarrow{P_1P_2} = -\mathbf{i}+3\mathbf{k}$; $\overrightarrow{P_2P_3} = 2\mathbf{i}+4\mathbf{j}-\mathbf{k}$

$$\overrightarrow{P_1P_2}\times\overrightarrow{P_2P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 3 \\ 2 & 4 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 4 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 0 \\ 2 & 4 \end{vmatrix}\mathbf{k} = -12\mathbf{i}+5\mathbf{j}-4\mathbf{k}$$

$A = \frac{1}{2}\|-12\mathbf{i}+5\mathbf{j}-4\mathbf{k}\| = \frac{\sqrt{185}}{2}$ sq. units

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51. $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&4&0\\2&2&2\end{vmatrix}=\begin{vmatrix}4&0\\2&2\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1&0\\2&2\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1&4\\2&2\end{vmatrix}\mathbf{k}=8\mathbf{i}+2\mathbf{j}-10\mathbf{k}$

$\mathbf{v}=|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|=|(\mathbf{i}+\mathbf{j})\cdot(8\mathbf{i}+2\mathbf{j}-10\mathbf{k})|=|8+2+0|=10$ cu. units

52. $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&4&1\\1&1&5\end{vmatrix}=\begin{vmatrix}4&1\\1&5\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&1\\1&5\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&4\\1&1\end{vmatrix}\mathbf{k}=19\mathbf{i}-4\mathbf{j}-3\mathbf{k}$

$\mathbf{v}=|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|=|(3\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(19\mathbf{i}-4\mathbf{j}-3\mathbf{k})|=|57-4-3|=50$ cu. units

53. $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2&6&-6\\5/2&3&1/2\end{vmatrix}=\begin{vmatrix}6&-6\\3&2/2\end{vmatrix}\mathbf{i}-\begin{vmatrix}-2&-6\\5/2&1/2\end{vmatrix}\mathbf{j}+\begin{vmatrix}-2&6\\5/2&3\end{vmatrix}\mathbf{k}=21\mathbf{i}-14\mathbf{j}-21\mathbf{k}$

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(4\mathbf{i}+6\mathbf{j})\cdot(21\mathbf{i}-14\mathbf{j}-21\mathbf{k})=84-84+0=0$. The vectors are coplanar.

54. The four points will be coplanar if the three vectors $\overrightarrow{P_1P_2}=\langle 3,-1,-1\rangle$, $\overrightarrow{P_2P_3}=\langle -3,-5,13\rangle$, and $\overrightarrow{P_3P_4}=\langle -8,7,-6\rangle$ are coplanar.

$\overrightarrow{P_2P_3}\times\overrightarrow{P_3P_4}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-3&-5&13\\-8&7&-6\end{vmatrix}=\begin{vmatrix}-5&13\\7&-6\end{vmatrix}\mathbf{i}-\begin{vmatrix}-3&13\\-8&-6\end{vmatrix}\mathbf{j}+\begin{vmatrix}-3&-5\\-8&7\end{vmatrix}\mathbf{k}=\langle -61,-122,-61\rangle$

$\overrightarrow{P_1P_2}\cdot(\overrightarrow{P_2P_3}\times\overrightarrow{P_3P_4})=\langle 3,-1,-1\rangle\cdot\langle -61,-122,-61\rangle=-183+122+61=0$

The four points are coplanar.

55. **(a)** Since $\theta=90°$, $\|\mathbf{a}\times\mathbf{b}\|=\|\mathbf{a}\|\,\|\mathbf{b}\|\,|\sin 90°|=6.4(5)=32$.

(b) The direction of $\mathbf{a}\times\mathbf{b}$ is into the fourth quadrant of the xy-plane or to the left of the plane determined by $\mathbf{a}$ and $\mathbf{b}$ as shown in Figure 6.54 in the text. It makes an angle of 30° with the positive x-axis.

(c) We identify $\mathbf{n}=(\sqrt{3}\,\mathbf{i}-\mathbf{j})/2$. Then $\mathbf{a}\times\mathbf{b}=32\mathbf{n}=16\sqrt{3}\,\mathbf{i}-16\mathbf{j}$.

56. Using Definition 7.4, $\mathbf{a}\times\mathbf{b}=\sqrt{27}\,(8)\sin 120°\mathbf{n}=24\sqrt{3}\,(\sqrt{3}/2)\mathbf{n}=36\mathbf{n}$. By the right-hand rule, $\mathbf{n}=\mathbf{j}$ or $\mathbf{n}=-\mathbf{j}$. Thus, $\mathbf{a}\times\mathbf{b}=36\mathbf{j}$ or $-36\mathbf{j}$.

57. **(a)** We note first that $\mathbf{a}\times\mathbf{b}=\mathbf{k}$, $\mathbf{b}\times\mathbf{c}=\frac{1}{2}(\mathbf{i}-\mathbf{k})$, $\mathbf{c}\times\mathbf{a}=\frac{1}{2}(\mathbf{j}-\mathbf{k})$, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\frac{1}{2}$, $\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\frac{1}{2}$, and $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=\frac{1}{2}$. Then

$$\mathbf{A}=\frac{\frac{1}{2}(\mathbf{i}-\mathbf{k})}{\frac{1}{2}}=\mathbf{i}-\mathbf{k},\quad \mathbf{B}=\frac{\frac{1}{2}(\mathbf{j}-\mathbf{k})}{\frac{1}{2}}=\mathbf{j}-\mathbf{k},\quad \text{and}\quad \mathbf{C}=\frac{\mathbf{k}}{\frac{1}{2}}=2\mathbf{k}.$$

(b) We need to compute $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. Using formula (10) in the text we have

$$\mathbf{B}\times\mathbf{C}=\frac{(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})}{[\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})][\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})]}=\frac{[(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}]\mathbf{a}-[(\mathbf{c}\times\mathbf{a})\cdot\mathbf{a}]\mathbf{b}}{[\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})][\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})]}$$
$$=\frac{\mathbf{a}}{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}\qquad \boxed{\text{since } (\mathbf{c}\times\mathbf{a})\cdot\mathbf{a}=0.}$$

Then

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}\cdot\frac{\mathbf{a}}{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}=\frac{1}{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}$$

and the volume of the unit cell of the reciprocal latrice is the reciprocal of the volume of the unit cell of the original lattice.

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58. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \begin{vmatrix} a_2 & a_3 \\ b_2+c_2 & b_3+c_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1+c_1 & b_3+c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix}\mathbf{k}$

$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_2c_3 - a_3c_2)\mathbf{i} - [(a_1b_3 - a_3b_1)\mathbf{j} + (a_1c_3 - a_3c_1)\mathbf{j}] + (a_1b_2 - a_2b_1)\mathbf{k} + (a_1c_2 - a_2c_1)\mathbf{k}$

$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} + (a_2c_3 - a_3c_2)\mathbf{i} - (a_1c_3 - a_3c_1)\mathbf{j} + (a_1c_2 - a_2c_1)\mathbf{k}$

$= \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$

59. $\mathbf{b}\times\mathbf{c} = (b_2c_3 - b_3c_2)\mathbf{i} - (b_1c_3 - b_3c_1)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k}$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = [a_2(b_1c_2 - b_2c_1) + a_3(b_1c_3 - b_3c_1)]\mathbf{i} - [a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - b_3c_2)]\mathbf{j}$

$+ [-a_1(b_1c_3 - b_3c_1) - a_2(b_2c_3 - b_3c_2)]\mathbf{k}$

$= (a_2b_1c_2 - a_2b_2c_1 + a_3b_1c_3 - a_3b_3c_1)\mathbf{i} - (a_1b_1c_2 - a_1b_2c_1 - a_3b_2c_3 + a_3b_3c_2)\mathbf{j}$

$- (a_1b_1c_3 - a_1b_3c_1 + a_2b_2c_3 - a_2b_3c_2)\mathbf{k}$

$(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = (a_1c_1 + a_2c_2 + a_3c_3)(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k})$

$= (a_2b_1c_2 - a_2b_2c_1 + a_3b_1c_3 - a_3b_3c_1)\mathbf{i} - (a_1b_1c_2 - a_1b_2c_1 - a_3b_2c_3 + a_3b_3c_2)\mathbf{j}$

$- (a_1b_1c_3 - a_1b_3c_1 + a_2b_2c_3 - a_2b_3c_2)\mathbf{k}$

60. The statement is false since $\mathbf{i}\times(\mathbf{i}\times\mathbf{j}) = \mathbf{i}\times\mathbf{k} = -\mathbf{j}$ and $(\mathbf{i}\times\mathbf{i})\times\mathbf{j} = \mathbf{0}\times\mathbf{j} = \mathbf{0}$.

61. Using equation 9 in the text,

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{and} \quad (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

The second determinant can be obtained from the first by an interchange of the second and third rows followed by an interchange of the new first and second rows. Using Theorem 8.11 in the text, we see that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.

62. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b})$

$= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{b}\cdot\mathbf{a})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} + (\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}$

$= [(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}] + [(\mathbf{b}\cdot\mathbf{a})\mathbf{c} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}] + [(\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}] = \mathbf{0}$

63. Since

$\|\mathbf{a}\times\mathbf{b}\|^2 = (a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2$

$= a_2^2b_3^2 - 2a_2b_3a_3b_2 + a_3^2b_2^2 + a_1^2b_3^2 - 2a_1b_3a_3b_1 + a_3^2b_1^2 + a_1^2b_2^2 - 2a_1b_2a_2b_1 + a_2^2b_1^2$

and

$\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$

$= a_1^2a_2^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + a_3^2b_3^2$

$- a_1^2b_1^2 - a_2^2b_2^2 - a_3^2b_3^2 - 2a_1b_1a_2b_2 - 2a_1b_1a_3b_3 - 2a_2b_2a_3b_3$

$= a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 - 2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3$

we see that $\|\mathbf{a}\times\mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2$.

64. No. For example $\mathbf{i}\times(\mathbf{i}+\mathbf{j}) = \mathbf{i}\times\mathbf{j}$ by the distributive law (**iii**) in the text, and the fact that $\mathbf{i}\times\mathbf{i} = \mathbf{0}$. But $\mathbf{i}+\mathbf{j}$ does not equal $\mathbf{j}$.

65. By the distributive law (**iii**) in the text:

$$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = (\mathbf{a}+\mathbf{b})\times\mathbf{a} - (\mathbf{a}+\mathbf{b})\times\mathbf{b} = \mathbf{a}\times\mathbf{a} + \mathbf{b}\times\mathbf{a} - \mathbf{a}\times\mathbf{b} - \mathbf{b}\times\mathbf{b} = 2\mathbf{b}\times\mathbf{a}$$

since $\mathbf{a}\times\mathbf{a} = \mathbf{0}$, $\mathbf{b}\times\mathbf{b} = \mathbf{0}$, and $-\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{a}$.

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Exercises 7.5

The equation of a line through P_1 and P_2 in 3-space with $\mathbf{r}_1 = \overrightarrow{OP_1}$ and $\mathbf{r}_2 = \overrightarrow{OP_2}$ can be expressed as $\mathbf{r} = \mathbf{r}_1 + t(k\mathbf{a})$ or $\mathbf{r} = \mathbf{r}_2 + t(k\mathbf{a})$ where $\mathbf{a} = \mathbf{r}_2 - \mathbf{r}_1$ and k is any non-zero scalar. Thus, the form of the equation of a line is not unique. (See the alternate solution to Problem 1.)

1. $\mathbf{a} = \langle 1-3, 2-5, 1-(-2)\rangle = \langle -2, -3, 3\rangle$; $\langle x, y, z\rangle = \langle 1, 2, 1\rangle + t\langle -2, -3, 3\rangle$
Alternate Solution: $\mathbf{a} = \langle 3-1, 5-2, -2-1\rangle = \langle 2, 3, -3\rangle$; $\langle x, y, z\rangle = \langle 3, 5, -2\rangle + t\langle 2, 3, -3\rangle$

2. $\mathbf{a} = \langle 0-(-2), 4-6, 5-3\rangle = \langle 2, -2, 2\rangle$; $\langle x, y, z\rangle = \langle 0, 4, 5\rangle + t\langle 2, -2, 2\rangle$

3. $\mathbf{a} = \langle 1/2-(-3/2), -1/2-5/2, 1-(-1/2)\rangle = \langle 2, -3, 3/2\rangle$; $\langle x, y, z\rangle = \langle 1/2, -1/2, 1\rangle + t\langle 2, -3, 3/2\rangle$

4. $\mathbf{a} = \langle 10-5, 2-(-3), -10-5\rangle = \langle 5, 5, -15\rangle$; $\langle x, y, z\rangle = \langle 10, 2, -10\rangle + t\langle 5, 5, -15\rangle$

5. $\mathbf{a} = \langle 1-(-4), 1-1, -1-(-1)\rangle = \langle 5, 0, 0\rangle$; $\langle x, y, z\rangle = \langle 1, 1, -1\rangle + t\langle 5, 0, 0\rangle$

6. $\mathbf{a} = \langle 3-5/2, 2-1, 1-(-2)\rangle = \langle 1/2, 1, 3\rangle$; $\langle x, y, z\rangle = \langle 3, 2, 1\rangle + t\langle 1/2, 1, 3\rangle$

7. $\mathbf{a} = \langle 2-6, 3-(-1), 5-8\rangle = \langle -4, 4, -3\rangle$; $x = 2-4t$, $y = 3+4t$, $z = 5-3t$

8. $\mathbf{a} = \langle 2-0, 0-4, 0-9\rangle = \langle 2, -4, -9\rangle$; $x = 2+2t$, $y = -4t$, $z = -9t$

9. $\mathbf{a} = \langle 1-3, 0-(-2), 0-(-7)\rangle = \langle -2, 2, 7\rangle$; $x = 1-2t$, $y = 2t$, $z = 7t$

10. $\mathbf{a} = \langle 0-(-2), 0-4, 5-0\rangle = \langle 2, -4, 5\rangle$; $x = 2t$, $y = -4t$, $z = 5+5t$

11. $\mathbf{a} = \langle 4-(-6), 1/2-(-1/4), 1/3-1/6\rangle = \langle 10, 3/4, 1/6\rangle$; $x = 4+10t$, $y = \dfrac{1}{2}+\dfrac{3}{4}t$, $z = \dfrac{1}{3}+\dfrac{1}{6}t$

12. $\mathbf{a} = \langle -3-4, 7-(-8), 9-(-1)\rangle = \langle -7, 15, 10\rangle$; $x = -3-7t$, $y = 7+15t$, $z = 9+10t$

13. $a_1 = 10-1 = 9$, $a_2 = 14-4 = 10$, $a_3 = -2-(-9) = 7$; $\dfrac{x-10}{9} = \dfrac{y-14}{10} = \dfrac{z+2}{7}$

14. $a_1 = 1-2/3 = 1/3$, $a_2 = 3-0 = 3$, $a_3 = 1/4-(-1/4) = 1/2$; $\dfrac{x-1}{1/3} = \dfrac{y-3}{3} = \dfrac{z-1/4}{1/2}$

15. $a_1 = -7-4 = -11$, $a_2 = 2-2 = 0$, $a_3 = 5-1 = 4$; $\dfrac{x+7}{-11} = \dfrac{z-5}{4}$, $y = 2$

16. $a_1 = 1-(-5) = 6$, $a_2 = 1-(-2) = 3$, $a_3 = 2-(-4) = 6$; $\dfrac{x-1}{6} = \dfrac{y-1}{3} = \dfrac{z-2}{6}$

17. $a_1 = 5-5 = 0$, $a_2 = 10-1 = 9$, $a_3 = -2-(-14) = 12$; $x = 5$, $\dfrac{y-10}{9} = \dfrac{z+2}{12}$

18. $a_1 = 5/6-1/3 = 1/2$; $a_2 = -1/4-3/8 = -5/8$; $a_3 = 1/5-1/10 = 1/10$
$\dfrac{x-5/6}{1/2} = \dfrac{y+1/4}{-5/8} = \dfrac{z-1/5}{1/10}$

19. parametric: $x = 4+3t$, $y = 6+t/2$, $z = -7-3t/2$; symmetric: $\dfrac{x-4}{3} = \dfrac{y-6}{1/2} = \dfrac{z+7}{-3/2}$

20. parametric: $x = 1-7t$, $y = 8-8t$, $z = -2$; symmetric: $\dfrac{x-1}{-7} = \dfrac{y-8}{-8}$, $z = -2$

21. parametric: $x = 5t$, $y = 9t$, $z = 4t$; symmetric: $\dfrac{x}{5} = \dfrac{y}{9} = \dfrac{z}{4}$

22. parametric: $x = 12t$, $y = -3-5t$, $z = 10-6t$; symmetric: $\dfrac{x}{12} = \dfrac{y+3}{-5} = \dfrac{z-10}{-6}$

23. Writing the given line in the form $x/2 = (y-1)/(-3) = (z-5)/6$, we see that a direction vector is $\langle 2, -3, 6\rangle$. Parametric equations for the line are $x = 6+2t$, $y = 4-3t$, $z = -2+6t$.

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24. A direction vector is $\langle 5, 1/3, -2\rangle$. Symmetric equations for the line are $(x-4)/5 = (y+11)/(1/3) = (z+7)/(-2)$.

25. A direction vector parallel to both the xz- and xy-planes is $\mathbf{i} = \langle 1, 0, 0\rangle$. Parametric equations for the line are $x = 2 + t$, $y = -2$, $z = 15$.

26. **(a)** Since the unit vector $\mathbf{j} = \langle 0, 1, 0\rangle$ lies along the y-axis, we have $x = 1$, $y = 2 + t$, $z = 8$.

(b) since the unit vector $\mathbf{k} = \langle 0, 0, 1\rangle$ is perpendicular to the xy-plane, we have $x = 1$, $y = 2$, $z = 8 + t$.

27. Both lines go through the points $(0, 0, 0)$ and $(6, 6, 6)$. Since two points determine a line, the lines are the same.

28. $\mathbf{a}$ and $\mathbf{f}$ are parallel since $\langle 9, -12, 6\rangle = -3\langle -3, 4, -2\rangle$. $\mathbf{c}$ and $\mathbf{d}$ are orthogonal since $\langle 2, -3, 4\rangle \cdot \langle 1, 4, 5/2\rangle = 0$.

29. In the xy-plane, $z = 9 + 3t = 0$ and $t = -3$. Then $x = 4 - 2(-3) = 10$ and $y = 1 + 2(-3) = -5$. The point is $(10, -5, 0)$. In the xz-plane, $y = 1+2t = 0$ and $t = -1/2$. Then $x = 4-2(-1/2) = 5$ and $z = 9+3(-1/2) = 15/2$. The point is $(5, 0, 15/2)$. In the yz-plane, $x = 4-2t = 0$ and $t = 2$. Then $y = 1+2(2) = 5$ and $z = 9+3(2) = 15$. The point is $(0, 5, 15)$.

30. The parametric equations for the line are $x = 1 + 2t$, $y = -2 + 3t$, $z = 4 + 2t$. In the xy-plane, $z = 4 + 2t = 0$ and $t = -2$. Then $x = 1 + 2(-2) = -3$ and $y = -2 + 3(-2) = -8$. The point is $(-3, -8, 0)$. In the xz-plane, $y = -2+3t = 0$ and $t = 2/3$. Then $x = 1+2(2/3) = 7/3$ and $z = 4+2(2/3) = 16/3$. The point is $(7/3, 0, 16/3)$. In the yz-plane, $x = 1 + 2t = 0$ and $t = -1/2$. Then $y = -2 + 3(-1/2) = -7/2$ and $z = 4 + 2(-1/2) = 3$. The point is $(0, -7/2, 3)$.

31. Solving the system $4 + t = 6 + 2s$, $5 + t = 11 + 4s$, $-1 + 2t = -3 + s$, or $t - 2s = 2$, $t - 4s = 6$, $2t - s = -2$ yields $s = -2$ and $t = -2$ in all three equations. Thus, the lines intersect at the point $x = 4 + (-2) = 2$, $y = 5 + (-2) = 3$, $z = -1 + 2(-2) = -5$, or $(2, 3, -5)$.

32. Solving the system $1 + t = 2 - s$, $2 - t = 1 + s$, $3t = 6s$, or $t + s = 1$, $t + s = 1$, $t - 2s = 0$ yields $s = 1/3$ and $t = 2/3$ in all three equations. Thus, the lines intersect at the point $x = 1 + 2/3 = 5/3$, $y = 2 - 2/3 = 4/3$, $z = 3(2/3) = 2$, or $(5/3, 4/3, 2)$.

33. The system of equations $2 - t = 4 + s$, $3 + t = 1 + s$, $1 + t = 1 - s$, or $t + s = -2$, $t - s = -2$, $t + s = 0$ has no solution since $-2 \neq 0$. Thus, the lines do not intersect.

34. Solving the system $3 - t = 2 + 2s$, $2 + t = -2 + 3s$, $8 + 2t = -2 + 8s$, or $t + 2s = 1$, $t - 3s = -4$, $2t - 8s = -10$ yields $s = 1$ and $t = -1$ in all three equations. Thus, the lines intersect at the point $x = 3 - (-1) = 4$, $y = 2 + (-1) = 1$, $z = 8 + 2(-1) = 6$, or $(4, 1, 6)$.

35. $\mathbf{a} = \langle -1, 2, -2\rangle$, $\mathbf{b} = \langle 2, 3, -6\rangle$, $\mathbf{a} \cdot \mathbf{b} = 16$, $\|\mathbf{a}\| = 3$, $\|\mathbf{b}\| = 7$; $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \dfrac{16}{3 \cdot 7}$;

$\theta = \cos^{-1}\dfrac{16}{21} \approx 40.37^\circ$

36. $\mathbf{a} = \langle 2, 7, -1\rangle$, $\mathbf{b} = \langle -2, 1, 4\rangle$, $\mathbf{a} \cdot \mathbf{b} = -1$, $\|\mathbf{a}\| = 3\sqrt{6}$, $\|\mathbf{b}\| = \sqrt{21}$;

$\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \dfrac{-1}{(3\sqrt{6})(\sqrt{21})} = -\dfrac{1}{9\sqrt{14}}$; $\theta = \cos^{-1}(-\dfrac{1}{9\sqrt{14}}) \approx 91.70^\circ$

37. A direction vector perpendicular to the given lines will be $\langle 1, 1, 1\rangle \times \langle -2, 1, -5\rangle = \langle -6, 3, 3\rangle$. Equations of the line are $x = 4 - 6t$, $y = 1 + 3t$, $z = 6 + 3t$.

38. The direction vectors of the given lines are $\langle 3, 2, 4\rangle$ and $\langle 6, 4, 8\rangle = 2\langle 3, 2, 4\rangle$. These are parallel, so we need a third vector parallel to the plane containing the lines which is not parallel to them. The point $(1, -1, 0)$ is on the first line and $(-4, 6, 10)$ is on the second line. A third vector is then $\langle 1, -1, 0\rangle - \langle -4, 6, 10\rangle = \langle 5, -7, -10\rangle$. Now a direction vector perpendicular to the plane is $\langle 3, 2, 4\rangle \times \langle 5, -7, -10\rangle = \langle 8, 50, -31\rangle$. Equations of the line through $(1, -1, 0)$ and perpendicular to the plane are $x = 1 + 8t$, $y = -1 + 50t$, $z = -31t$.

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39. $2(x-5)-3(y-1)+4(z-3)=0$; $2x-3y+4z=19$

40. $4(x-1)-2(y-2)+0(z-5)=0$; $4x-2y=0$

41. $-5(x-6)+0(y-10)+3(z+7)=0$; $-5x+3z=-51$

42. $6x-y+3z=0$

43. $6(x-1/2)+8(y-3/4)-4(z+1/2)=0$; $6x+8y-4z=11$

44. $-(x+1)+(y-1)-(z-0)=0$; $-x+y-z=2$

45. From the points $(3,5,2)$ and $(2,3,1)$ we obtain the vector $\mathbf{u}=\mathbf{i}+2\mathbf{j}+\mathbf{k}$. From the points $(2,3,1)$ and $(-1,-1,4)$ we obtain the vector $\mathbf{v}=3\mathbf{i}+4\mathbf{j}-3\mathbf{k}$. From the points $(-1,-1,4)$ and (x,y,z) we obtain the vector $\mathbf{w}=(x+1)\mathbf{i}+(y+1)\mathbf{j}+(z-4)\mathbf{k}$. Then, a normal vector is

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&1\\3&4&-3\end{vmatrix}=-10\mathbf{i}+6\mathbf{j}-2\mathbf{k}.$$

A vector equation of the plane is $-10(x+1)+6(y+1)-2(z-4)=0$ or $5x-3y+z=2$.

46. From the points $(0,1,0)$ and $(0,1,1)$ we obtain the vector $\mathbf{u}=\mathbf{k}$. From the points $(0,1,1)$ and $(1,3,-1)$ we obtain the vector $\mathbf{v}=\mathbf{i}+2\mathbf{j}-2\mathbf{k}$. From the points $(1,3,-1)$ and (x,y,z) we obtain the vector $\mathbf{w}=(x-1)\mathbf{i}+(y-3)\mathbf{j}+(z+1)\mathbf{k}$. Then, a normal vector is

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\1&2&-2\end{vmatrix}=-2\mathbf{i}+\mathbf{j}.$$

A vector equation of the plane is $-2(x-1)+(y-3)+0(z+1)=0$ or $-2x+y=1$.

47. From the points $(0,0,0)$ and $(1,1,1)$ we obtain the vector $\mathbf{u}=\mathbf{i}+\mathbf{j}+\mathbf{k}$. From the points $(1,1,1)$ and $(3,2,-1)$ we obtain the vector $\mathbf{v}=2\mathbf{i}+\mathbf{j}-2\mathbf{k}$. From the points $(3,2,-1)$ and (x,y,z) we obtain the vector $\mathbf{w}=(x-3)\mathbf{i}+(y-2)\mathbf{j}+(z+1)\mathbf{k}$. Then, a normal vector is

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&1\\2&1&-2\end{vmatrix}=-3\mathbf{i}+4\mathbf{j}-\mathbf{k}.$$

A vector equation of the plane is $-3(x-3)+4(y-2)-(z+1)=0$ or $-3x+4y-z=0$.

48. The three points are not colinear and all satisfy $x=0$, which is the equation of the plane.

49. From the points $(1,2,-1)$ and $(4,3,1)$ we obtain the vector $\mathbf{u}=3\mathbf{i}+\mathbf{j}+2\mathbf{k}$. From the points $(4,3,1)$ and $(7,4,3)$ we obtain the vector $\mathbf{v}=3\mathbf{i}+\mathbf{j}+2\mathbf{k}$. From the points $(7,4,3)$ and (x,y,z) we obtain the vector $\mathbf{w}=(x-7)\mathbf{i}+(y-4)\mathbf{j}+(z-3)\mathbf{k}$. Since $\mathbf{u}\times\mathbf{v}=\mathbf{0}$, the points are colinear.

50. From the points $(2,1,2)$ and $(4,1,0)$ we obtain the vector $\mathbf{u}=2\mathbf{i}-2\mathbf{k}$. From the points $(4,1,0)$ and $(5,0,-5)$ we obtain the vector $\mathbf{v}=\mathbf{i}-\mathbf{j}-5\mathbf{k}$. From the points $(5,0,-5)$ and (x,y,z) we obtain the vector $\mathbf{w}=(x-5)\mathbf{i}+y\mathbf{j}+(z+5)\mathbf{k}$. Then, a normal vector is

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&0&-2\\1&-1&-5\end{vmatrix}=-2\mathbf{i}+8\mathbf{j}-2\mathbf{k}.$$

A vector equation of the plane is $-2(x-5)+8y-2(z+5)=0$ or $x-4y+z=0$.

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51. A normal vector to $x + y - 4z = 1$ is $\langle 1, 1, -4\rangle$. The equation of the parallel plane is $(x - 2) + (y - 3) - 4(z + 5) = 0$ or $x + y - 4z = 25$.

52. A normal vector to $5x-y+z = 6$ is $\langle 5, -1, 1, \rangle$. The equation of the parallel plane is $5(x-0)-(y-0)+(z-0) = 0$ or $5x - y + z = 0$.

53. A normal vector to the xy-plane is $\langle 0, 0, 1\rangle$. The equation of the parallel plane is $z - 12 = 0$ or $z = 12$.

54. A normal vector is $\langle 0, 1, 0\rangle$. The equation of the plane is $y + 5 = 0$ or $y = -5$.

55. Direction vectors of the lines are $\langle 3, -1, 1\rangle$. and $\langle 4, 2, 1\rangle$. A normal vector to the plane is $\langle 3, -1, 1\rangle \times \langle 4, 2, 1\rangle = \langle -3, 1, 10\rangle$. A point on the first line, and thus in the plane, is $\langle 1, 1, 2\rangle$. The equation of the plane is $-3(x - 1) + (y - 1) + 10(z - 2) = 0$ or $-3x + y + 10z = 18$.

56. Direction vectors of the lines are $\langle 2, -1, 6\rangle$ and $\langle 1, 1, -3\rangle$. A normal vector to the plane is $\langle 2, -1, 6\rangle \times \langle 1, 1, -3\rangle = \langle -3, 12, 3\rangle$. A point on the first line, and thus in the plane, is $(1, -1, 5)$. The equation of the plane is $-3(x - 1) + 12(y + 1) + 3(z - 5) = 0$ or $-x + 4y + z = 0$.

57. A direction vector for the two lines is $\langle 1, 2, 1\rangle$. Points on the lines are $(1, 1, 3)$ and $(3, 0, -2)$. Thus, another vector parallel to the plane is $\langle 1-3, 1-0, 3+2\rangle = \langle -2, 1, 5\rangle$. A normal vector to the plane is $\langle 1, 2, 1\rangle \times \langle -2, 1, 5\rangle = \langle 9, -7, 5\rangle$. Using the point $(3, 0, -2)$ in the plane, the equation of the plane is $9(x - 3) - 7(y - 0) + 5(z + 2) = 0$ or $9x - 7y + 5z = 17$.

58. A direction vector for the line is $\langle 3, 2, -2\rangle$. Letting $t = 0$, we see that the origin is on the line and hence in the plane. Thus, another vector parallel to the plane is $\langle 4 - 0, 0 - 0, -6 - 0\rangle = \langle 4, 0, -6\rangle$. A normal vector to the plane is $\langle 3, 2, -2\rangle \times \langle 4, 0, -6\rangle = \langle -12, 10, -8\rangle$. The equation of the plane is $-12(x-0)+10(y-0)-8(z-0) = 0$ or $6x - 5y + 4z = 0$.

59. A direction vector for the line, and hence a normal vector to the plane, is $\langle -3, 1, -1/2\rangle$. The equation of the plane is $-3(x - 2) + (y - 4) - \frac{1}{2}(z - 8) = 0$ or $-3x + y - \frac{1}{2}z = -6$.

60. A normal vector to the plane is $\langle 2 - 1, 6 - 0, -3 + 2\rangle = \langle 1, 6, -1\rangle$. The equation of the plane is $(x - 1) + 6(y - 1) - (z - 1) = 0$ or $x + 6y - z = 6$.

61. Normal vectors to the planes are **(a)** $\langle 2, -1, 3\rangle$, **(b)** $\langle 1, 2, 2\rangle$, **(c)** $\langle 1, 1, -3/2\rangle$, **(d)** $\langle -5, 2, 4\rangle$, **(e)** $\langle -8, -8, 12\rangle$, **(f)** $\langle -2, 1, -3\rangle$. Parallel planes are **(c)** and **(e)**, and **(a)** and **(f)**. Perpendicular planes are **(a)** and **(d)**, **(b)** and **(c)**, **(b)** and **(e)**, and **(d)** and **(f)**.

62. A normal vector to the plane is $\langle -7, 2, 3\rangle$. This is a direction vector for the line and the equations of the line are $x = -4 - 7t$, $y = 1 + 2t$, $z = 7 + 3t$.

63. A direction vector of the line is $\langle -6, 9, 3\rangle$, and the normal vectors of the planes are **(a)** $\langle 4, 1, 2\rangle$, **(b)** $\langle 2, -3, 1\rangle$, **(c)** $\langle 10, -15, -5\rangle$, **(d)** $\langle -4, 6, 2\rangle$. Vectors **(c)** and **(d)** are multiples of the direction vector and hence the corresponding planes are perpendicular to the line.

64. A direction vector of the line is $\langle -2, 4, 1\rangle$, and normal vectors to the planes are **(a)** $\langle 1, -1, 3\rangle$, **(b)** $\langle 6, -3, 0\rangle$, **(c)** $\langle 1, -2, 5\rangle$, **(d)** $\langle -2, 1, -2\rangle$. Since the dot product of each normal vector with the direction vector is non-zero, none of the planes are parallel to the line.

65. Letting $z = t$ in both equations and solving $5x - 4y = 8 + 9t$, $x + 4y = 4 - 3t$, we obtain $x = 2 + t$, $y = \frac{1}{2} - t$, $z = t$.

66. Letting $y = t$ in both equations and solving $x - z = 2 - 2t$, $3x + 2z = 1 + t$, we obtain $x = 1 - \frac{3}{5}t$, $y = t$, $z = -1 + \frac{7}{5}t$ or, letting $t = 5s$, $x = 1 - 3s$, $y = 5s$, $z = -1 + 7s$.

67. Letting $z = t$ in both equations and solving $4x - 2y = 1 + t$, $x + y = 1 - 2t$, we obtain $x = \frac{1}{2} - \frac{1}{2}t$, $y = \frac{1}{2} - \frac{3}{2}t$, $z = t$.

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68. Letting $z = t$ and using $y = 0$ in the first equation, we obtain $x = -\frac{1}{2}t$, $y = 0$, $z = t$.

69. Substituting the parametric equations into the equation of the plane, we obtain $2(1 + 2t) - 3(2 - t) + 2(-3t) = -7$ or $t = -3$. Letting $t = -3$ in the equation of the line, we obtain the point of intersection $(-5, 5, 9)$.

70. Substituting the parametric equations into the equation of the plane, we obtain $(3 - 2t) + (1 + 6t) + 4(2 - \frac{1}{2}t) = 12$ or $2t = 0$. Letting $t = 0$ in the equation of the line, we obtain the point of intersection $(3, 1, 2)$.

71. Substituting the parametric equations into the equation of the plane, we obtain $1 + 2 - (1 + t) = 8$ or $t = -6$. Letting $t = -6$ in the equation of the line, we obtain the point of intersection $(1, 2, -5)$.

72. Substituting the parametric equations into the equation of the plane, we obtain $4 + t - 3(2 + t) + 2(1 + 5t) = 0$ or $t = 0$. Letting $t = 0$ in the equation of the line, we obtain the point of intersection $(4, 2, 1)$.

In Problems 73 and 74, the cross product of the normal vectors to the two planes will be a vector parallel to both planes, and hence a direction vector for a line parallel to the two planes.

73. Normal vectors are $\langle 1, 1, -4\rangle$ and $\langle 2, -1, 1\rangle$. A direction vector is

$$\langle 1, 1, -4\rangle \times \langle 2, -1, 1\rangle = \langle -3, -9, -3\rangle = -3\langle 1, 3, 1\rangle.$$

Equations of the line are $x = 5 + t$, $y = 6 + 3t$, $z = -12 + t$.

74. Normal vectors are $\langle 2, 0, 1\rangle$ and $\langle -1, 3, 1\rangle$. A direction vector is

$$\langle 2, 0, 1\rangle \times \langle -1, 3, 1\rangle = \langle -3, -3, 6\rangle = -3\langle 1, 1, -2\rangle.$$

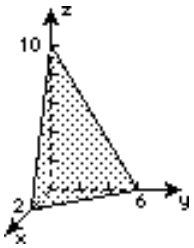
Equations of the line are $x = -3 + t$, $y = 5 + t$, $z = -1 - 2t$.

In Problems 75 and 76, the cross product of the direction vector of the line with the normal vector of the given plane will be a normal vector to the desired plane.

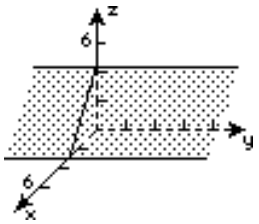
75. A direction vector of the line is $\langle 3, -1, 5\rangle$ and a normal vector to the given plane is $\langle 1, 1, 1\rangle$. A normal vector to the desired plane is $\langle 3, -1, 5\rangle \times \langle 1, 1, 1\rangle = \langle -6, 2, 4\rangle$. A point on the line, and hence in the plane, is $(4, 0, 1)$. The equation of the plane is $-6(x - 4) + 2(y - 0) + 4(z - 1) = 0$ or $3x - y - 2z = 10$.

76. A direction vector of the line is $\langle 3, 5, 2\rangle$ and a normal vector to the given plane is $\langle 2, -4, -1\rangle$. A normal vector to the desired plane is $\langle -3, 5, 2\rangle \times \langle 2, -4, -1\rangle = \langle 3, 1, 2\rangle$. A point on the line, and hence in the plane, is $(2, -2, 8)$. The equation of the plane is $3(x - 2) + (y + 2) + 2(z - 8) = 0$ or $3x + y + 2z = 20$.

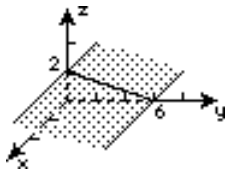
77.



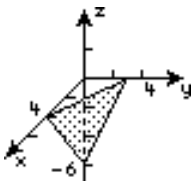
78.



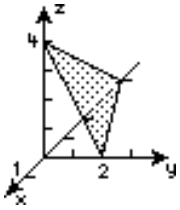
79.



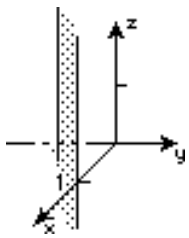
80.



81.



82.



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Exercises 7.6

1. Not a vector space. Axiom (**vi**) is not satisfied.

2. Not a vector space. Axiom (**i**) is not satisfied.

3. Not a vector space. Axiom (**x**) is not satisfied.

4. A vector space

5. A vector space

6. A vector space

7. Not a vector space. Axiom (**ii**) is not satisfied.

8. A vector space

9. A vector space

10. Not a vector space. Axiom (**i**) is not satisfied.

11. A subspace

12. Not a subspace. Axiom (**i**) is not satisfied.

13. Not a subspace. Axiom (**ii**) is not satisfied.

14. A subspace

15. A subspace

16. A subspace

17. A subspace

18. A subspace

19. Not a subspace. Neither axioms (**i**) nor(**ii**) are satisfied.

20. A subspace

21. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be in S. Then

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (at_1, bt_1, ct_1) + (at_2, bt_2, ct_2) = (a(t_1 + t_2), b(t_1 + t_2), c(t_1 + t_2))$$

is in S. Also, for (x, y, z) in S then $k(x, y, z) = (kx, ky, kz) = (a(kt), b(kt), c(kt))$ is also in S.

22. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be in S. Then $ax_1 + by_1 + cz_1 = 0$ and $ax_2 + by_2 + cz_2 = 0$. Adding gives $a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = 0$ and so $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ is in S. Also, for (x, y, z) then $ax + by + cz = 0$ implies $k(ax + by + cz) = k \cdot 0 = 0$ and $a(kx) + b(ky) + c(kz) = 0$. this means $k(x, y, z) = (kx, ky, kz)$ is in S.

23. **(a)** $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$ if and only if $c_1 + c_2 + c_3 = 0$, $c_2 + c_3 = 0$, $c_3 = 0$. The only solution of this system is $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

(b) Solving the system $c_1 + c_2 + c_3 = 3$, $c_2 + c_3 = -4$, $c_3 = 8$ gives $c_1 = 7$, $c_2 = -12$, $c_3 = 8$. Thus $\mathbf{a} = 7\mathbf{u}_1 - 12\mathbf{u}_2 + 8\mathbf{u}_3$.

24. **(a)** The assumption $c_1p_1 + c_2p_2 = 0$ is equivalent to $(c_1 + c_2)x + (c_1 - c_2) = 0$. Thus $c_1 + c_2 = 0$, $c_1 - c_2 = 0$. The only solution of this system is $c_1 = 0$, $c_2 = 0$.

(b) Solving the system $c_1 + c_2 = 5$, $c_1 - c_2 = 2$ gives $c_1 = \frac{7}{2}$, $c_2 = \frac{3}{2}$. Thus $p(x) = \frac{7}{2}p_1(x) + \frac{3}{2}p_2(x)$

25. Linearly dependent since $\langle -6, 12\rangle = -\frac{3}{2}\langle 4, -8\rangle$

26. Linearly dependent since $2\langle 1, 1\rangle + 3\langle 0, 1\rangle + (-1)\langle 2, 5\rangle = \langle 0, 0\rangle$

27. Linearly independent

28. Linearly dependent since for all x $(1) \cdot 1 + (-2)(x + 1) + (1)(x + 1)^2 + (-1)x^2 = 0$.

29. f is discontinuous at $x = -1$ and at $x = -3$.

30. $(x, \sin x) = \displaystyle\int_0^{2\pi} x \sin x \, dx = (-x \cos x + \sin x) \Big|_0^{2\pi} = -2\pi$

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31. $\|x\|^2 = \displaystyle\int_0^{2\pi} x^2\,dx = \frac{1}{3}x^3\,\Big|_0^{2\pi} = \frac{8}{3}\pi^3$ and so $\|x\| = 2\sqrt{\dfrac{2\pi^3}{3}}$. Now

$$\|\sin x\|^2 = \int_0^{2\pi} \sin^2 x\,dx = \frac{1}{2}\int_0^{2\pi}(1-\cos 2x)\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\Big|_0^{2\pi} = \pi$$

and so $\|\sin x\| = \sqrt{\pi}$.

32. A basis could be 1, x, $e^x\cos 3x$, $e^x \sin 3x$.

33. We need to show that $\text{Span}\{x_1, x_2, \ldots, x_n\}$ is closed under vector addition and scalar multiplication. Suppose $\mathbf{u}$ and $\mathbf{v}$ are in $\text{Span}\{x_1, x_2, \ldots, x_n\}$. Then $\mathbf{u} = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ and $\mathbf{v} = b_1x_1 + b_2x_2 + \cdots + b_nx_n$, so that

$$\mathbf{u} + \mathbf{v} = (a_1 + b_1)x_1 + (a_2 + b_2)x_2 + \cdots + (a_n + b_n)x_n,$$

which is in $\text{Span}\{x_1, x_2, \ldots, x_n\}$. Also, for any real number k,

$$k\mathbf{u} = k(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = ka_1x_1 + ka_2x_2 + \cdots + ka_nx_n,$$

which is in $\text{Span}\{x_1, x_2, \ldots, x_n\}$. Thus, $\text{Span}\{x_1, x_2, \ldots, x_n\}$ is a subspace of $\mathbf{V}$.

Chapter 7 Review Exercises

1. True

2. False; the points must be non-collinear.

3. False; since a normal to the plane is $\langle 2, 3, -4\rangle$ which is not a multiple of the direction vector $\langle 5, -2, 1\rangle$ of the line.

4. True **5.** True **6.** True **7.** True **8.** True **9.** True

10. True; since $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$ are both normal to the plane and hence parallel (unless $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ or $\mathbf{c}\times\mathbf{d} = \mathbf{0}$.)

11. $9\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

12. orthogonal

13. $-5(\mathbf{k}\times\mathbf{j}) = -5(-\mathbf{i}) = 5\mathbf{i}$

14. $\mathbf{i}\cdot(\mathbf{i}\times\mathbf{j}) = \mathbf{i}\times\mathbf{k} = \mathbf{0}$

15. $\sqrt{(-12)^2 + 4^2 + 6^2} = 14$

16. $(-1-20)\mathbf{i} - (-2-0)\mathbf{j} + (8-0)\mathbf{k} = -21\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$

17. $-6\mathbf{i} + \mathbf{j} - 7\mathbf{k}$

18. The coordinates of $(1, -2, -10)$ satisfy the given equation.

19. Writing the line in parametric form, we have $x = 1+t$, $y = -2+3t$, $z = -1+2t$. Substituting into the equation of the plane yields $(1+t)+2(-2+3t)-(-1+2t) = 13$ or $t = 3$. Thus, the point of intersection is $x = 1+3 = 4$, $y = -2+3(3) = 7$, $z = -1+2(3) = 5$, or $(4, 7, 5)$.

20. $|\mathbf{a}| = \sqrt{4^2 + 3^2 + (-5)^2} = 5\sqrt{2}$; $\mathbf{u} = -\dfrac{1}{5\sqrt{2}}(4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}) = -\dfrac{4}{5\sqrt{2}}\mathbf{i} - \dfrac{3}{5\sqrt{2}}\mathbf{j} + \dfrac{1}{\sqrt{2}}\mathbf{k}$

21. $x_2 - 2 = 3$, $x_2 = 5$; $y_2 - 1 = 5$, $y_2 = 6$; $z_2 - 7 = -4$, $z_2 = 3$; $P_2 = (5, 6, 3)$

22. $(5, 1/2, 5/2)$

23. $(7.2)(10)\cos 135^\circ = -36\sqrt{2}$

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24. $2\mathbf{b} = \langle -2, 4, 2\rangle$; $4\mathbf{c} = \langle 0, -8, 8\rangle$; $\mathbf{a} \cdot (2\mathbf{b} + 4\mathbf{c}) = \langle 3, 1, 0\rangle \cdot \langle -2, -4, 10\rangle = -10$

25. 12, −8, 6

26. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{1}{\sqrt{2}\sqrt{2}} = \dfrac{1}{2}$; $\theta = 60°$

27. $A = \dfrac{1}{2}|5\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}| = \dfrac{3\sqrt{10}}{2}$

28. From $3(x-3) + 0(y-6) + (1)(z-(-2)) = 0$ we obtain $3x + z = 7$.

29. $|-5-(-3)| = 2$

30. parallel: $-2c = 5$, $c = -5/2$; orthogonal: $1(-2) + 3(-6) + c(5) = 0$, $c = 4$

31. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{k} = \mathbf{i} - \mathbf{j} - 3\mathbf{k}$

A unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ is

$$\frac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\times\mathbf{b}\|} = \frac{1}{\sqrt{1+1+9}}(\mathbf{i}-\mathbf{j}-3\mathbf{k}) = \frac{1}{\sqrt{11}}\mathbf{i} - \frac{1}{\sqrt{11}}\mathbf{j} - \frac{3}{\sqrt{11}}\mathbf{k}.$$

32. $\|\mathbf{a}\| = \sqrt{1/4 + 1/4 + 1/6} = \dfrac{3}{4}$; $\cos\alpha = \dfrac{1/2}{3/4} = \dfrac{2}{3}$, $\alpha \approx 48.19°$; $\cos\beta = \dfrac{1/2}{3/4} = \dfrac{2}{3}$, $\beta \approx 48.19°$;
$\cos\gamma = \dfrac{-1/4}{3/4} = -\dfrac{1}{3}$, $\gamma \approx 109.47°$

33. $\text{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a}\cdot\mathbf{b}/\|\mathbf{b}\| = \langle 1, 2, -2\rangle \cdot \langle 4, 3, 0\rangle/5 = 2$

34. $\text{comp}_{\mathbf{a}}\mathbf{b} = \mathbf{b}\cdot\mathbf{a}/\|\mathbf{a}\| = \langle 4, 3, 0\rangle \cdot \langle 1, 2, -2\rangle/3 = 10/3$
$\text{proj}_{\mathbf{a}}\mathbf{b} = (\text{comp}_{\mathbf{a}}\mathbf{b})\mathbf{a}/\|\mathbf{a}\| = (10/3)\langle 1, 2, -2\rangle/3 = \langle 10/9, 20/9, -20/9\rangle$

35. Using the result of Problem 34,

$$\text{proj}_{\mathbf{a}^\perp}\mathbf{b} = b - \text{proj}_{\mathbf{a}}\mathbf{b} = \langle 4, 3, 0\rangle - \langle 10/9, 20/9, -20/9\rangle = \langle 26/9, 7/9, 20/9\rangle.$$

36. $\text{comp}_{\mathbf{b}}(\mathbf{a}-\mathbf{b}) = (\mathbf{a}-\mathbf{b})\cdot\mathbf{b}/\|\mathbf{b}\| = \langle -3, -1, -2\rangle \cdot \langle 4, 3, 0\rangle/5 = -3$
$\text{proj}_{\mathbf{b}}(\mathbf{a}-\mathbf{b}) = (\text{comp}_{\mathbf{b}}(\mathbf{a}-\mathbf{b}))\mathbf{b}/\|\mathbf{b}\| = -3\langle 4, 3, 0\rangle/5 = \langle -12/5, -9/5, 0\rangle$
$\text{proj}_{\mathbf{b}^\perp}(\mathbf{a}-\mathbf{b}) = (\mathbf{a}-\mathbf{b}) - \text{proj}_{\mathbf{b}}(\mathbf{a}-\mathbf{b}) = \langle -3, -1, -2\rangle - \langle -12/5, -9/5, 0\rangle = \langle -3/5, 4/5, -10/5\rangle$

37. Let $\mathbf{a} = \langle a, b, c\rangle$ and $\mathbf{r} = \langle x, y, z\rangle$. Then

(a) $(\mathbf{r}-\mathbf{a})\cdot\mathbf{r} = \langle x-a, y-b, z-c\rangle \cdot \langle x, y, z\rangle = x^2 - ax + y^2 - by + z^2 - zc = 0$ implies

$(x - \dfrac{a}{2})^2 + (y - \dfrac{b}{2})^2 + (z - \dfrac{c}{2})^2 = \dfrac{a^2+b^2+c^2}{4}$. The surface is a sphere.

(b) $(\mathbf{r}-\mathbf{a})\cdot\mathbf{a} = \langle x-a, y-b, z-c\rangle \cdot \langle a, b, c\rangle = a(x-a) + b(y-b) + c(z-c) = 0$
The surface is a plane.

38. $\langle 4, 2, -2\rangle - \langle 2, 4, -3\rangle = \langle 2, -2, 1\rangle$; $\langle 2, 4, -3\rangle - \langle 6, 7, -5\rangle = \langle -4, -3, 2\rangle$; $\langle 2, -2, 1\rangle \cdot \langle -4, -3, 2\rangle = 0$
The points are the vertices of a right triangle.

39. A direction vector of the given line is $\langle 4, -2, 6\rangle$. A parallel line containing $(7, 3, -5)$ is $(x-7)/4 = (y-3)/(-2) = (z+5)/6$.

40. A normal to the plane is $\langle 8, 3, -4\rangle$. The line with this direction vector and through $(5, -9, 3)$ is $x = 5 + 8t$, $y = -9 + 3t$, $z = 3 - 4t$.

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41. The direction vectors are $\langle -2,3,1\rangle$ and $\langle 2,1,1\rangle$. Since $\langle -2,3,1\rangle \cdot \langle 2,1,1\rangle = 0$, the lines are orthogonal. Solving $1-2t = x = 1+2s$, $3t = y = -4+s$, we obtain $t=-1$ and $s=1$. The point $(3,-3,0)$ obtained by letting $t=-1$ and $s=1$ is common to the two lines, so they do intersect.

42. Vectors in the plane are $\langle 2,3,1\rangle$ and $\langle 1,0,2\rangle$. A normal vector is $\langle 2,3,1\rangle \times \langle 1,0,2\rangle = \langle 6,-3,-3\rangle = 3\langle 2,-1,-1\rangle$. An equation of the plane is $2x-y-z=0$

43. The lines are parallel with direction vector $\langle 1,4,-2\rangle$. Since $(0,0,0)$ is on the first line and $(1,1,3)$ is on the second line, the vector $\langle 1,1,3\rangle$ is in the plane. A normal vector to the plane is thus $\langle 1,4,-2\rangle \times \langle 1,1,3\rangle = \langle 14,-5,-3\rangle$. An equation of the plane is $14x-5y-3z=0$.

44. Letting $z=t$ in the equations of the plane and solving $-x+y = 4+8t$, $3x-y=-2t$, we obtain $x=2+3t$, $y=6+11t$, $z=t$. Thus, a normal to the plane is $\langle 3,11,1\rangle$ and an equation of the plane is

$$3(x-1)+11(y-7)+(z+1)=0 \quad \text{or} \quad 3x+11y+z=79.$$

45. $\mathbf{F} = 10\dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{10}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = 5\sqrt{2}\,\mathbf{i}+5\sqrt{2}\,\mathbf{j}$; $\mathbf{d} = \langle 7,4,0\rangle - \langle 4,1,0\rangle = 3\mathbf{i}+3\mathbf{j}$

$W = \mathbf{F}\cdot\mathbf{d} = 15\sqrt{2}+15\sqrt{2} = 30\sqrt{2}$ N-m

46. $\mathbf{F} = 5\sqrt{2}\,\mathbf{i}+5\sqrt{2}\,\mathbf{j}+50\mathbf{i} = (5\sqrt{2}+50)\mathbf{i}+5\sqrt{2}\,\mathbf{j}$; $\mathbf{d} = 3\mathbf{i}+3\mathbf{j}$

$W = 15\sqrt{2}+150+15\sqrt{2} = 30\sqrt{2}+150$ N-m ≈ 192.4 N-m

47. Since $\mathbf{F}_2 = 200(\mathbf{i}+\mathbf{j})/\sqrt{2} = 100\sqrt{2}\,\mathbf{i}+100\sqrt{2}\,\mathbf{j}$, $\mathbf{F}_3 = \mathbf{F}_2-\mathbf{F}_1 = (100\sqrt{2}-200)\mathbf{i}+100\sqrt{2}\,\mathbf{j}$ and

$\|\mathbf{F}_3\| = \sqrt{(100\sqrt{2}-200)^2+(100\sqrt{2})^2} = 200\sqrt{2-\sqrt{2}} \approx 153$ lb.

48. Let $\|\mathbf{F}_1\| = F_1$ and $\|\mathbf{F}_2\| = F_2$. Then $\mathbf{F}_1 = F_1[(\cos 45^\circ)\mathbf{i}+(\sin 45^\circ)\mathbf{j}]$ and $\mathbf{F}_2 = F_2[(\cos 120^\circ)\mathbf{i}+(\sin 120^\circ)\mathbf{j}]$, or $\mathbf{F}_1 = F_1(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j})$ and $\mathbf{F}_2 = F_2(-\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j})$. Since $\mathbf{w}+\mathbf{F}_1+\mathbf{F}_2 = \mathbf{0}$,

$$F_1(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j})+F_2(-\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}) = 50\mathbf{j}, \qquad (\frac{1}{\sqrt{2}}F_1-\frac{1}{2}F_2)\mathbf{i}+(\frac{1}{\sqrt{2}}F_1+\frac{\sqrt{3}}{2}F_2)\mathbf{j} = 50\mathbf{j}$$

and

$$\frac{1}{\sqrt{2}}F_1-\frac{1}{2}F_2 = 0, \qquad \frac{1}{\sqrt{2}}F_1+\frac{\sqrt{3}}{2}F_2 = 50.$$

Solving, we obtain $F_1 = 25(\sqrt{6}-\sqrt{2}\,) \approx 25.9$ lb and $F_2 = 50(\sqrt{3}-1) \approx 36.6$ lb.

49. Not a vector space. Axiom (**viii**) is not satisfied.

50. The vectors are linearly independent. The only solution of the system

$$c_1 = 0, \qquad c_1+2c_2+c_3 = 0, \qquad 2c_1+3c_2-c_3 = 0$$

is $c_1=0$, $c_2=0$, $c_3=0$.

51. Let p_1 and p_2 be in P_n such that $\dfrac{d^2p_1}{dx^2} = 0$ and $\dfrac{d^2p_2}{dx^2} = 0$. Since

$$0 = \frac{d^2p_1}{dx^2}+\frac{d^2p_2}{dx^2} = \frac{d^2}{dx^2}(p_1+p_2) \quad \text{and} \quad 0 = k\,\frac{d^2p_1}{dx^2} = \frac{d^2}{dx^2}(kp_1)$$

we conclude that the set of polynomials with the given property is a subspace of P_n. A basis for the subspace is 1, x.

52. The intersection $W_1 \cap W_2$ is a subspace of V. If x and y are in $W_1 \cap W_2$ then $\mathbf{x}$ and $\mathbf{y}$ are in each subspace and so $\mathbf{x}+\mathbf{y}$ is in each subspace. That is, $\mathbf{x}+\mathbf{y}$ is in $W_1 \cap W_2$. Similarly, if $\mathbf{x}$ is in $W_1 \cap W_2$ then $\mathbf{x}$ is in each subspace and so $k\mathbf{x}$ is in each subspace. That is, $k\mathbf{x}$ is in $W_1 \cap W_2$ for any scalar k.

The union $W_1 \cup W_2$ is generally not a subspace. For example, $W_1 = \{\langle x,y\rangle \mid y = x\}$ and $W_2 = \{\langle x,y\rangle \mid y = 2x\}$ are subspaces of R^2. Now $\langle 1,1\rangle$ is in W_1 and $\langle 1,2\rangle$ is in W_2 but $\langle 1,1\rangle + \langle 1,2\rangle = \langle 2,3\rangle$ is not in $W_1 \cup W_2$.

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8 Matrices

Exercises 8.1

1. 2×4 **2.** 3×2 **3.** 3×3 **4.** 1×3 **5.** 3×4

6. 8×1 **7.** Not equal **8.** Not equal **9.** Not equal **10.** Not equal

11. Solving $x = y - 2$, $y = 3x - 2$ we obtain $x = 2$, $y = 4$.

12. Solving $x^2 = 9$, $y = 4x$ we obtain $x = 3$, $y = 12$ and $x = -3$, $t = -12$.

13. $c_{23} = 2(0) - 3(-3) = 9$; $c_{12} = 2(3) - 3(-2) = 12$

14. $c_{23} = 2(1) - 3(0) = 2$; $c_{12} = 2(-1) - 3(0) = -2$

15. **(a)** $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 4-2 & 5+6 \\ -6+8 & 9-10 \end{bmatrix} = \begin{bmatrix} 2 & 11 \\ 2 & -1 \end{bmatrix}$

(b) $\mathbf{B} - \mathbf{A} = \begin{bmatrix} -2-4 & 6-5 \\ 8+6 & -10-9 \end{bmatrix} = \begin{bmatrix} -6 & 1 \\ 14 & -19 \end{bmatrix}$

(c) $2\mathbf{A} + 3\mathbf{B} = \begin{bmatrix} 8 & 10 \\ -12 & 18 \end{bmatrix} + \begin{bmatrix} -6 & 18 \\ 24 & -30 \end{bmatrix} = \begin{bmatrix} 2 & 28 \\ 12 & -12 \end{bmatrix}$

16. **(a)** $\mathbf{A} - \mathbf{B} = \begin{bmatrix} -2-3 & 0+1 \\ 4-0 & 1-2 \\ 7+4 & 3+2 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 4 & -1 \\ 11 & 5 \end{bmatrix}$

(b) $\mathbf{B} - \mathbf{A} = \begin{bmatrix} 3+2 & -1-0 \\ 0-4 & 2-1 \\ -4-7 & -2-3 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -4 & 1 \\ -11 & -5 \end{bmatrix}$

(c) $2(\mathbf{A} + \mathbf{B}) = 2\begin{bmatrix} 1 & -1 \\ 4 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 8 & 6 \\ 6 & 2 \end{bmatrix}$

17. **(a)** $\mathbf{AB} = \begin{bmatrix} -2-9 & 12-6 \\ 5+12 & -30+8 \end{bmatrix} = \begin{bmatrix} -11 & 6 \\ 17 & -22 \end{bmatrix}$

(b) $\mathbf{BA} = \begin{bmatrix} -2-30 & 3+24 \\ 6-10 & -9+8 \end{bmatrix} = \begin{bmatrix} -32 & 27 \\ -4 & -1 \end{bmatrix}$

(c) $\mathbf{A}^2 = \begin{bmatrix} 4+15 & -6-12 \\ -10-20 & 15+16 \end{bmatrix} = \begin{bmatrix} 19 & -18 \\ -30 & 31 \end{bmatrix}$

(d) $\mathbf{B}^2 = \begin{bmatrix} 1+18 & -6+12 \\ -3+6 & 18+4 \end{bmatrix} = \begin{bmatrix} 19 & 6 \\ 3 & 22 \end{bmatrix}$

18. **(a)** $\mathbf{AB} = \begin{bmatrix} -4+4 & 6-12 & -3+8 \\ -20+10 & 30-30 & -15+20 \\ -32+12 & 48-36 & -24+24 \end{bmatrix} = \begin{bmatrix} 0 & -6 & 5 \\ -10 & 0 & 5 \\ -20 & 12 & 0 \end{bmatrix}$

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(b) $\mathbf{BA} = \begin{bmatrix} -4+30-24 & -16+60-36 \\ 1-15+16 & 4-30+24 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 2 & -2 \end{bmatrix}$

19. (a) $\mathbf{BC} = \begin{bmatrix} 9 & 24 \\ 3 & 8 \end{bmatrix}$

(b) $\mathbf{A}(\mathbf{BC}) = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 9 & 24 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ -6 & -16 \end{bmatrix}$

(c) $\mathbf{C}(\mathbf{BA}) = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(d) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 5 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} -4 & -5 \\ 8 & 10 \end{bmatrix}$

20. (a) $\mathbf{AB} = \begin{bmatrix} 5 & -6 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} = (-16)$

(b) $\mathbf{BA} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 5 & -6 & 7 \end{bmatrix} = \begin{bmatrix} 15 & -18 & 21 \\ 20 & -24 & 28 \\ -5 & 6 & -7 \end{bmatrix}$

(c) $(\mathbf{BA})\mathbf{C} = \begin{bmatrix} 15 & -18 & 21 \\ 20 & -24 & 28 \\ -5 & 6 & -7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 78 & 54 & 99 \\ 104 & 72 & 132 \\ -26 & -18 & -33 \end{bmatrix}$

(d) Since $\mathbf{AB}$ is 1×1 and $\mathbf{C}$ is 3×3 the product $(\mathbf{AB})\mathbf{C}$ is not defined.

21. (a) $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 4 & 8 & -10 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ -10 \end{bmatrix} = (180)$

(b) $\mathbf{B}^T\mathbf{B} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 10 \\ 8 & 16 & 20 \\ 10 & 20 & 25 \end{bmatrix}$

(c) $\mathbf{A} + \mathbf{B}^T = \begin{bmatrix} 4 \\ 8 \\ -10 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -5 \end{bmatrix}$

22. (a) $\mathbf{A} + \mathbf{B}^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 5 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 5 & 11 \end{bmatrix}$

(b) $2\mathbf{A}^T - \mathbf{B}^T = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}$

(c) $\mathbf{A}^T(\mathbf{A}-\mathbf{B}) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} -3 & -7 \\ -6 & -14 \end{bmatrix}$

23. (a) $(\mathbf{AB})^T = \begin{bmatrix} 7 & 10 \\ 38 & 75 \end{bmatrix}^T = \begin{bmatrix} 7 & 38 \\ 10 & 75 \end{bmatrix}$

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(b) $\mathbf{B}^T\mathbf{A}^T = \begin{bmatrix} 5 & -2 \\ 10 & -5 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 38 \\ 10 & 75 \end{bmatrix}$

24. (a) $\mathbf{A}^T + \mathbf{B} = \begin{bmatrix} 5 & -4 \\ 9 & 6 \end{bmatrix} + \begin{bmatrix} -3 & 11 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 2 & 8 \end{bmatrix}$

(b) $2\mathbf{A} + \mathbf{B}^T = \begin{bmatrix} 10 & 18 \\ -8 & 12 \end{bmatrix} + \begin{bmatrix} -3 & -7 \\ 11 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 3 & 14 \end{bmatrix}$

25. $\begin{bmatrix} -4 \\ 8 \end{bmatrix} - \begin{bmatrix} 4 \\ 16 \end{bmatrix} + \begin{bmatrix} -6 \\ 9 \end{bmatrix} = \begin{bmatrix} -14 \\ 1 \end{bmatrix}$

26. $\begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} + \begin{bmatrix} -5 \\ -5 \\ 15 \end{bmatrix} + \begin{bmatrix} -6 \\ -8 \\ 10 \end{bmatrix} = \begin{bmatrix} -5 \\ -10 \\ 22 \end{bmatrix}$

27. $\begin{bmatrix} -19 \\ 18 \end{bmatrix} - \begin{bmatrix} 19 \\ 20 \end{bmatrix} = \begin{bmatrix} -38 \\ -2 \end{bmatrix}$

28. $\begin{bmatrix} -7 \\ 17 \\ -6 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 8 \\ -6 \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \\ 4 \end{bmatrix}$

29. 4×5

30. 3×2

31. $\mathbf{A}^T = \begin{bmatrix} 2 & -3 \\ 4 & 2 \end{bmatrix}$; $(\mathbf{A}^T)^T = \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix} = \mathbf{A}$

32. $(\mathbf{A} + \mathbf{B})^T = \begin{bmatrix} 6 & -6 \\ 14 & 10 \end{bmatrix} = \mathbf{A}^T + \mathbf{B}^T$

33. $(\mathbf{AB})^T = \begin{bmatrix} 16 & 40 \\ -8 & -20 \end{bmatrix}^T = \begin{bmatrix} 16 & -8 \\ 40 & -20 \end{bmatrix}$; $\mathbf{B}^T\mathbf{A}^T = \begin{bmatrix} 4 & 2 \\ 10 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -8 \\ 40 & -20 \end{bmatrix}$

34. $(6\mathbf{A})^T = \begin{bmatrix} 12 & -18 \\ 24 & 12 \end{bmatrix} = 6\mathbf{A}^T$

35. $\mathbf{B} = \mathbf{A}\mathbf{A}^T = \begin{bmatrix} 2 & 1 \\ 6 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 6 & 2 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 & 9 \\ 15 & 39 & 27 \\ 9 & 27 & 29 \end{bmatrix} = \mathbf{B}^T$

36. Using Problem 33 we have $(\mathbf{A}\mathbf{A}^T)^T = (\mathbf{A}^T)^T\mathbf{A}^T = \mathbf{A}\mathbf{A}^T$, so that $\mathbf{A}\mathbf{A}^T$ is symmetric.

37. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\mathbf{AB} = \mathbf{0}$.

38. We see that $\mathbf{A} \neq \mathbf{B}$, but $\mathbf{AC} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix} = \mathbf{BC}$.

39. Since $(\mathbf{A}+\mathbf{B})^2 = (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) \neq \mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2$, and $\mathbf{AB} \neq \mathbf{BA}$ in general, $(\mathbf{A}+\mathbf{B})^2 \neq \mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2$.

40. Since $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2$, and $\mathbf{AB} \neq \mathbf{BA}$ in general, $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) \neq \mathbf{A}^2 - \mathbf{B}^2$.

41. $a_{11}x_1 + a_{12}x_2 = b_1$; $a_{21}x_1 + a_{22}x_2 = b_2$

42. $\begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 9 \end{bmatrix}$

43. $\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by/2 & bx/2 + cy \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax^2 + bxy/2 + bxy/2 + cy^2 \end{bmatrix} = \begin{bmatrix} ax^2 + bxy + cy^2 \end{bmatrix}$

44. $\begin{bmatrix} 0 & -\partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & -\partial/\partial x \\ -\partial/\partial y & \partial/\partial x & 0 \end{bmatrix}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} -\partial Q/\partial z + \partial R/\partial y \\ \partial P/\partial z - \partial R/\partial x \\ -\partial P/\partial y + \partial Q/\partial x \end{bmatrix} = \text{curl } \mathbf{F}$

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45. **(a)** $M_Y \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\gamma + y\sin\gamma \\ -x\sin\gamma + y\cos\gamma \\ z \end{bmatrix} = \begin{bmatrix} x_Y \\ y_Y \\ z_Y \end{bmatrix}$

(b) $M_R = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}; \qquad M_P \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$

(c) $M_P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30^\circ & \sin 30^\circ \\ 0 & -\sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2}(\sqrt{3}+1) \\ \frac{1}{2}(\sqrt{3}-1) \end{bmatrix}$

$$M_R M_P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & 0 & -\sin 45^\circ \\ 0 & 1 & 0 \\ \sin 45^\circ & 0 & \cos 45^\circ \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2}(\sqrt{3}+1) \\ \frac{1}{2}(\sqrt{3}-1) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2}(\sqrt{3}+1) \\ \frac{1}{2}(\sqrt{3}-1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}(3\sqrt{2}-\sqrt{6}) \\ \frac{1}{2}(\sqrt{3}+1) \\ \frac{1}{4}(\sqrt{2}+\sqrt{6}) \end{bmatrix}$$

$$M_Y M_R M_P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 60^\circ & \sin 60^\circ & 0 \\ -\sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4}(3\sqrt{2}-\sqrt{6}) \\ \frac{1}{2}(\sqrt{3}+1) \\ \frac{1}{4}(\sqrt{2}+\sqrt{6}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4}(3\sqrt{2}-\sqrt{6}) \\ \frac{1}{2}(\sqrt{3}+1) \\ \frac{1}{4}(\sqrt{2}+\sqrt{6}) \end{bmatrix} = \begin{bmatrix} \frac{1}{8}(3\sqrt{2}-\sqrt{6}+6+2\sqrt{3}) \\ \frac{1}{8}(-3\sqrt{6}+3\sqrt{2}+2\sqrt{3}+2) \\ \frac{1}{4}(\sqrt{2}+\sqrt{6}) \end{bmatrix}$$

46. **(a)** $\mathbf{LU} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix} = \mathbf{A}$

(b) $\mathbf{LU} = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 4 & 1 \end{bmatrix} = \mathbf{A}$

(c) $\mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 10 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -21 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 6 & 1 \end{bmatrix} = \mathbf{A}$

(d) $\mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \mathbf{A}$

47. **(a)** $\mathbf{AB} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} = \begin{bmatrix} 17 & 43 \\ 3 & 75 \\ -14 & 51 \end{bmatrix}$

since

$$A_{11}B_1 + A_{12}B_2 = \begin{bmatrix} 13 & 25 \\ -9 & 49 \end{bmatrix} + \begin{bmatrix} 4 & 18 \\ 12 & 26 \end{bmatrix} = \begin{bmatrix} 17 & 43 \\ 3 & 75 \end{bmatrix}$$

and

$$A_{21}B_1 + A_{22}B_2 = [\,-24 \quad 34\,] + [\,10 \quad 17\,] = [\,-14 \quad 51\,].$$

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(b) It is easier to enter smaller strings of numbers and the chance of error is decreased. Also, if the large matrix has submatrices consisting of all zeros or diagonal matrices, these are easily entered without listing all of the entries.

Exercises 8.2

1. $\left[\begin{array}{cc|c} 1 & -1 & 11 \\ 4 & 3 & -5 \end{array}\right] \xrightarrow{-4R_1+R_2} \left[\begin{array}{cc|c} 1 & -1 & 11 \\ 0 & 7 & -49 \end{array}\right] \xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{cc|c} 1 & -1 & 11 \\ 0 & 1 & -7 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -7 \end{array}\right]$

The solution is $x_1 = 4$, $x_2 = -7$.

2. $\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 1 & -1 & -2 \end{array}\right] \xrightarrow{R_{12}} \left[\begin{array}{cc|c} 1 & -1 & -2 \\ 3 & -2 & 4 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 1 & 10 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 10 \end{array}\right]$

The solution is $x_1 = 8$, $x_2 = 10$.

3. $\left[\begin{array}{cc|c} 9 & 3 & -5 \\ 2 & -1 & -1 \end{array}\right] \xrightarrow{\frac{1}{9}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & -\frac{5}{9} \\ 2 & 1 & -1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & -\frac{5}{9} \\ 0 & \frac{1}{3} & \frac{1}{9} \end{array}\right] \xrightarrow{3R_2} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & -\frac{5}{9} \\ 0 & 1 & \frac{1}{3} \end{array}\right]$

$\xrightarrow{-\frac{1}{3}R_2+R_1} \left[\begin{array}{cc|c} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & \frac{1}{3} \end{array}\right]$

The solution is $x_1 = -\frac{2}{3}$, $x_2 = \frac{1}{3}$.

4. $\left[\begin{array}{cc|c} 10 & 15 & 1 \\ 3 & 2 & -1 \end{array}\right] \xrightarrow{\frac{1}{10}R_1} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{1}{10} \\ 3 & 2 & -1 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{1}{10} \\ 0 & -\frac{5}{2} & -\frac{13}{10} \end{array}\right] \xrightarrow{-\frac{2}{5}R_2} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{1}{10} \\ 0 & 1 & \frac{13}{25} \end{array}\right]$

$\xrightarrow{-\frac{3}{2}R_2+R_1} \left[\begin{array}{cc|c} 1 & 0 & -\frac{17}{25} \\ 0 & 1 & \frac{13}{25} \end{array}\right]$

The solution is $x_1 = -\frac{17}{25}$, $x_2 = \frac{13}{25}$.

5. $\left[\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 2 & 3 & 5 & 7 \\ 1 & -2 & 3 & -11 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 5 & 7 & 13 \\ 0 & -1 & 4 & -8 \end{array}\right] \xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 1 & \frac{7}{5} & \frac{13}{5} \\ 0 & -1 & 4 & -8 \end{array}\right]$

$\xrightarrow[R_2+R_3]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{7}{5} & \frac{13}{5} \\ 0 & 0 & \frac{27}{5} & -\frac{27}{5} \end{array}\right] \xrightarrow{\frac{5}{27}R_3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{7}{5} & \frac{13}{5} \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[-\frac{7}{5}R_3+R_2]{-\frac{2}{5}R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$

The solution is $x_1 = 0$, $x_2 = 4$, $x_3 = -1$.

6. $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 2 & 9 \\ 1 & -1 & 1 & 3 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 4 & 9 \\ 0 & -3 & 2 & 3 \end{array}\right] \xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{4}{3} & -3 \\ 0 & -3 & 2 & 3 \end{array}\right]$

$\xrightarrow[3R_2+R_3]{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{3} & 6 \\ 0 & 1 & -\frac{4}{3} & -3 \\ 0 & 0 & -2 & -6 \end{array}\right] \xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{3} & 6 \\ 0 & 1 & -\frac{4}{3} & -3 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[\frac{4}{3}R_3+R_2]{-\frac{5}{3}R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

The solution is $x_1 = 1$, $x_2 = 1$, $x_3 = 3$.

7. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$

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Since $x_3 = 0$, setting $x_2 = t$ we obtain $x_1 = -t$, $x_2 = t$, $x_3 = 0$.

8. $\left[\begin{array}{ccc|c} 1 & 2 & -4 & 9 \\ 5 & -1 & 2 & 1 \end{array}\right] \xrightarrow{-5R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 9 \\ 0 & -11 & 22 & -44 \end{array}\right] \xrightarrow{-\frac{1}{11}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 9 \\ 0 & 1 & -2 & 4 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 4 \end{array}\right]$

If $x_3 = t$, the solution is $x_1 = 1$, $x_2 = 4 + 2t$, $x_3 = t$

9. $\left[\begin{array}{ccc|c} 1 & -1 & -1 & 8 \\ 1 & -1 & 1 & 3 \\ -1 & 1 & 1 & 4 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 8 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & 12 \end{array}\right]$

Since the bottom row implies $0 = 12$, the system is inconsistent.

10. $\left[\begin{array}{cc|c} 3 & 1 & 4 \\ 4 & 3 & -3 \\ 2 & -1 & 11 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{array}\right]$

The solution is $x_1 = 3$, $x_2 = -5$.

11. $\left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

The solution is $x_1 = x_2 = x_3 = 0$.

12. $\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 2 & 4 & 5 & 0 \\ 6 & 0 & -3 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The solution is $x_1 = \frac{1}{2}t$, $x_2 = -\frac{3}{2}t$, $x_3 = t$.

13. $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & -3 & -1 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right]$

The solution is $x_1 = -2$, $x_2 - 2$, $x_3 = 4$.

14. $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 3 & -1 & 2 & 5 \\ 2 & 1 & 1 & 1 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 1 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & -2 \end{array}\right]$

Since the bottom row implies $0 = -2$, the system is inconsistent.

15. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & -1 & -1 \\ 3 & 1 & 1 & 5 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

If $x_3 = t$ the solution is $x_1 = 1$, $x_2 = 2 - t$, $x_3 = t$.

16. $\left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ -3 & -2 & 1 & -7 \\ 2 & 3 & 1 & 8 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

If $x_3 = t$ the solution is $x_1 = 1 + t$, $x_2 = 2 - t$, $x_3 = t$.

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17. $\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 2 & 1 & 1 & 3 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

The solution is $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$.

18. $\left[\begin{array}{cccc|c} 2 & 1 & 1 & 0 & 3 \\ 3 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 3 \\ 4 & 5 & -2 & 1 & 16 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

The solution is $x_1 = 1$, $x_2 = 2$, $x_3 = -1$, $x_4 = 0$.

19. $\left[\begin{array}{cccc|c} 1 & 3 & 5 & -1 & 1 \\ 0 & 1 & 1 & -1 & 4 \\ 1 & 2 & 5 & -4 & -2 \\ 1 & 4 & 6 & -2 & 6 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{cccc|c} 1 & 3 & 5 & -1 & 1 \\ 0 & 1 & 1 & -1 & 4 \\ 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

Since the bottom row implies $0 = 1$, the system is inconsistent.

20. $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 4 & 9 & 1 & 12 & 0 \\ 3 & 9 & 6 & 21 & 0 \\ 1 & 3 & 1 & 9 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 8 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

If $x_4 = t$ the solution is $x_1 = 19t$, $x_2 = -10t$, $x_3 = 2t$, $x_4 = t$.

21. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4.280 \\ 0.2 & -0.1 & -0.5 & -1.978 \\ 4.1 & 0.3 & 0.12 & 1.686 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4.28 \\ 0 & 1 & 2.333 & 9.447 \\ 0 & 0 & 1 & 4.1 \end{array}\right]$

The solution is $x_1 = 0.3$, $x_2 = -0.12$, $x_3 = 4.1$.

22. $\left[\begin{array}{ccc|c} 2.5 & 1.4 & 4.5 & 2.6170 \\ 1.35 & 0.95 & 1.2 & 0.7545 \\ 2.7 & 3.05 & -1.44 & -1.4292 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0.56 & 1.8 & 1.0468 \\ 0 & 1 & -6.3402 & -3.3953 \\ 0 & 0 & 1 & 0.28 \end{array}\right]$

The solution is $x_1 = 1.45$, $x_2 = -1.62$, $x_3 = 0.28$.

23. From $x_1\text{Na} + x_2\text{H}_2\text{O} \rightarrow x_3\text{NaOH} + x_4\text{H}_2$ we obtain the system $x_1 = x_3$, $2x_2 = x_3 + 2x_4$, $x_2 = x_3$. We see that $x_1 = x_2 = x_3$, so the second equation becomes $2x_1 = x_1 + 2x_4$ or $x_1 = 2x_4$. A solution of the system is $x_1 = x_2 = x_3 = 2t$, $x_4 = t$. Letting $t = 1$ we obtain the balanced equation $2\text{Na} + 2\text{H}_2\text{O} \rightarrow 2\text{NaOH} + \text{H}_2$.

24. From $x_1\text{KClO}_3 \rightarrow x_2\text{KCl} + x_3\text{O}_2$ we obtain the system $x_1 = x_2$, $x_1 = x_2$, $3x_1 = 2x_3$. Letting $x_3 = t$ we see that a solution of the system is $x_1 = x_2 = \frac{2}{3}t$, $x_3 = t$. Taking $t = 3$ we obtain the balanced equation

$$2\text{KClO}_3 \rightarrow 2\text{KCl} + 3\text{O}_2.$$

25. From $x_1\text{Fe}_3\text{O}_4 + x_2\text{C} \rightarrow x_3\text{Fe} + x_4\text{CO}$ we obtain the system $3x_1 = x_3$, $4x_1 = x_4$, $x_2 = x_4$. Letting $x_1 = t$ we see that $x_3 = 3t$ and $x_2 = x_4 = 4t$. Taking $t = 1$ we obtain the balanced equation

$$\text{Fe}_3\text{O}_4 + 4\text{C} \rightarrow 3\text{Fe} + 4\text{CO}.$$

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26. From $x_1C_5H_8 + x_2O_2 \rightarrow x_3CO_2 + x_4H_2O$ we obtain the system $5x_1 = x_3$, $8x_1 = 2x_4$, $2x_2 = 2x_3 + x_4$. Letting $x_1 = t$ we see that $x_3 = 5t$, $x_4 = 4t$, and $x_2 = 7t$. Taking $t = 1$ we obtain the balanced equation

$$C_5H_8 + 7O_2 \rightarrow 5CO_2 + 4H_2O.$$

27. From $x_1Cu + x_2HNO_3 \rightarrow x_3Cu(NO_3)_2 + x_4H_2O + x_5NO$ we obtain the system

$$x_1 = 3, \qquad x_2 = 2x_4, \qquad x_2 = 2x_3 + x_5, \qquad 3x_2 = 6x_3 + x_4 + x_5.$$

Letting $x_4 = t$ we see that $x_2 = 2t$ and

$$\begin{aligned} 2t &= 2x_3 + x_5 \\ 6t &= 6x_3 + t + x_5 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 2x_3 + x_5 &= 2t \\ 6x_3 + x_5 &= 5t. \end{aligned}$$

Then $x_3 = \frac{3}{4}t$ and $x_5 = \frac{1}{2}t$. Finally, $x_1 = x_3 = \frac{3}{4}t$. Taking $t = 4$ we obtain the balanced equation

$$3Cu + 8HNO_3 \rightarrow 3Cu(NO_3)_2 + 4H_2O + 2NO.$$

28. From $x_1Ca_3(PO_4)_2 + x_2H_3PO_4 \rightarrow x_3Ca(H_2PO_4)_2$ we obtain the system

$$3x_1 = x_3, \qquad 2x_1 + x_2 = 2x_3, \qquad 8x_1 + 4x_2 = 8x_3, \qquad 3x_2 = 4x_3.$$

Letting $x_1 = t$ we see from the first equation that $x_3 = 3t$ and from the fourth equation that $x_2 = 4t$. These choices also satisfy the second and third equations. Taking $t = 1$ we obtain the balanced equation

$$Ca_3(PO_4)_2 + 4H_3PO_4 \rightarrow 3Ca(H_2PO_4)_2.$$

29. The system of equations is

$$\begin{aligned} -i_1 + i_2 - i_3 &= 0 \\ 10 - 3i_1 + 5i_3 &= 0 \\ 27 - 6i_2 - 5i_3 &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} -i_1 + i_2 - i_3 &= 0 \\ 3i_1 - 5i_3 &= 10 \\ 6i_2 + 5i_3 &= 27 \end{aligned}$$

Gaussian elimination gives

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 3 & 0 & -5 & 10 \\ 0 & 6 & 5 & 27 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -8/3 & 10/3 \\ 0 & 0 & 1 & 1/3 \end{array}\right].$$

The solution is $i_1 = \dfrac{35}{9}$, $i_2 = \dfrac{38}{9}$, $i_3 = \dfrac{1}{3}$.

30. The system of equations is

$$\begin{aligned} i_1 - i_2 - i_3 &= 0 \\ 52 - i_1 - 5i_2 &= 0 \\ -10i_3 + 5i_2 &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} i_1 - i_2 - i_3 &= 0 \\ i_1 + 5i_2 &= 52 \\ 5i_2 - 10i_3 &= 0 \end{aligned}$$

Gaussian elimination gives

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & 5 & 0 & 52 \\ 0 & 5 & -10 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1/6 & 26/3 \\ 0 & 0 & 1 & 4 \end{array}\right].$$

The solution is $i_1 = 12$, $i_2 = 8$, $i_3 = 4$.

31. Interchange row 1 and row in $\mathbf{I}_3$.

32. Multiply row 3 by c in $\mathbf{I}_3$.

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33. Add c times row 2 to row 3 in $\mathbf{I}_3$.

34. Add row 4 to row 1 in $\mathbf{I}_4$.

35. $\mathbf{EA} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

36. $\mathbf{EA} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ca_{31} & ca_{32} & ca_{33} \end{bmatrix}$

37. $\mathbf{EA} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ca_{21} + a_{31} & ca_{22} + a_{32} & ca_{23} + a_{33} \end{bmatrix}$

38. $\mathbf{E}_1\mathbf{E}_2\mathbf{A} = \mathbf{E}_1 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ca_{21} + a_{31} & ca_{22} + a_{32} & ca_{23} + a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ ca_{21} + a_{31} & ca_{22} + a_{32} & ca_{23} + a_{33} \end{bmatrix}$

39. The system is equivalent to

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}.$$

Letting

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix} \mathbf{X}$$

we have

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}.$$

This implies $y_1 = 2$ and $\frac{1}{2}y_1 + y_2 = 1 + y_2 = 6$ or $y_2 = 5$. Then

$$\begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix},$$

which implies $3x_2 = 5$ or $x_2 = \frac{5}{3}$ and $2x_1 - 2x_2 = 2x_1 - \frac{10}{3} = 2$ or $x_1 = \frac{8}{3}$. The solution is $\mathbf{X} = \left(\frac{8}{3}, \frac{5}{3}\right)$.

40. The system is equivalent to

$$\begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 0 & -\frac{1}{3} \end{bmatrix} \mathbf{X} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Letting

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 0 & -\frac{1}{3} \end{bmatrix} \mathbf{X}$$

we have

$$\begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This implies $y_1 = 1$ and $\frac{2}{3}y_1 + y_2 = \frac{2}{3} + y_2 = -1$ or $y_2 = -\frac{5}{3}$. Then

$$\begin{bmatrix} 6 & 2 \\ 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{5}{3} \end{bmatrix},$$

which implies $-\frac{1}{3}x_2 = -\frac{5}{3}$ or $x_2 = 5$ and $6x_1 + 2x_2 = 6x_1 + 10 = 1$ or $x_1 = -\frac{3}{2}$. The solution is $\mathbf{X} = \left(-\frac{3}{2}, 5\right)$.

41. The system is equivalent to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 10 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -21 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

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Letting

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -21 \end{bmatrix} \mathbf{X}$$

we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 10 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

This implies $y_1 = 2$, $y_2 = -1$, and $2y_1 + 10y_2 + y_3 = 4 - 10 + y_3 = 1$ or $y_3 = 7$. Then

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix},$$

which implies $-21x_3 = 7$ or $x_3 = -\frac{1}{3}$, $x_2 + 2x_3 = x_2 - \frac{2}{3} = -1$ or $x_2 = -\frac{1}{3}$, and $x_1 - 2x_2 + x_3 = x_1 + \frac{2}{3} - \frac{1}{3} = 2$ or $x_1 = \frac{5}{3}$. The solution is $\mathbf{X} = (\frac{5}{3}, -\frac{1}{3}, -\frac{1}{3})$.

42. The system is equivalent to

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}.$$

Letting

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{X}$$

we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}.$$

This implies $y_1 = 0$, $3y_1 + y_2 = y_2 = 1$, and $y_1 + y_2 + y_3 = 0 + 1 + y_3 = 4$ or $y_3 = 3$. Then

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix},$$

which implies $x_3 = 3$, $-2x_2 - x_3 = -2x_2 - 3 = 1$ or $x_2 = -2$, and $x_1 + x_2 + x_3 = x_1 - 2 + 3 = 0$ or $x_1 = -1$. The solution is $\mathbf{X} = (-1, -2, 3)$.

43. Using the **Solve** function in *Mathematica* we find

(a) $x_1 = -0.0717393 - 1.43084c$, $x_2 = -0.332591 + 0.855709c$, $x_3 = c$, where c is any real number

(b) $x_1 = c/3$, $x_2 = 5c/6$, $x_3 = c$, where c is any real number

(c) $x_1 = -3.76993$, $x_2 = -1.09071$, $x_3 = -4.50461$, $x_4 = -3.12221$

(d) $x_1 = \frac{8}{3} - \frac{7}{3}b + \frac{2}{3}c$, $x_2 = \frac{2}{3} - \frac{1}{3}b - \frac{1}{3}c$, $x_3 = -3$, $x_4 = b$, $x_5 = c$, where b and c are any real numbers.

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Exercises 8.3

1. $\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$; The rank is 2.

2. $\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$; The rank is 1.

3. $\begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; The rank is 1.

4. $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 4 \\ -1 & 0 & 3 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$; The rank is 3.

5. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 1 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$; The rank is 3.

6. $\begin{bmatrix} 3 & -1 & 2 & 0 \\ 6 & 2 & 4 & 5 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 0 & \frac{5}{4} \end{bmatrix}$; The rank is 2.

7. $\begin{bmatrix} 1 & -2 \\ 3 & -6 \\ 7 & -1 \\ 4 & 5 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$; The rank is 2.

8. $\begin{bmatrix} 1 & -2 & 3 & 4 \\ 1 & 4 & 6 & 8 \\ 0 & 1 & 0 & 0 \\ 2 & 5 & 6 & 8 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$; The rank is 3.

9. $\begin{bmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 1 & 0 & 5 & 1 \\ 2 & 1 & \frac{2}{3} & 3 & \frac{1}{3} \\ 6 & 6 & 6 & 12 & 0 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{3}{2} & \frac{1}{6} \\ 0 & 1 & \frac{4}{3} & 1 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; The rank is 3.

10. $\begin{bmatrix} 1 & -2 & 1 & 8 & -1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 3 & -1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 & -1 & 2 & 10 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & -2 & 1 & 8 & -1 & 1 & 2 & 6 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & -2 & 1 & 8 & -1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 3 & -1 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 9 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; The rank is 4.

11. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 5 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$;

Since the rank of the matrix is 3 and there are 3 vectors, the vectors are linearly independent.

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12. $\begin{bmatrix} 2 & 6 & 3 \\ 1 & -1 & 4 \\ 3 & 2 & 1 \\ 2 & 5 & 4 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -\frac{5}{8} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Since the rank of the matrix is 3 and there are 4 vectors, the vectors are linearly dependent.

13. $\begin{bmatrix} 1 & -1 & 3 & -1 \\ 1 & -1 & 4 & 2 \\ 1 & -1 & 5 & 7 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & -1 & 3 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Since the rank of the matrix is 3 and there are 3 vectors, the vectors are linearly independent.

14. $\begin{bmatrix} 2 & 1 & 1 & 5 \\ 2 & 2 & 1 & 1 \\ 3 & -1 & 6 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -7 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Since the rank of the matrix is 4 and there are 4 vectors, the vectors are linearly independent.

15. Since the number of unknowns is $n = 8$ and the rank of the coefficient matrix is $r = 3$, the solution of the system has $n - r = 5$ parameters.

16. **(a)** The maximum possible rank of $\mathbf{A}$ is the number of rows in $\mathbf{A}$, which is 4.

(b) The system is inconsistent if $\text{rank}(\mathbf{A}) < \text{rank}(\mathbf{A}/\mathbf{B}) = 2$ and consistent if $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}/\mathbf{B}) = 2$.

(c) The system has $n = 6$ unknowns and the rank of $\mathbf{A}$ is $r = 3$, so the solution of the system has $n - r = 3$ parameters.

17. Since $2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$ we conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent. Thus, the rank of $\mathbf{A}$ is at most 2.

18. Since the rank of $\mathbf{A}$ is $r = 3$ and the number of equations is $n = 6$, the solution of the system has $n - r = 3$ parameters. Thus, the solution of the system is not unique.

19. The system consists of 4 equations, so the rank of the coefficient matrix is at most 4, and the maximum number of linearly independent rows is 4. However, the maximum number of linearly independent columns is the same as the maximum number of linearly independent rows. Thus, the coefficient matrix has at most 4 linearly independent columns. Since there are 5 column vectors, they must be linearly dependent.

20. Using the **RowReduce** in *Mathematica* we find that the reduced row-echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{834}{2215} & -\frac{261}{443} \\ 0 & 1 & 0 & 0 & 0 & \frac{1818}{2215} & \frac{282}{443} \\ 0 & 0 & 1 & 0 & 0 & \frac{13}{443} & -\frac{6}{443} \\ 0 & 0 & 0 & 1 & 0 & \frac{4214}{2215} & -\frac{130}{443} \\ 0 & 0 & 0 & 0 & 1 & -\frac{6079}{2215} & \frac{677}{443} \end{bmatrix}.$$

We conclude that the system is consistent and the solution is $x_1 = -\frac{226}{443} - \frac{834}{2215}c$, $x_2 = \frac{282}{443} - \frac{1818}{2215}c$, $x_3 = -\frac{6}{443} - \frac{13}{443}c$, $x_4 = -\frac{130}{443} - \frac{4214}{2215}c$, $x_5 = \frac{677}{433} + \frac{6079}{2215}c$, $x_6 = c$.

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Exercises 8.4

1. $M_{12} = \begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} = 9$

2. $M_{32} = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix}$

3. $C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} = 1$

4. $C_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 4 \\ -2 & 5 \end{vmatrix} = 18$

5. $M_{33} = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = 2$

6. $M_{41} = \begin{vmatrix} 2 & 4 & 0 \\ 2 & -2 & 3 \\ 1 & 0 & -1 \end{vmatrix} = 24$

7. $C_{34} = (-1)^{3+4} \begin{vmatrix} 0 & 2 & 4 \\ 1 & 2 & -2 \\ 1 & 1 & 1 \end{vmatrix} = 10$

8. $C_{23} = (-1)^{2+3} \begin{vmatrix} 0 & 2 & 0 \\ 5 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 22$

9. -7 **10.** 2 **11.** 17 **12.** $-1/2$

13. $(1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4$

14. $(-3-\lambda)(5-\lambda) - 8 = \lambda^2 - 2\lambda - 23$

15. $\begin{vmatrix} 0 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 5 & 8 \end{vmatrix} = -3 \begin{vmatrix} 2 & 0 \\ 5 & 8 \end{vmatrix} = -48$

16. $\begin{vmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 5 \begin{vmatrix} -3 & 0 \\ 0 & 2 \end{vmatrix} = 5(-3)(2) = -30$

17. $\begin{vmatrix} 3 & 0 & 2 \\ 2 & 7 & 1 \\ 2 & 6 & 4 \end{vmatrix} = 3 \begin{vmatrix} 7 & 1 \\ 6 & 4 \end{vmatrix} + 2 \begin{vmatrix} 2 & 7 \\ 2 & 6 \end{vmatrix} = 3(22) + 2(-2) = 62$

18. $\begin{vmatrix} 1 & -1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & 9 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 1 & 9 \end{vmatrix} - 2 \begin{vmatrix} -1 & -1 \\ 1 & 9 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 2 & -2 \end{vmatrix} = 20 - 2(-8) + 4 = 40$

19. $\begin{vmatrix} 4 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} = 4 \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} - 5 \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$

20. $\begin{vmatrix} \frac{1}{4} & 6 & 0 \\ \frac{1}{3} & 8 & 0 \\ \frac{1}{2} & 9 & 0 \end{vmatrix} = 0$, expanding along the third column.

21. $\begin{vmatrix} -2 & -1 & 4 \\ -3 & 6 & 1 \\ -3 & 4 & 8 \end{vmatrix} = -2 \begin{vmatrix} 6 & 1 \\ 4 & 8 \end{vmatrix} + 3 \begin{vmatrix} -1 & 4 \\ 4 & 8 \end{vmatrix} - 3 \begin{vmatrix} -1 & 4 \\ 6 & 1 \end{vmatrix} = -2(44) + 3(-24) - 3(-25) = -85$

22. $\begin{vmatrix} 3 & 5 & 1 \\ -1 & 2 & 5 \\ 7 & -4 & 10 \end{vmatrix} = 3 \begin{vmatrix} 2 & 5 \\ -4 & 10 \end{vmatrix} - 5 \begin{vmatrix} -1 & 5 \\ 7 & 10 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 7 & -4 \end{vmatrix} = 3(40) - 5(-45) + (-10) = 335$

23. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} y & z \\ 3 & 4 \end{vmatrix} - \begin{vmatrix} x & z \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} x & y \\ 2 & 3 \end{vmatrix} = (4y - 3z) - (4x - 2z) + (3x - 2y) = -x + 2y - z$

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24. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 2+x & 3+y & 4+z \end{vmatrix} = \begin{vmatrix} y & z \\ 3+y & 4+z \end{vmatrix} - \begin{vmatrix} x & z \\ 2+x & 4+z \end{vmatrix} + \begin{vmatrix} x & y \\ 2+x & 3+y \end{vmatrix}$

$= (4y + yz - 3z - yz) - (4x + xz - 2z - xz) + (3x + xy - 2y - xy) = -x + 2y - z$

25. $\begin{vmatrix} 1 & 1 & -3 & 0 \\ 1 & 5 & 3 & 2 \\ 1 & -2 & 1 & 0 \\ 4 & 8 & 0 & 0 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & -3 \\ 1 & -2 & 1 \\ 4 & 8 & 0 \end{vmatrix} = 2(4) \begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} - 2(8) \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = 8(-5) - 16(4) = -104$

26. $\begin{vmatrix} 2 & 1 & -2 & 1 \\ 0 & 5 & 0 & 4 \\ 1 & 6 & 1 & 0 \\ 5 & -1 & 1 & 1 \end{vmatrix} = 5 \begin{vmatrix} 2 & -2 & 1 \\ 1 & 1 & 0 \\ 5 & 1 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & 1 & -2 \\ 1 & 6 & 1 \\ 5 & -1 & 1 \end{vmatrix} = 5(0) + 4(80) = 320$

27. Expanding along the first column in the original matrix and each succeeding minor, we obtain $3(1)(2)(4)(2) = 48$.

28. Expanding along the bottom row we obtain

$$-1 \begin{vmatrix} 2 & 0 & 0 & -2 \\ 1 & 6 & 0 & 5 \\ 1 & 2 & -1 & 1 \\ 2 & 1 & -2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 & 0 & 0 \\ 1 & 1 & 6 & 0 \\ 1 & 0 & 2 & -1 \\ 2 & 0 & 1 & -2 \end{vmatrix} = -1(-48) + 0 = 48.$$

29. Solving $\lambda^2 - 2\lambda - 15 - 20 = \lambda^2 - 2\lambda - 35 = (\lambda - 7)(\lambda + 5) = 0$ we obtain $\lambda = 7$ and -5.

30. Solving $-\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda(\lambda - 2)(\lambda - 1) = 0$ we obtain $\lambda = 0$, 1, and 2.

Exercises 8.5

1. Theorem 8.11

2. Theorem 8.14

3. Theorem 8.14

4. Theorem 8.12 and 8.11

5. Theorem 8.12 (twice)

6. Theorem 8.11 (twice)

7. Theorem 8.10

8. Theorem 8.12 and 8.9

9. Theorem 8.8

10. Theorem 8.11 (twice)

11. $\det \mathbf{A} = -5$

12. $\det \mathbf{B} = 2(3)(5) = 30$

13. $\det \mathbf{C} = -5$

14. $\det \mathbf{D} = 5$

15. $\det \mathbf{A} = 6(\frac{2}{3})(-4)(-5) = 80$

16. $\det \mathbf{B} = -a_{13}a_{22}a_{31}$

17. $\det \mathbf{C} = (-5)(7)(3) = -105$

18. $\det \mathbf{D} = 4(7)(-2) = -56$

19. $\det \mathbf{A} = 14 = \det \mathbf{A}^T$

20. $\det \mathbf{A} = 96 = \det{}^T$

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21. $\det \mathbf{AB} = \begin{vmatrix} 0 & -2 & 2 \\ 10 & 7 & 23 \\ 8 & 4 & 16 \end{vmatrix} = -80 = 20(-4) = \det \mathbf{A} \det \mathbf{B}$

22. From Problem 21, $(\det \mathbf{A})^2 = \det \mathbf{A}^2 = \det \mathbf{I} = 1$, so $\det \mathbf{A} = \pm 1$.

23. Using Theorems 8.14, 8.12, and 8.9, $\det \mathbf{A} = \begin{vmatrix} a & 1 & 2 \\ b & 1 & 2 \\ c & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} = 0.$

24. Using Theorems 8.14 and 8.9,

$$\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix} = (x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

25. $\begin{vmatrix} 1 & 1 & 5 \\ 4 & 3 & 6 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 0 & -1 & -14 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 0 & -1 & -14 \\ 0 & 0 & 15 \end{vmatrix} = 1(-1)(15) = -15$

26. $\begin{vmatrix} 2 & 4 & 5 \\ 4 & 2 & 0 \\ 8 & 7 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 5 \\ 0 & -6 & -10 \\ 0 & -9 & -22 \end{vmatrix} = -2 \begin{vmatrix} 2 & 4 & 5 \\ 0 & 3 & 5 \\ 0 & -9 & -22 \end{vmatrix} = -2 \begin{vmatrix} 2 & 4 & 5 \\ 0 & 3 & 5 \\ 0 & 0 & -7 \end{vmatrix} = -2(2)(3)(-7) = 84$

27. $\begin{vmatrix} -1 & 2 & 3 \\ 4 & -5 & -2 \\ 9 & -9 & 6 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 3 \\ 0 & 3 & 10 \\ 0 & 9 & 33 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 3 \\ 0 & 3 & 10 \\ 0 & 0 & 3 \end{vmatrix} = -1(3)(3) = -9$

28. $\begin{vmatrix} -2 & 2 & -6 \\ 5 & 0 & 1 \\ 1 & -2 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & 2 \\ 5 & 0 & 1 \\ -2 & 2 & -6 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & 2 \\ 0 & 10 & -9 \\ 0 & -2 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & 2 \\ 0 & 10 & -9 \\ 0 & 0 & -\frac{19}{5} \end{vmatrix} = -1(10)(-\frac{19}{5}) = 38$

29. $\begin{vmatrix} 1 & -2 & 2 & 1 \\ 2 & 1 & -2 & 3 \\ 3 & 4 & -8 & 1 \\ 3 & -11 & 12 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 2 & 1 \\ 0 & 5 & -6 & 1 \\ 0 & 10 & -14 & -2 \\ 0 & -5 & 6 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 2 & 1 \\ 0 & 5 & -6 & 1 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 1(5)(-2)(0) = 0$

30. $\begin{vmatrix} 0 & 1 & 4 & 5 \\ 2 & 5 & 0 & 1 \\ 1 & 2 & 2 & 0 \\ 3 & 1 & 3 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 2 & 0 \\ 2 & 5 & 0 & 1 \\ 0 & 1 & 4 & 5 \\ 3 & 1 & 3 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 1 & 4 & 5 \\ 0 & -5 & -3 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & -23 & 7 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & -23 & \frac{37}{2} \end{vmatrix}$

$$= -(1)(1)(8)(\frac{37}{2}) = -148$$

31. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 2 & 3 & 6 & 7 \\ 1 & 5 & 8 & 20 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 3 & 5 & 16 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -1 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 8 \end{vmatrix} = 1(1)(2)(8) = 16$

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32. $$\begin{vmatrix} 2 & 9 & 1 & 8 \\ 1 & 3 & 7 & 4 \\ 0 & 1 & 6 & 5 \\ 3 & 1 & 4 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 7 & 4 \\ 2 & 9 & 1 & 8 \\ 0 & 1 & 6 & 5 \\ 3 & 1 & 4 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 7 & 4 \\ 0 & 3 & -13 & 0 \\ 0 & 1 & 6 & 5 \\ 0 & -8 & -17 & -10 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 7 & 4 \\ 0 & 1 & 6 & 5 \\ 0 & 3 & -13 & 0 \\ 0 & -8 & -17 & -10 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 7 & 4 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & -31 & -15 \\ 0 & 0 & 31 & 30 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 7 & 4 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & -31 & -15 \\ 0 & 0 & 0 & 15 \end{vmatrix} = 1(1)(-31)(15) = -465$$

33. We first use the second row to reduce the third row. Then we use the first row to reduce the second row.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 0 & b^2 - ab & c^2 - ac \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b - a & c - a \\ 0 & b(b-a) & c(c-a) \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & b & c \end{vmatrix}.$$

Expanding along the first row gives $(b-a)(c-a)(c-b)$.

34. In order, we use the third row to reduce the fourth row, the second row to reduce the third row, and the first row to reduce the second row. We then pull out a common factor from each column.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & b^2 - ab & c^2 - ac & d^2 - ac \\ 0 & b^3 - ab^2 & c^3 - ac^2 & d^3 - ad^2 \end{vmatrix} = (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & b & c & d \\ 0 & b^2 & c^2 & d^2 \end{vmatrix}.$$

Expanding along the first column and using Problem 33 we obtain $(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$.

35. Since $C_{11} = 4$, $C_{12} = 5$, and $C_{13} = -6$, we have $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} = (-1)(4) + 2(5) + 1(-6) = 0$. Since $C_{12} = 5$, $C_{22} = -7$, and $C_{23} = -3$, we have $a_{13}C_{12} + a_{23}C_{22} + a_{33}C_{32} = 2(5) + 1(-7) + 1(-3) = 0$.

36. Since $C_{11} + -7$, $C_{12} = -8$, and $C_{13} = -10$ we have $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} = -2(-7) + 3(-8) - 1(-10) = 0$. Since $C_{12} = -8$, $C_{22} = -19$, and $C_{32} = -7$ we have $a_{13}C_{12} + a_{23}C_{22} + a_{33}C_{32} = 5(-8) - 1(-19) - 3(-7) = 0$.

37. $\det(\mathbf{A} + \mathbf{B}) = \begin{vmatrix} 10 & 0 \\ 0 & -3 \end{vmatrix} = -30$; $\det \mathbf{A} + \det \mathbf{B} = 10 - 31 = -21$

38. $\det(2\mathbf{A}) = 2^5 \det \mathbf{A} = 32(-7) = -224$

39. Factoring -1 out of each row we see that $\det(-\mathbf{A}) = (-1)^5 \det \mathbf{A} = -\det \mathbf{A}$. Then $-\det \mathbf{A} = \det(-\mathbf{A}) = \det \mathbf{A}^T = \det \mathbf{A}$ and $\det \mathbf{A} = 0$.

40. **(a)** Cofactors: $25! \approx 1.55(10^{25})$; Row reduction: $25^3/3 \approx 5.2(10^3)$

(b) Cofactors: about 90 billion centuries; Row reduction: about $\frac{1}{10}$ second

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Exercises 8.6

1. $\mathbf{AB} = \begin{bmatrix} 3-2 & -1+1 \\ 6-6 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $\mathbf{AB} = \begin{bmatrix} 2-1 & -1+1 & -2+2 \\ 6-6 & -3+4 & 6-6 \\ 2+1-3 & -1-1+2 & 2+2-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. $\det \mathbf{A} = 9$. $\mathbf{A}$ is nonsingular. $\mathbf{A}^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & \frac{1}{9} \\ -\frac{4}{9} & \frac{5}{9} \end{bmatrix}$

4. $\det \mathbf{A} = 5$. $\mathbf{A}$ is nonsingular. $\mathbf{A}^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -4 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{4}{5} & \frac{1}{15} \end{bmatrix}$

5. $\det \mathbf{A} = 12$. $\mathbf{A}$ is nonsingular. $\mathbf{A}^{-1} = \frac{1}{12}\begin{bmatrix} 2 & 0 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

6. $\det \mathbf{A} = -3\pi^2$. $\mathbf{A}$ is nonsingular. $\mathbf{A}^{-1} = -\frac{1}{3\pi^2}\begin{bmatrix} \pi & \pi \\ \pi & -2\pi \end{bmatrix} = \begin{bmatrix} -\frac{1}{3\pi} & -\frac{1}{3\pi} \\ -\frac{1}{3\pi} & \frac{2}{3\pi} \end{bmatrix}$

7. $\det \mathbf{A} = -16$. $\mathbf{A}$ is nonsingular. $\mathbf{A}^{-1} = -\frac{1}{16}\begin{bmatrix} 8 & -8 & -8 \\ 2 & -4 & 6 \\ -6 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{8} \\ \frac{3}{8} & -\frac{1}{4} & \frac{1}{8} \end{bmatrix}$

8. $\det \mathbf{A} = 0$. $\mathbf{A}$ is singular.

9. $\det \mathbf{A} = -30$. $\mathbf{A}$ is nonsingular. $\mathbf{A}^{-1} = -\frac{1}{30}\begin{bmatrix} -14 & 13 & 16 \\ -2 & 4 & -2 \\ -4 & -7 & -4 \end{bmatrix} = \begin{bmatrix} \frac{7}{15} & -\frac{13}{30} & -\frac{8}{15} \\ \frac{1}{15} & -\frac{2}{15} & \frac{1}{15} \\ \frac{2}{15} & \frac{7}{30} & \frac{2}{15} \end{bmatrix}$

10. $\det \mathbf{A} = 78$. $\mathbf{A}$ is nonsingular. $\mathbf{A}^{-1} = \frac{1}{78}\begin{bmatrix} 8 & 20 & 2 \\ -2 & -5 & 19 \\ 12 & -9 & 3 \end{bmatrix}\begin{bmatrix} \frac{4}{39} & \frac{10}{39} & \frac{1}{39} \\ -\frac{1}{39} & -\frac{5}{78} & \frac{19}{78} \\ \frac{2}{13} & -\frac{3}{26} & \frac{1}{26} \end{bmatrix}$

11. $\det \mathbf{A} = -36$. $\mathbf{A}$ is nonsingular. $\mathbf{A}^{-1} = -\frac{1}{36}\begin{bmatrix} -12 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 18 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$

12. $\det \mathbf{A} = 16$. $\mathbf{A}$ is nonsingular. $\mathbf{A}^{-1} = \frac{1}{16}\begin{bmatrix} 0 & 0 & 2 \\ 8 & 0 & 0 \\ 0 & 16 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{8} \\ \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

13. $\det \mathbf{A} = 27$. $\mathbf{A}$ is nonsingular. $\mathbf{A}^{-1} = \frac{1}{27}\begin{bmatrix} 6 & 21 & -9 & -36 \\ -1 & 1 & 6 & -3 \\ 10 & 17 & -6 & -51 \\ 4 & -4 & 3 & 12 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{7}{9} & -\frac{1}{3} & -\frac{4}{3} \\ -\frac{1}{27} & \frac{1}{27} & \frac{2}{9} & -\frac{1}{9} \\ \frac{10}{27} & \frac{17}{27} & -\frac{2}{9} & -\frac{17}{9} \\ \frac{4}{27} & -\frac{4}{27} & \frac{1}{9} & \frac{4}{9} \end{bmatrix}$

14. $\det \mathbf{A} = -6$. $\mathbf{A}$ is nonsingular. $\mathbf{A}^{-1} = -\frac{1}{6}\begin{bmatrix} 0 & 1 & -3 & 3 \\ 0 & 1 & 3 & -9 \\ 0 & -2 & 0 & 0 \\ -6 & -1 & -3 & 15 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{6} & -\frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{3} & 0 & 0 \\ 1 & \frac{1}{6} & \frac{1}{2} & -\frac{5}{2} \end{bmatrix}$

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15. $\left[\begin{array}{cc|cc} 6 & -2 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{4}R_2]{\frac{1}{6}R_1} \left[\begin{array}{cc|cc} 1 & -\frac{1}{3} & \frac{1}{6} & 0 \\ 0 & 1 & 0 & \frac{1}{4} \end{array}\right] \xrightarrow{\frac{1}{3}R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{6} & \frac{1}{12} \\ 0 & 1 & 0 & \frac{1}{4} \end{array}\right]$; $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{12} \\ 0 & \frac{1}{4} \end{bmatrix}$

16. $\left[\begin{array}{cc|cc} 8 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{array}\right] \xrightarrow[2R_2]{\frac{1}{8}R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{8} & 0 \\ 0 & 1 & 0 & 2 \end{array}\right]$; $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & 2 \end{bmatrix}$

17. $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right] \xrightarrow{-5R_1+R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -12 & -5 & 1 \end{array}\right] \xrightarrow{-\frac{1}{12}R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{5}{12} & -\frac{1}{12} \end{array}\right] \xrightarrow{-3R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & \frac{5}{12} & -\frac{1}{12} \end{array}\right]$;

$\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{5}{12} & -\frac{1}{12} \end{bmatrix}$

18. $\left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ -2 & 4 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ -2 & 4 & 0 & 1 \end{array}\right] \xrightarrow{2R_1+R_2} \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{\frac{3}{2}R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 2 & \frac{3}{2} \\ 0 & 1 & 1 & 1 \end{array}\right]$;

$\mathbf{A}^{-1} = \begin{bmatrix} 2 & \frac{3}{2} \\ 1 & 1 \end{bmatrix}$

19. $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$; $\mathbf{A}$ is singular.

20. $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{9} & -\frac{1}{9} & \frac{2}{9} \\ 0 & 1 & 0 & -\frac{2}{9} & -\frac{5}{9} & \frac{1}{9} \\ 0 & 0 & 1 & -\frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \end{array}\right]$; $\mathbf{A}^{-1} = \begin{bmatrix} \frac{5}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{2}{9} & -\frac{5}{9} & \frac{1}{9} \\ -\frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \end{bmatrix}$

21. $\left[\begin{array}{ccc|ccc} 4 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_{13}} \left[\begin{array}{ccc|ccc} -1 & -2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 3 & 1 & 0 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & 0 \end{array}\right]$;

$\mathbf{A}^{-1} = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & 0 \end{bmatrix}$

22. $\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 2 & -2 & 0 & 1 & 0 \\ 8 & 10 & -6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right]$; $\mathbf{A}$ is singular.

23. $\left[\begin{array}{ccc|ccc} -1 & 3 & 0 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|ccc} 1 & -3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 6 & -3 \\ 0 & 1 & 0 & 2 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$;

$\mathbf{A}^{-1} = \begin{bmatrix} 5 & 6 & -3 \\ 2 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

24. $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & \frac{5}{8} \\ 0 & 1 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{8} \end{array}\right]$; $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -2 & \frac{5}{8} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{8} \end{bmatrix}$

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25. $\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$

$$\xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{6} & \frac{7}{6} \\ 0 & 1 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]; \quad \mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{6} & \frac{7}{6} \\ 1 & \frac{1}{3} & \frac{1}{3} & -\frac{4}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

26. $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{interchange}]{\text{row}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]; \quad \mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

27. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ -1 & \frac{10}{3} \end{bmatrix}$

28. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} -1 & -4 & 20 \\ 2 & 6 & -30 \\ 3 & 6 & -32 \end{bmatrix}$

29. $\mathbf{A} = (\mathbf{A}^{-1})^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$

30. $\mathbf{A}^T = \begin{bmatrix} 1 & 2 \\ 4 & 10 \end{bmatrix}$; $(\mathbf{A}^T)^{-1} = \begin{bmatrix} 5 & -1 \\ -2 & \frac{1}{2} \end{bmatrix}$; $\mathbf{A}^{-1} = \begin{bmatrix} 5 & -2 \\ -1 & \frac{1}{2} \end{bmatrix}$; $(\mathbf{A}^{-1})^T = \begin{bmatrix} 5 & -1 \\ -2 & \frac{1}{2} \end{bmatrix}$

31. Multiplying $\begin{bmatrix} 4 & -3 \\ x & -4 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ x & -4 \end{bmatrix} = \begin{bmatrix} 16-3x & 0 \\ 0 & 16-3x \end{bmatrix}$ we see that $x = 5$.

32. $\mathbf{A}^{-1} = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

33. **(a)** $\mathbf{A}^T = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} = \mathbf{A}^{-1}$ **(b)** $\mathbf{A}^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \mathbf{A}^{-1}$

34. Since $\det\mathbf{A} \cdot \det\mathbf{A}^{-1} = \det\mathbf{A}\mathbf{A}^{-1} = \det\mathbf{I} = 1$, we see that $\det\mathbf{A}^{-1} = 1/\det\mathbf{A}$. If $\mathbf{A}$ is orthogonal, $\det\mathbf{A} = \det\mathbf{A}^T = \det\mathbf{A}^{-1} = 1/\det\mathbf{A}$ and $(\det\mathbf{A})^2 = 1$, so $\det\mathbf{A} = \pm 1$.

35. Since $\mathbf{A}$ and $\mathbf{B}$ are nonsingular, $\det\mathbf{AB} = \det\mathbf{A} \cdot \det\mathbf{B} \neq 0$, and $\mathbf{AB}$ is nonsingular.

36. Suppose $\mathbf{A}$ is singular. Then $\det\mathbf{A} = 0$, $\det\mathbf{AB} = \det\mathbf{A} \cdot \det\mathbf{B} = 0$, and $\mathbf{AB}$ is singular.

37. Since $\det\mathbf{A} \cdot \det\mathbf{A}^{-1} = \det\mathbf{A}\mathbf{A}^{-1} = \det\mathbf{I} = 1$, $\det\mathbf{A}^{-1} = 1/\det\mathbf{A}$.

38. Suppose $\mathbf{A}^2 = \mathbf{A}$ and $\mathbf{A}$ is nonsingular. Then $\mathbf{A}^2\mathbf{A}^{-1} = \mathbf{A}\mathbf{A}^{-1}$, and $\mathbf{A} = \mathbf{I}$. Thus, if $\mathbf{A}^2 = \mathbf{A}$, either $\mathbf{A}$ is singular or $\mathbf{A} = \mathbf{I}$.

39. If $\mathbf{A}$ is nonsingular, then $\mathbf{A}^{-1}$ exists, and $\mathbf{AB} = \mathbf{0}$ implies $\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{0}$, so $\mathbf{B} = \mathbf{0}$.

40. If $\mathbf{A}$ is nonsingular, $\mathbf{A}^{-1}$ exists, and $\mathbf{AB} = \mathbf{AC}$ implies $\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{AC}$, so $\mathbf{B} = \mathbf{C}$.

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41. No, consider $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

42. $\mathbf{A}$ is nonsingular if $a_{11}a_{22}a_{33} = 0$ or a_{11}, a_{22}, and a_{33} are all nonzero.

$$\mathbf{A}^{-1} = \begin{bmatrix} 1/a_{11} & 0 & 0 \\ 0 & 1/a_{22} & 0 \\ 0 & 0 & 1/a_{33} \end{bmatrix}$$

For any diagonal matrix, the inverse matrix is obtaining by taking the reciprocals of the diagonal entries and leaving all other entries 0.

43. $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$; $\mathbf{A}^{-1}\begin{bmatrix} 4 \\ 14 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$; $x_1 = 6,\ x_2 = -2$

44. $\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$; $\mathbf{A}^{-1}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$; $x_1 = \dfrac{1}{2},\ x_2 = -\dfrac{3}{2}$

45. $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{16} & \frac{3}{8} \\ -\frac{1}{8} & \frac{1}{4} \end{bmatrix}$; $\mathbf{A}^{-1}\begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{2} \end{bmatrix}$; $x_1 = \dfrac{3}{4},\ x_2 = -\dfrac{1}{2}$

46. $\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$; $\mathbf{A}^{-1}\begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -11 \\ \frac{15}{2} \end{bmatrix}$; $x_1 = -11,\ x_2 = \dfrac{15}{2}$

47. $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ -1 & 1 & 0 \\ \frac{6}{5} & -\frac{1}{5} & -\frac{1}{5} \end{bmatrix}$; $\mathbf{A}^{-1}\begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}$; $x_1 = 2,\ x_2 = 4,\ x_3 = -6$

48. $\mathbf{A}^{-1} = \begin{bmatrix} \frac{5}{12} & -\frac{1}{12} & \frac{1}{4} \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ -\frac{1}{12} & \frac{5}{12} & -\frac{1}{4} \end{bmatrix}$; $\mathbf{A}^{-1}\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{3}{2} \end{bmatrix}$; $x_1 = -\dfrac{1}{2},\ x_2 = 0,\ x_3 = \dfrac{3}{2}$

49. $\mathbf{A}^{-1} = \begin{bmatrix} -2 & -3 & 2 \\ \frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{5}{4} & \frac{7}{4} & -1 \end{bmatrix}$; $\mathbf{A}^{-1}\begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 21 \\ 1 \\ -11 \end{bmatrix}$; $x_1 = 21,\ x_2 = 1,\ x_3 = -11$

50. $\mathbf{A}^{-1} = \begin{bmatrix} 2 & -1 & 1 & 1 \\ -1 & 2 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$; $\mathbf{A}^{-1}\begin{bmatrix} 2 \\ 1 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -4 \end{bmatrix}$; $x_1 = 1,\ x_2 = 2,\ x_3 = -1,\ x_4 = -4$

51. $\begin{bmatrix} 7 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$; $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{10} & \frac{1}{10} \\ -\frac{3}{20} & \frac{7}{20} \end{bmatrix}$; $\mathbf{X} = \mathbf{A}^{-1}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{9}{10} \\ \frac{13}{20} \end{bmatrix}$; $\mathbf{X} = \mathbf{A}^{-1}\begin{bmatrix} 10 \\ 50 \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \end{bmatrix}$;

$\mathbf{X} = \mathbf{A}^{-1}\begin{bmatrix} 0 \\ -20 \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$

52. $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$; $\mathbf{A}^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}$; $\mathbf{X} = \mathbf{A}^{-1}\begin{bmatrix} -1 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -12 \\ -52 \\ 23 \end{bmatrix}$;

$\mathbf{X} = \mathbf{A}^{-1}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ -3 \end{bmatrix}$; $= \mathbf{A}^{-1}\begin{bmatrix} 0 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 27 \\ -11 \end{bmatrix}$

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53. $\det \mathbf{A} = 18 \neq 0$, so the system has only the trivial solution.

54. $\det \mathbf{A} = 0$, so the system has a nontrivial solution.

55. $\det \mathbf{A} = 0$, so the system has a nontrivial solution.

56. $\det \mathbf{A} = 12 \neq 0$, so the system has only the trivial solution.

57. **(a)** $$\begin{bmatrix} 1 & 1 & 1 \\ -R_1 & R_2 & 0 \\ 0 & -R_2 & R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ E_2 - E_1 \\ E_3 - E_2 \end{bmatrix}$$

(b) $\det \mathbf{A} = R_1R_2 + R_1R_3 + R_2R_3 > 0$, so $\mathbf{A}$ is nonsingular.

(c) $$\mathbf{A}^{-1} = \frac{1}{R_1R_2 + R_1R_3 + R_2R_3} \begin{bmatrix} R_2R_3 & -R_2 - R_3 & -R_2 \\ R_1R_3 & R_3 & -R_1 \\ R_1R_2 & R_2 & R_1 + R_2 \end{bmatrix};$$

$$\mathbf{A}^{-1} \begin{bmatrix} 0 \\ E_2 - E_1 \\ E_3 - E_2 \end{bmatrix} = \frac{1}{R_1R_2 + R_1R_3 + R_2R_3} \begin{bmatrix} R_2E_1 - R_2E_3 + R_3E_1 - R_3E_2 \\ R_1E_2 - R_1E_3 - R_3E_1 + R_3E_2 \\ -R_1E_2 + R_1E_3 - R_2E_1 + R_2E_3 \end{bmatrix}$$

58. **(a)** We write the equations in the form

$$\begin{aligned} -4u_1 + u_2 + u_4 &= -200 \\ u_1 - 4u_2 + u_3 &= -300 \\ u_2 - 4u_3 + u_4 &= -300 \\ u_1 + u_3 - 4u_4 &= -200. \end{aligned}$$

In matrix form this becomes $$\begin{bmatrix} -4 & 1 & 0 & 1 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -200 \\ -300 \\ -300 \\ -200 \end{bmatrix}.$$

(b) $$\mathbf{A}^{-1} = \begin{bmatrix} -\frac{7}{24} & -\frac{1}{12} & -\frac{1}{24} & -\frac{1}{12} \\ -\frac{1}{12} & -\frac{7}{24} & -\frac{1}{12} & -\frac{1}{24} \\ -\frac{1}{24} & -\frac{1}{12} & -\frac{7}{24} & -\frac{1}{12} \\ -\frac{1}{12} & -\frac{1}{24} & -\frac{1}{12} & -\frac{7}{24} \end{bmatrix}; \quad \mathbf{A}^{-1} \begin{bmatrix} -200 \\ -300 \\ -300 \\ -200 \end{bmatrix} = \begin{bmatrix} \frac{225}{2} \\ \frac{275}{2} \\ \frac{275}{2} \\ \frac{225}{2} \end{bmatrix}; \quad u_1 = u_4 = \frac{225}{2}, \quad u_2 = u_3 = \frac{275}{2}$$

Exercises 8.7

1. $\det \mathbf{A} = 10$, $\det \mathbf{A}_1 = -6$, $\det \mathbf{A}_2 = 12$; $x_1 = \frac{-6}{10} = -\frac{3}{5}$, $x_2 = \frac{12}{10} = \frac{6}{5}$

2. $\det \mathbf{A} = -3$, $\det \mathbf{A}_1 = -6$, $\det \mathbf{A}_2 = -6$; $x_1 = \frac{-6}{-3} = 2$, $x_2 = \frac{-6}{-3} = 2$

3. $\det \mathbf{A} = 0.3$, $\det \mathbf{A}_1 = 0.03$, $\det \mathbf{A}_2 = -0.09$; $x_1 = \frac{0.03}{0.3} = 0.1$, $x_2 = \frac{-0.09}{0.3} = -0.3$

4. $\det \mathbf{A} = -0.015$, $\det \mathbf{A}_1 = -0.00315$, $\det \mathbf{A}_2 = -0.00855$; $x_1 = \frac{-0.00315}{-0.015} = 0.21$, $x_2 = \frac{-0.00855}{-0.015} = 0.57$

5. $\det \mathbf{A} = 1$, $\det \mathbf{A}_1 = 4$, $\det \mathbf{A}_2 = -7$; $x = 4$, $y = -7$

6. $\det \mathbf{A} = -70$, $\det \mathbf{A}_1 = -14$, $\det \mathbf{A}_2 = 35$; $r = \frac{-14}{-70} = \frac{1}{5}$, $s = \frac{35}{-70} = -\frac{1}{2}$

7. $\det \mathbf{A} = 11$, $\det \mathbf{A}_1 = -44$, $\det \mathbf{A}_2 = 44$, $\det \mathbf{A}_3 = -55$; $x_1 = \frac{-44}{11} = -4$, $x_2 = \frac{44}{11} = 4$, $x_3 = \frac{-55}{11} = -5$

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8. $\det \mathbf{A} = -63,\ \det \mathbf{A}_1 = 173,\ \det \mathbf{A}_2 = -136,\ \det \mathbf{A}_3 = -\frac{61}{2};\ x_1 = -\frac{173}{63},\ x_2 = \frac{136}{63},\ x_3 = \frac{61}{126}$

9. $\det \mathbf{A} = -12,\ \det \mathbf{A}_1 = -48,\ \det \mathbf{A}_2 = -18,\ \det \mathbf{A}_3 = -12;\ u = \frac{48}{12} = 4,\ v = \frac{18}{12} = \frac{3}{2},\ w = 1$

10. $\det \mathbf{A} = 1,\ \det \mathbf{A}_1 = -2,\ \det \mathbf{A}_2 = 2,\ \det \mathbf{A}_3 = 5;\ x = -2,\ y = 2,\ z = 5$

11. $\det \mathbf{A} = 6 - 5k,\ \det \mathbf{A}_1 = 12 - 7k,\ \det \mathbf{A}_2 = 6 - 7k;\ x_1 = \dfrac{12 - 7k}{6 - 5k},\ x_2 = \dfrac{6 - 7k}{6 - 5k}$. The system is inconsistent for $k = 6/5$.

12. **(a)** $\det \mathbf{A} = \epsilon - 1,\ \det \mathbf{A}_1 = \epsilon - 2,\ \det \mathbf{A}_2 = 1;\ x_1 = \dfrac{\epsilon - 2}{\epsilon - 1} = \dfrac{\epsilon - 1 - 1}{\epsilon - 1} = 1 - \dfrac{1}{\epsilon - 1},\ x_2 = \dfrac{1}{\epsilon - 1}$

(b) When $\epsilon = 1.01$, $x_1 = -99$ and $x_2 = 100$. When $\epsilon = 0.99$, $x_1 = 101$ and $x_2 = -100$.

13. $\det \mathbf{A} \approx 0.6428,\ \det \mathbf{A}_1 \approx 289.8,\ \det \mathbf{A}_2 \approx 271.9;\ x_1 \approx \frac{289.8}{0.6428} \approx 450.8,\ x_2 \approx \frac{271.9}{0.6428} \approx 423$

14. We have $(\sin 30°)F + (\sin 30°)(0.5N) + N \sin 60° = 400$ and $(\cos 30°)F + (\cos 30°)(0.5N) - N \cos 60° = 0$. The system is

$$(\sin 30°)F + (0.5 \sin 30° + \sin 60°)N = 400$$
$$(\cos 30°)F + (0.5 \cos 30° - \cos 60°)N = 0.$$

$\det \mathbf{A} \approx -1,\ \det \mathbf{A}_1 \approx -26.795,\ \det \mathbf{A}_2 \approx -346.41;\ F \approx 26.795,\ N \approx 346.41$

15. The system is

$$i_1 + i_2 - i_3 = 0$$
$$r_1 i_1 - r_2 i_2 = E_1 - E_2$$
$$r_2 i_2 + R i_3 = E_2$$

$\det \mathbf{A} = -r_1 R - r_2 R - r_1 r_2,\ \det A_3 = -r_1 E_2,\ -r_2 E_1;\ i_3 = \dfrac{r_1 E_2 + r_2 E_1}{r_1 R + r_2 R + r_1 r_2}$

Exercises 8.8

1. $\mathbf{K}_3$ since $\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} = (-1) \begin{bmatrix} -2 \\ 5 \end{bmatrix};\quad \lambda = -1$

2. $\mathbf{K}_1$ and $\mathbf{K}_2$ since $\begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 - \sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -2 + 2\sqrt{2} \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1 \\ 2 - \sqrt{2} \end{bmatrix},\quad \lambda = \sqrt{2}$

$$\begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 + \sqrt{2} \\ 2 \end{bmatrix} = \begin{bmatrix} 2 + 2\sqrt{2} \\ 2\sqrt{2} \end{bmatrix} = \sqrt{2} \begin{bmatrix} 2 + \sqrt{2} \\ 2 \end{bmatrix};\quad \lambda = \sqrt{2}$$

3. $\mathbf{K}_3$ since $\begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -5 \\ 10 \end{bmatrix};\quad \lambda = 0$

4. $\mathbf{K}_2$ since $\begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 + 2i \\ -1 \end{bmatrix} = \begin{bmatrix} -4 + 4i \\ -2i \end{bmatrix} = 2i \begin{bmatrix} 2 + 2i \\ -1 \end{bmatrix};\quad \lambda = 2i$

5. $\mathbf{K}_2$ and $\mathbf{K}_3$ since $\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ -12 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix};\quad \lambda = 3$

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix};\quad \lambda = 1$$

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6. $\mathbf{K}_2$ since $\begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 9 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}; \quad \lambda = 3$

7. We solve $\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -1-\lambda & 2 \\ -7 & 8-\lambda \end{vmatrix} = (\lambda - 6)(\lambda - 1) = 0.$

For $\lambda_1 = 6$ we have $\left[\begin{array}{cc|c} -7 & 2 & 0 \\ -7 & 2 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & -2/7 & 0 \\ 0 & 0 & 0 \end{array}\right]$

so that $k_1 = \frac{2}{7}k_2$. If $k_2 = 7$ then $\mathbf{K}_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$. For $\lambda_2 = 1$ we have

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -7 & 7 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = k_2$. If $k_2 = 1$ then $\mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

8. We solve $\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = \lambda(\lambda - 3) = 0.$

For $\lambda_1 = 0$ we have $\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

so that $k_1 = -\frac{1}{2}k_2$. If $k_2 = 2$ then $\mathbf{K}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. For $\lambda_2 = 3$ we have

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = k_2$. If $k_2 = 1$ then $\mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

9. We solve $\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -8-\lambda & -1 \\ 16 & -\lambda \end{vmatrix} = (\lambda + 4)^2 = 0.$

For $\lambda_1 = \lambda_2 = -4$ we have $\left[\begin{array}{cc|c} -4 & -1 & 0 \\ 16 & 4 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 1/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

so that $k_1 = -\frac{1}{4}k_2$. If $k_2 = 4$ then $\mathbf{K}_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

10. We solve $\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 1 \\ 1/4 & 1-\lambda \end{vmatrix} = (\lambda - 3/2)(\lambda - 1/2) = 0.$

For $\lambda_1 = 3/2$ we have $\left[\begin{array}{cc|c} -1/2 & 1 & 0 \\ 1/4 & -1/2 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

so that $k_1 = 2k_2$. If $k_2 = 1$ then $\mathbf{K}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. If $\lambda_2 = 1/2$ then

$$\left[\begin{array}{cc|c} 1/2 & 1 & 0 \\ 1/4 & 1/2 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = -2k_2$. If $k_2 = 1$ then $\mathbf{K}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

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11. We solve $\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} -1-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix} = \lambda^2+9 = (\lambda-3i)(\lambda+3i) = 0.$

For $\lambda_1 = 3i$ we have $\left[\begin{array}{cc|c} -1-3i & 2 & 0 \\ -5 & 1-3i & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & -(1/5)+(3/5)i & 0 \\ 0 & 0 & 0 \end{array}\right]$

so that $k_1 = \left(\frac{1}{5} - \frac{3}{5}i\right)k_2$. If $k_2 = 5$ then $\mathbf{K}_1 = \begin{bmatrix} 1-3i \\ 5 \end{bmatrix}$. For $\lambda_2 = -3i$ we have

$$\left[\begin{array}{cc|c} -1+3i & 2 & 0 \\ -5 & 1+3i & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{5}-\frac{3}{5}i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = \left(\frac{1}{5} + \frac{3}{5}i\right)k_2$. If $k_2 = 5$ then $\mathbf{K}_2 = \begin{bmatrix} 1+3i \\ 5 \end{bmatrix}$.

12. We solve $\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2-2\lambda+2 = 0.$

For $\lambda_1 = 1-i$ we have $\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

so that $k_1 = -ik_2$. If $k_2 = 1$ then $\mathbf{K}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ and $\mathbf{K}_2 = \overline{\mathbf{K}}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

13. We solve $\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & 8 \\ 0 & -5-\lambda \end{vmatrix} = (\lambda-4)(\lambda+5) = 0.$

For $\lambda_1 = 4$ we have $\left[\begin{array}{cc|c} 0 & 8 & 0 \\ 0 & -9 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

so that $k_2 = 0$. If $k_1 = 1$ then $\mathbf{K}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. For $\lambda_2 = -5$ we have

$$\left[\begin{array}{cc|c} 9 & 8 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & \frac{8}{9} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = -\frac{8}{9}k_2$. If $k_2 = 9$ then $\mathbf{K}_2 = \begin{bmatrix} -8 \\ 9 \end{bmatrix}$.

14. We solve $\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 7-\lambda & 0 \\ 0 & 13-\lambda \end{vmatrix} = (\lambda-7)(\lambda-13) = 0.$

For $\lambda_1 = 7$ we have $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 6 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

so that $k_2 = 0$. If $k_1 = 1$ then $\mathbf{K}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. For $\lambda_2 = 13$ we have

$$\left[\begin{array}{cc|c} -6 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = 0$. If $k_2 = 1$ then $\mathbf{K}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

15. We solve $\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 5-\lambda & -1 & 0 \\ 0 & -5-\lambda & 9 \\ 5 & -1 & -\lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & -1 & 0 \\ 4-\lambda & -5-\lambda & 9 \\ 4-\lambda & -1 & -\lambda \end{vmatrix} = \lambda(4-\lambda)(\lambda+4) = 0.$

For $\lambda_1 = 0$ we have $\left[\begin{array}{ccc|c} 5 & -1 & 0 & 0 \\ 0 & -5 & 9 & 0 \\ 5 & -1 & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -9/25 & 0 \\ 0 & 1 & -9/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

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so that $k_1 = \frac{9}{25}k_3$ and $k_2 = \frac{9}{5}k_3$. If $k_3 = 25$ then $\mathbf{K}_1 = \begin{bmatrix} 9 \\ 45 \\ 25 \end{bmatrix}$. If $\lambda_2 = 4$ then

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -9 & 9 & 0 \\ 5 & -1 & -4 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = k_3$ and $k_2 = k_3$. If $k_3 = 1$ then $\mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. If $\lambda_3 = -4$ then

$$\left[\begin{array}{ccc|c} 9 & -1 & 0 & 0 \\ 0 & -1 & 9 & 0 \\ 5 & -1 & 4 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = k_3$ and $k_2 = 9k_3$. If $k_3 = 1$ then $\mathbf{K}_3 = \begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix}$.

16. We solve $\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 4 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda)(1-\lambda) = 0.$

For $\lambda_1 = 1$ we have $\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

so that $k_1 = 0$ and $k_2 = 0$. If $k_3 = 1$ then $\mathbf{K}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. If $\lambda_2 = 2$ then

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & -1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = 0$ and $k_3 = 0$. If $k_2 = 1$ then $\mathbf{K}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. If $\lambda_3 = 3$ then

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 4 & 0 & -2 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = \frac{1}{2}k_3$ and $k_2 = 0$. If $k_3 = 2$ then $\mathbf{K}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

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17. We solve $$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 4 & 0 \\ -1 & -4-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = -(\lambda+2)^3 = 0.$$

For $\lambda_1 = \lambda_2 = \lambda_3 = -2$ we have $$\left[\begin{array}{ccc|c} 2 & 4 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = -2k_2$. If $k_2 = 1$ and $k_3 = 1$ then

$$\mathbf{K}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

18. We solve $$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 6 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 6 & 0 \\ 0 & 3-\lambda & 3-\lambda \\ 0 & 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)^2 = 0.$$

For $\lambda_1 = 3$ we have $$\left[\begin{array}{ccc|c} -2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = 3k_3$ and $k_2 = k_3$. If $k_3 = 1$ then $\mathbf{K}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. For $\lambda_2 = \lambda_3 = 1$ we have

$$\left[\begin{array}{ccc|c} 0 & 6 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_2 = 0$ and $k_3 = 0$. If $k_1 = 1$ then $\mathbf{K}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

19. We solve $$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 0 & -1 \\ 0 & -\lambda & 0 \\ 1 & 1 & -1-\lambda \end{vmatrix} = -(\lambda+1)(\lambda^2+1) = 0.$$

For $\lambda_1 = -1$ we have $$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = k_3$ and $k_2 = -k_3$. If $k_3 = 1$ then $\mathbf{K}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. For $\lambda_2 = i$ we have

$$\left[\begin{array}{ccc|c} -i & 0 & -1 & 0 \\ 1 & -i & 0 & 0 \\ 1 & 1 & -1-i & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -i & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = ik_3$ and $k_2 = k_3$. If $k_3 = 1$ then $\mathbf{K}_2 = \begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{K}_3 = \overline{\mathbf{K}}_2 = \begin{bmatrix} -i \\ 1 \\ 1 \end{bmatrix}$.

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20. We solve

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & -1 & 0 \\ 5 & 2-\lambda & 4 \\ 0 & 1 & 2-\lambda \end{vmatrix} = -\lambda^3+6\lambda^2-13\lambda+10 = (\lambda-2)(-\lambda^2+4\lambda-5)$$

$$= (\lambda-2)(\lambda-(2+i))(\lambda-(2-i)) = 0.$$

For $\lambda_1 = 2$ we have $\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 5 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 4/5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

so that $k_1 = -\frac{4}{5}k_3$ and $k_2 = 0$. If $k_3 = 5$ then $\mathbf{K}_1 = \begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}$. For $\lambda_2 = 2+i$ we have

$$\left[\begin{array}{ccc|c} -i & -1 & 0 & 0 \\ 5 & -i & 4 & 0 \\ 0 & 1 & -i & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & -i & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = ik_2$ and $k_2 = ik_3$. If $k_3 = i$ then $\mathbf{K}_2 = \begin{bmatrix} -i \\ -1 \\ i \end{bmatrix}$. For $\lambda_3 = 2-i$ we have

$$\left[\begin{array}{ccc|c} i & -1 & 0 & 0 \\ 5 & i & 4 & 0 \\ 0 & 1 & i & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = -ik_2$ and $k_2 = -ik_3$. If $k_3 = i$ then $\mathbf{K}_3 = \begin{bmatrix} -1 \\ 1 \\ i \end{bmatrix}$.

21. We solve $\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 5-\lambda & 6 \\ 0 & 0 & -7-\lambda \end{vmatrix} = -(\lambda-1)(\lambda-5)(\lambda+7) = 0.$

For $\lambda_1 = 1$ we have $\left[\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 4 & 6 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

so that $k_2 = k_3 = 0$. If $k_1 = 1$ then $\mathbf{K}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. For $\lambda_2 = 5$ we have

$$\left[\begin{array}{ccc|c} -4 & 2 & 3 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & -12 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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so that $k_3 = 0$ and $k_2 = 2k_1$. If $k_1 = 1$ then $\mathbf{K}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. For $\lambda_3 = -7$ we have

$$\left[\begin{array}{ccc|c} 8 & 2 & 3 & 0 \\ 0 & 12 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = -\frac{1}{4}k_3$ and $k_2 = -\frac{1}{2}k_3$. If $k_3 = 4$ then $\mathbf{K}_3 = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$.

22. We solve $$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = -\lambda^2(\lambda - 1) = 0.$$

For $\lambda_1 = \lambda_2 = 0$ we have $$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_3 = 0$. If $k_1 = 1$ and $k_2 = 0$ then $\mathbf{K}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and if $k_1 = 0$ and $k_2 = 1$ then $\mathbf{K}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. For $\lambda_3 = 1$ we have

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = k_2 = 0$. If $k_3 = 1$ then $\mathbf{K}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

23. The eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ are

$$\lambda_1 = 4, \quad \lambda_2 = 6, \quad \mathbf{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the eigenvalues and eigenvectors of $\mathbf{A}^{-1} = \dfrac{1}{24}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$ are

$$\lambda_1 = \frac{1}{4}, \quad \lambda_2 = \frac{1}{6}, \quad \mathbf{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

24. The eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$ are

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3, \quad \mathbf{K}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{K}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}.$$

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and the eigenvalues and eigenvectors of $\mathbf{A}^{-1} = \frac{1}{6}\begin{bmatrix} 4 & -6 & 2 \\ -1 & 9 & -2 \\ -4 & 12 & -2 \end{bmatrix}$ are

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1}{2}, \quad \lambda_3 = \frac{1}{3}, \quad \mathbf{K}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{K}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}.$$

25. Since $\det \mathbf{A} = \begin{vmatrix} 6 & 0 \\ 3 & 0 \end{vmatrix} = 0$ the matrix is singular. Now from

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 6-\lambda & 0 \\ 3 & -\lambda \end{vmatrix} = \lambda(\lambda - 6)$$

we see $\lambda = 0$ is an eigenvalue.

26. Since $\det \mathbf{A} = \begin{vmatrix} 1 & 0 & 1 \\ 4 & -4 & 5 \\ 7 & -4 & 8 \end{vmatrix} = 0$ the matrix is singular. Now from

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 4 & -4-\lambda & 5 \\ 7 & -4 & 8-\lambda \end{vmatrix} = -\lambda(\lambda^2 - 5\lambda - 15)$$

we see $\lambda = 0$ is an eigenvalue.

27. **(a)** Since $p + 1 - p = 1$ and $q + 1 - q = 1$, the first matrix $\mathbf{A}$ is stochastic. Since $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$, $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$, and $\frac{1}{6} + \frac{1}{3} + \frac{1}{2} = 1$, the second matrix $\mathbf{A}$ is stochastic.

(b) The matrix from part (a) is shown with its eigenvalues and corresponding eigenvectors.

$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$; eigenvalues: $1, \frac{1}{6} - \frac{1}{12}\sqrt{2}, \frac{1}{6} + \frac{1}{12}\sqrt{2}$;

eigenvectors: $(1,1,1), \left(-\frac{3(-1+\sqrt{2})}{-6+\sqrt{2}}, \frac{2(2+\sqrt{2})}{-6+\sqrt{2}}, 1\right), \left(-\frac{3(1+\sqrt{2})}{6+\sqrt{2}}, \frac{2(-2+\sqrt{2})}{6+\sqrt{2}}, 1\right)$

Further examples indicate that 1 is always an eigenvalue with corresponding eigenvector $(1,1,1)$. To prove this, let $\mathbf{A}$ be a stochastic matrix and $\mathbf{K} = (1,1,1)$. Then

$$\mathbf{AK} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} a_{11} + \cdots + a_{1n} \\ \vdots \\ a_{n1} + \cdots + a_{nn} \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 1\mathbf{K},$$

and 1 is an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $(1,1,1)$.

(c) For the 3×3 matrix in part (a) we have

$$\mathbf{A}^2 = \begin{bmatrix} \frac{3}{8} & \frac{7}{24} & \frac{1}{3} \\ \frac{1}{3} & \frac{11}{36} & \frac{13}{36} \\ \frac{5}{18} & \frac{23}{72} & \frac{29}{72} \end{bmatrix}, \quad \mathbf{A}^3 = \begin{bmatrix} \frac{49}{144} & \frac{29}{96} & \frac{103}{288} \\ \frac{71}{216} & \frac{11}{36} & \frac{79}{216} \\ \frac{5}{16} & \frac{67}{216} & \frac{163}{432} \end{bmatrix}.$$

These powers of $\mathbf{A}$ are also stochastic matrices. To prove that this is true in general for 2×2 matrices, we prove the more general theorem that any product of 2×2 stochastic matrices is stochastic. Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

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be stochastic matrices. Then

$$\mathbf{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$$

The sums of the rows are

$$\begin{aligned} a_{11}b_{11} + a_{12}b_{21} + a_{11}b_{12} + a_{12}b_{22} &= a_{11}(b_{11} + b_{12}) + a_{12}(b_{21} + b_{22}) \\ &= a_{11}(1) + a_{12}(1) = a_{11} + a_{12} = 1 \\ a_{21}b_{11} + a_{22}b_{21} + a_{21}b_{12} + a_{22}b_{22} &= a_{21}(b_{11} + b_{12}) + a_{22}(b_{21} + b_{22}) \\ &= a_{21}(1) + a_{22}(1) = a_{21} + a_{22} = 1. \end{aligned}$$

Thus, the product matrix **AB** is stochastic. It follows that any power of a 2×2 matrix is stochastic. The proof in the case of an $n \times n$ matrix is very similar.

Exercises 8.9

1. The characteristic equation is $\lambda^2 - 6\lambda + 13 = 0$. Then

$$\mathbf{A}^2 - 6\mathbf{A} + 13\mathbf{I} = \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix} - \begin{bmatrix} 6 & -12 \\ 24 & 30 \end{bmatrix} + \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

2. The characteristic equation is $-\lambda^3 + \lambda^2 + 4\lambda - 1$. Then

$$-\mathbf{A}^3 + \mathbf{A}^2 + \mathbf{A} - \mathbf{I} = -\begin{bmatrix} 2 & 6 & 13 \\ 4 & 5 & 17 \\ 1 & 5 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 5 \\ 0 & 4 & 5 \\ 1 & 1 & 4 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3. The characteristic equation is $\lambda^2 - 3\lambda - 10 = 0$, with eigenvalues -2 and 5. Substituting the eigenvalues into $\lambda^m = c_0 + c_1\lambda$ generates

$$\begin{aligned} (-2)^m &= c_0 - 2c_1 \\ 5^m &= c_0 + 5c_1. \end{aligned}$$

Solving the system gives

$$c_0 = \frac{1}{7}[5(-2)^m + 2(5)^m], \qquad c_1 = \frac{1}{7}[-(-2)^m + 5^m].$$

Thus

$$\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} = \begin{bmatrix} \frac{1}{7}[3(-1)^m 2^{m+1} + 5^m] & \frac{3}{7}[-(-2)^m + 5^m] \\ \frac{2}{7}[-(-2)^m + 5^m] & \frac{1}{7}[(-2)^m + 6(5)^m] \end{bmatrix}$$

and

$$\mathbf{A}^3 = \begin{bmatrix} 11 & 57 \\ 38 & 106 \end{bmatrix}.$$

4. The characteristic equation is $\lambda^2 - 10\lambda + 16 = 0$, with eigenvalues 2 and 8. Substituting the eigenvalues into $\lambda^m = c_0 + c_1\lambda$ generates

$$\begin{aligned} 2^m &= c_0 + 2c_1 \\ 8^m &= c_0 + 8c_1. \end{aligned}$$

Solving the system gives

$$c_0 = \frac{1}{3}(2^{m+2} - 8^m), \qquad c_1 = \frac{1}{6}(-2^m + 8^m).$$

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Thus

$$\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} = \begin{bmatrix} \frac{1}{2}(2^m + 8^m) & \frac{1}{2}(2^m - 8^m) \\ \frac{1}{2}(2^m - 8^m) & \frac{1}{2}(2^m + 8^m) \end{bmatrix}$$

and

$$\mathbf{A}^4 = \begin{bmatrix} 2056 & -2040 \\ -2040 & 2056 \end{bmatrix}.$$

5. The characteristic equation is $\lambda^2 - 8\lambda - 20 = 0$, with eigenvalues -2 and 10. Substituting the eigenvalues into $\lambda^m = c_0 + c_1\lambda$ generates

$$(-2)^m = c_0 - 2c_1$$
$$10^m = c_0 + 10c_1.$$

Solving the system gives

$$c_0 = \frac{1}{6}[5(-2)^m + 10^m], \qquad c_1 = \frac{1}{12}[-(-2)^m + 10^m].$$

Thus

$$\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} = \begin{bmatrix} \frac{1}{6}[(-2)^m + 2^m 5^{m+1}] & \frac{5}{12}[-(-2)^m + 10^m] \\ \frac{1}{3}[-(-2)^m + 10^m] & \frac{1}{6}[5(-2)^m + 10^m] \end{bmatrix}$$

and

$$\mathbf{A}^5 = \begin{bmatrix} 83328 & 41680 \\ 33344 & 16640 \end{bmatrix}.$$

6. The characteristic equation is $\lambda^2 + 4\lambda + 3 = 0$, with eigenvalues -3 and -1. Substituting the eigenvalues into $\lambda^m = c_0 + c_1\lambda$ generates

$$(-3)^m = c_0 - 3c_1$$
$$(-1)^m = c_0 - c_1.$$

Solving the system gives

$$c_0 = \frac{1}{2}[-(-3)^m + 3(-1)^m], \qquad c_1 = \frac{1}{2}[-(-3)^m + (-1)^m].$$

Thus

$$\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} = \begin{bmatrix} (-1)^m & -(-3)^m + (-1)^m \\ 0 & (-3)^m \end{bmatrix}$$

and

$$\mathbf{A}^6 = \begin{bmatrix} 1 & -728 \\ 0 & 729 \end{bmatrix}.$$

7. The characteristic equation is $-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$, with eigenvalues -1, 1, and 2. Substituting the eigenvalues into $\lambda^m = c_0 + c_1\lambda + c_2\lambda^2$ generates

$$(-1)^m = c_0 - c_1 + c_2$$
$$1 = c_0 + c_1 + c_2$$
$$2^m = c_0 + 2c_1 + 4c_2.$$

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Solving the system gives

$$c_0 = \frac{1}{3}[3 + (-1)^m - 2^m],$$
$$c_1 = \frac{1}{2}[1 - (-1)^m],$$
$$c_2 = \frac{1}{6}[-3 + (-1)^m + 2^{m+1}].$$

Thus

$$\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} + c_2\mathbf{A}^2 = \begin{bmatrix} 1 & -1 + 2^m & -1 + 2^m \\ 0 & \frac{1}{3}[(-1)^m + 2^{m+1}] & -\frac{2}{3}[(-1)^m - 2^m] \\ 0 & \frac{1}{3}[-(-1)^m + 2^m] & \frac{1}{3}[2(-1)^m + 2^m] \end{bmatrix}$$

and

$$\mathbf{A}^{10} = \begin{bmatrix} 1 & 1023 & 1023 \\ 0 & 683 & 682 \\ 0 & 341 & 342 \end{bmatrix}.$$

8. The characteristic equation is $-\lambda^3 - \lambda^2 + 2\lambda + 2 = 0$, with eigenvalues -1, $-\sqrt{2}$, and $\sqrt{2}$. Substituting the eigenvalues into $\lambda^m = c_0 + c_1\lambda + c_2\lambda^2$ generates

$$(-1)^m = c_0 - c_1 + c_2$$
$$(-\sqrt{2}\,)^m = c_0 - \sqrt{2}c_1 + 2c_2$$
$$(\sqrt{2}\,)^m = c_0 + \sqrt{2}c_1 + 2c_2.$$

Solving the system gives

$$c_0 = [2 - (\sqrt{2}\,)^{m-1} - (\sqrt{2}\,)^{m-2}](-1)^m + (\sqrt{2} - 1)(\sqrt{2}\,)^{m-2},$$
$$c_1 = \frac{1}{2}[1 - (-1)^m](\sqrt{2}\,)^{m-1},$$
$$c_2 = (-1)^{m+1} + \frac{1}{2}(1 + \sqrt{2}\,)(-1)^m(\sqrt{2}\,)^{m-1} + \frac{1}{2}(\sqrt{2} - 1)(\sqrt{2}\,)^{m-1}.$$

Thus $\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} + c_2\mathbf{A}^2$ and

$$\mathbf{A}^6 = \begin{bmatrix} 1 & 0 & 7 \\ 7 & 8 & -7 \\ 0 & 0 & 8 \end{bmatrix}.$$

9. The characteristic equation is $-\lambda^3 + 3\lambda^2 + 6\lambda - 8 = 0$, with eigenvalues -2, 1, and 4. Substituting the eigenvalues into $\lambda^m = c_0 + c_1\lambda + c_2\lambda^2$ generates

$$(-2)^m = c_0 - 2c_1 + 4c_2$$
$$1 = c_0 + c_1 + c_2$$
$$4^m = c_0 + 4c_1 + 16c_2.$$

Solving the system gives

$$c_0 = \frac{1}{9}[8 + (-1)^m 2^{m+1} - 4^m],$$
$$c_1 = \frac{1}{18}[4 - 5(-2)^m + 4^m],$$
$$c_2 = \frac{1}{18}[-2 + (-2)^m + 4^m].$$

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Thus

$$\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} + c_2\mathbf{A}^2 = \begin{bmatrix} \frac{1}{9}[(-2)^m + (-1)^m 2^{m+1} + 3\cdot 2^{2m+1}] & \frac{1}{3}[-(-2)^m + 4^m] & 0 \\ -\frac{2}{3}[(-2)^m - 4^m] & \frac{1}{3}[(-1)^m 2^{m+1} + 4^m] & 0 \\ \frac{1}{3}[-3 + (-2)^m + 2^{2m+1}] & \frac{1}{3}[-(-2)^m + 4^m] & 1 \end{bmatrix}$$

and

$$\mathbf{A}^{10} = \begin{bmatrix} 699392 & 349184 & 0 \\ 698368 & 350208 & 0 \\ 699391 & 349184 & 1 \end{bmatrix}.$$

10. The characteristic equation is $-\lambda^3 - \frac{3}{2}\lambda^2 + \frac{3}{2}\lambda + 1 = 0$, with eigenvalues -2, $-\frac{1}{2}$, and 1. Substituting the eigenvalues into $\lambda^m = c_0 + c_1\lambda + c_2\lambda^2$ generates

$$\begin{aligned} (-2)^m &= c_0 - 2c_1 + 4c_2 \\ \left(-\frac{1}{2}\right)^m &= c_0 - \frac{1}{2}c_1 + \frac{1}{4}c_2 \\ 1 &= c_0 + c_1 + c_2. \end{aligned}$$

Solving the system gives

$$\begin{aligned} c_0 &= \frac{1}{9}[2^{-m}[(-4)^m + 8(-1)^m + 2^{m+1} - (-1)^m 2^{2m+1}], \\ c_1 &= -\frac{1}{9}2^{-m}[(-4)^m + 4(-1)^m - 5\cdot 2^m], \\ c_2 &= \frac{2}{9}[1 + (-2)^m - (-1)^m 2^{m-1}]. \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{A}^m &= c_0\mathbf{I} + c_1\mathbf{A} + c_2\mathbf{A}^2 \\ &= \begin{bmatrix} \frac{1}{3}2^{-m}[2(-1)^m + 2^m] & \frac{1}{3}[\left[-1 + \left(-\frac{1}{2}\right)^m\right] & 0 \\ \frac{2}{3}\left[-1 + \left(-\frac{1}{2}\right)^m\right] & \frac{1}{3}\left[2 + \left(-\frac{1}{2}\right)^m\right] & 0 \\ -\frac{1}{9}2^{-m}[7(-4)^m - 6(-1)^m - 3\cdot 2^m + (-1)^m 2^{2m+1}] & \frac{1}{3}\left[-1 + \left(-\frac{1}{2}\right)^m\right] & \frac{1}{3}[(-2)^m + (-1)^m 2^{m+1}] \end{bmatrix} \end{aligned}$$

and

$$\mathbf{A}^8 = \begin{bmatrix} \frac{43}{128} & -\frac{85}{256} & 0 \\ -\frac{85}{128} & \frac{171}{256} & 0 \\ -\frac{32725}{128} & -\frac{85}{256} & 256 \end{bmatrix}.$$

11. The characteristic equation is $\lambda^2 - 8\lambda + 16 = 0$, with eigenvalues 4 and 4. Substituting the eigenvalues into $\lambda^m = c_0 + c_1\lambda$ generates

$$\begin{aligned} 4^m &= c_0 + 4c_1 \\ 4^{m-1}m &= c_1. \end{aligned}$$

Solving the system gives

$$c_0 = -4^m(m-1), \qquad c_1 = 4^{m-1}m.$$

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$$\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} = \begin{bmatrix} 4^{m-1}(3m+4) & 3\cdot 4^{m-1}m \\ -3\cdot 4^{m-1}m & 4^{m-1}(-3m+4) \end{bmatrix}$$

and

$$\mathbf{A}^6 = \begin{bmatrix} 22528 & 18432 \\ -18432 & -14336 \end{bmatrix}.$$

12. The characteristic equation is $-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$, with eigenvalues -3, -3, and 5. Substituting the eigenvalues into $\lambda^m = c_0 + c_1\lambda + c_2\lambda^2$ generates

$$\begin{aligned} (-3)^m &= c_0 - 3c_1 + 9c_2 \\ (-3)^{m-1}m &= c_1 - 6c_2 \\ 5^m &= c_0 + 5c_1 + 25c_2. \end{aligned}$$

Solving the system gives

$$\begin{aligned} c_0 &= \frac{1}{64}[73(-3)^m - 2(-1)^m 3^{m+2} + 9\cdot 5^m - 40(-3)^m m], \\ c_1 &= \frac{1}{96}[-(-1)^m 3^{m+2} + 9\cdot 5^m - 8(-3)^m m], \\ c_2 &= \frac{1}{64}[-(-3)^m + 5^m - 8(-3)^{m-1}m]. \end{aligned}$$

Thus

$$\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} + c_2\mathbf{A}^2$$

$$= \begin{bmatrix} \frac{1}{32}[31(-3)^m - (-1)^m 3^{m+1} + 4\cdot 5^m] & \frac{1}{16}[-(-3)^m - (-1)^m 3^{m+1} + 4\cdot 5^m] & \frac{1}{32}[(-3)^m + (-1)^m 3^{m+1} - 4\cdot 5^m] \\ \frac{1}{16}[-(-3)^m - (-1)^m 3^{m+1} + 4\cdot 5^m] & \frac{1}{8}[7(-3)^m - (-1)^m 3^{m+1} + 4\cdot 5^m] & \frac{1}{16}[(-3)^m + (-1)^m 3^{m+1} - 4\cdot 5^m] \\ \frac{3}{32}[(-3)^m + (-1)^m 3^{m+1} - 4\cdot 5^m] & \frac{3}{16}[(-3)^m + (-1)^m 3^{m+1} - 4\cdot 5^m] & \frac{1}{32}[29(-3)^m - (-1)^m 3^{m+2} + 12\cdot 5^m] \end{bmatrix}$$

and

$$\mathbf{A}^5 = \begin{bmatrix} 178 & 842 & -421 \\ 842 & 1441 & -842 \\ -1263 & -2526 & 1020 \end{bmatrix}.$$

13. **(a)** The characteristic equation is $\lambda^2 - 4\lambda = \lambda(\lambda - 4) = 0$, so 0 is an eigenvalue. Since the matrix satisfies the characteristic equation, $\mathbf{A}^2 = 4\mathbf{A}$, $\mathbf{A}^3 = 4\mathbf{A}^2 = 4^2\mathbf{A}$, $\mathbf{A}^4 = 4^2\mathbf{A}^2 = 4^3\mathbf{A}$, and, in general,

$$\mathbf{A}^m = 4^m\mathbf{A} = \begin{bmatrix} 4^m & 4^m \\ 3(4)^m & 3(4)^m \end{bmatrix}.$$

(b) The characteristic equation is $\lambda^2 = 0$, so 0 is an eigenvalue. Since the matrix satisfies the characteristic equation, $\mathbf{A}^2 = \mathbf{0}$, $\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \mathbf{0}$, and, in general, $\mathbf{A}^m = \mathbf{0}$.

(c) The characteristic equation is $-\lambda^3 + 5\lambda^2 - 6\lambda = 0$, with eigenvalues 0, 2, and 3. Substituting $\lambda = 0$ into $\lambda^m = c_0 + c_1\lambda + c_2\lambda^2$ we find that $c_0 = 0$. Using the nonzero eigenvalues, we find

$$\begin{aligned} 2^m &= 2c_1 + 4c_2 \\ 3^m &= 3c_1 + 9c_2. \end{aligned}$$

Solving the system gives

$$c_1 = \frac{1}{6}[9(2)^m - 4(3)^m], \qquad c_2 = \frac{1}{6}[-3(2)^m + 2(3)^m].$$

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Thus $\mathbf{A}^m = c_1\mathbf{A} + c_2\mathbf{A}^2$ and

$$\mathbf{A}^m = \begin{bmatrix} 2(3)^{m-1} & 3^{m-1} & 3^{m-1} \\ \frac{1}{6}[9(2)^m - 4(3)^m] & \frac{1}{6}[3(2)^m - 2(3)^m] & \frac{1}{6}[-3(2)^m - 2(3)^m] \\ \frac{1}{6}[-9(2)^m + 8(3)^m] & \frac{1}{6}[-3(2)^m + 4(3)^m] & \frac{1}{6}[3(2)^m + 4(3)^m] \end{bmatrix}.$$

14. (a) Let

$$\mathbf{X}_{n-1} = \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\mathbf{X}_n = \mathbf{A}\mathbf{X}_{n-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} x_{n-1} + y_{n-1} \\ x_{n-1} \end{bmatrix}.$$

(b) The characteristic equation of $\mathbf{A}$ is $\lambda^2 - \lambda - 1 = 0$, with eigenvalues $\lambda_1 = \frac{1}{2}(1 - \sqrt{5}\,)$ and $\lambda_2 = \frac{1}{2}(1 + \sqrt{5}\,)$. From $\lambda^m = c_0 + c_1\lambda$ we get $\lambda_1^m = c_0 + c_1\lambda_1$ and $\lambda_2^m = c_0 + c_1\lambda_2$. Solving this system gives

$$c_0 = (\lambda_2\lambda_1^m - \lambda_1\lambda_2^m)/(\lambda_2 - \lambda_1) \quad \text{and} \quad c_1 = (\lambda_2^m - \lambda_1^m)/(\lambda_2 - \lambda_1).$$

Thus

$$\begin{aligned} \mathbf{A}^m &= c_0\mathbf{I} + c_1\mathbf{A} \\ &= \frac{1}{2^{m+1}\sqrt{5}} \begin{bmatrix} (1+\sqrt{5}\,)^{m+1} - (1-\sqrt{5}\,)^{m+1} & 2(1+\sqrt{5}\,)^m - 2(1-\sqrt{5}\,)^m \\ 2(1+\sqrt{5}\,)^m - 2(1-\sqrt{5}\,)^m & (1+\sqrt{5}\,)(1-\sqrt{5}\,)^m - (1-\sqrt{5}\,)(1+\sqrt{5}\,)^m \end{bmatrix}. \end{aligned}$$

(c) From part (a), $\mathbf{X}_2 = \mathbf{A}\mathbf{X}_1$, $\mathbf{X}_3 = \mathbf{A}\mathbf{X}_2 = \mathbf{A}^2\mathbf{X}_1$, $\mathbf{X}_4 = \mathbf{A}\mathbf{X}_3 = \mathbf{A}^3\mathbf{X}_1$, and, in general, $\mathbf{X}_n = \mathbf{A}^{n-1}\mathbf{X}_1$. With

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{we have} \quad \mathbf{X}_{12} = \mathbf{A}^{11}\mathbf{X}_1 = \begin{bmatrix} 144 & 89 \\ 89 & 55 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 233 \\ 144 \end{bmatrix},$$

so the number of adult pairs is 233. With

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{we have} \quad \mathbf{A}^{11}\mathbf{X}_1 = \begin{bmatrix} 144 & 89 \\ 89 & 55 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 144 \\ 89 \end{bmatrix},$$

so the number of baby pairs is 144. With

$$\mathbf{X}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{we have} \quad \mathbf{A}^{11}\mathbf{X}_1 = \begin{bmatrix} 144 & 89 \\ 89 & 55 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 377 \\ 233 \end{bmatrix},$$

so the total number of pairs is 377.

15. The characteristic equation of $\mathbf{A}$ is $\lambda^2 - 5\lambda + 10 = 0$, so $\mathbf{A}^2 - 5\mathbf{A} + 10\mathbf{I} = \mathbf{0}$ and $\mathbf{I} = -\frac{1}{10}\mathbf{A}^2 + \frac{1}{2}\mathbf{A}$. Multiplying by $\mathbf{A}^{-1}$ we find

$$\mathbf{A}^{-1} = -\frac{1}{10}\mathbf{A} + \frac{1}{2}\mathbf{I} = -\frac{1}{10}\begin{bmatrix} 2 & -4 \\ 1 & 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & \frac{2}{5} \\ -\frac{1}{10} & \frac{1}{5} \end{bmatrix}.$$

16. The characteristic equation of $\mathbf{A}$ is $-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$, so $-\mathbf{A}^3 + 2\mathbf{A}^2 + \mathbf{A} - 2\mathbf{I} = \mathbf{0}$ and $\mathbf{I} = -\frac{1}{2}\mathbf{A}^3 + \mathbf{A}^2 + \frac{1}{2}\mathbf{A}$. Multiplying by $\mathbf{A}^{-1}$ we find

$$\mathbf{A}^{-1} = -\frac{1}{2}\mathbf{A}^2 + \mathbf{A} + \frac{1}{2}\mathbf{I} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \end{bmatrix}.$$

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17. **(a)** Since

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \quad \text{we see that} \quad \mathbf{A}^m = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

for all integers $m \geq 2$. Thus $\mathbf{A}$ is not nilpotent.

(b) Since $\mathbf{A}^2 = \mathbf{0}$, the matrix is nilpotent with index 2.

(c) Since $\mathbf{A}^3 = \mathbf{0}$, the matrix is nilpotent with index 3.

(d) Since $\mathbf{A}^2 = \mathbf{0}$, the matrix is nilpotent with index 2.

(e) Since $\mathbf{A}^4 = \mathbf{0}$, the matrix is nilpotent with index 4.

(f) Since $\mathbf{A}^4 = \mathbf{0}$, the matrix is nilpotent with index 4.

18. **(a)** If $\mathbf{A}^m = \mathbf{0}$ for some m, then $(\det \mathbf{A})^m = \det \mathbf{A}^m = \det \mathbf{0} = 0$, and $\mathbf{A}$ is a singular matrix.

(b) By (1) of Section 8.8 we have $\mathbf{AK} = \lambda\mathbf{K}$, $\mathbf{A}^2\mathbf{K} = \lambda\mathbf{AK} = \lambda^2\mathbf{K}$, $\mathbf{A}^3\mathbf{K} = \lambda^2\mathbf{AK} = \lambda^3\mathbf{K}$, and, in general, $\mathbf{A}^m\mathbf{K} = \lambda^m\mathbf{K}$. If $\mathbf{A}$ is nilpotent with index m, then $\mathbf{A}^m = \mathbf{0}$ and $\lambda^m = 0$.

Exercises 8.10

1. **(a)–(b)**

$$\begin{bmatrix} 0 & 0 & -4 \\ 0 & -4 & 0 \\ -4 & 0 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix} = -4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda_1 = -4$$

$$\begin{bmatrix} 0 & 0 & -4 \\ 0 & -4 & 0 \\ -4 & 0 & 15 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}; \quad \lambda_2 = -1$$

$$\begin{bmatrix} 0 & 0 & -4 \\ 0 & -4 & 0 \\ -4 & 0 & 15 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 16 \\ 0 \\ -64 \end{bmatrix} = 16 \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}; \quad \lambda_3 = 16$$

(c) $\mathbf{K}_1^T\mathbf{K}_2 = [\,0 \quad 1 \quad 0\,] \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = 0; \quad \mathbf{K}_1^T\mathbf{K}_3 = [\,0 \quad 1 \quad 0\,] \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = 0; \quad \mathbf{K}_2^T\mathbf{K}_3 = [\,4 \quad 0 \quad 1\,] \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = 4 - 4 = 0$

2. **(a)–(b)**

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}; \quad \lambda_1 = 2$$

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}; \quad \lambda_2 = 2$$

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \quad \lambda_3 = -1$$

(c) $\mathbf{K}_1^T\mathbf{K}_2 = [\,-2 \quad 1 \quad 1\,] \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 1 - 1 = 0; \quad \mathbf{K}_1^T\mathbf{K}_3 = [\,-2 \quad 1 \quad 1\,] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -2 + 1 + 1 = 0$

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$$\mathbf{K}_2^T\mathbf{K}_3 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 - 1 = 0$$

3. (**a**)–(**b**) $\begin{bmatrix} 5 & 13 & 0 \\ 13 & 5 & 0 \\ 0 & 0 & -8 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 9\sqrt{2} \\ 9\sqrt{2} \\ 0 \end{bmatrix} = 18\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}; \quad \lambda_1 = 18$

$$\begin{bmatrix} 5 & 13 & 0 \\ 13 & 5 & 0 \\ 0 & 0 & -8 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix} = \begin{bmatrix} -\frac{8\sqrt{2}}{3} \\ \frac{8\sqrt{3}}{3} \\ -\frac{8\sqrt{3}}{3} \end{bmatrix} = (-8)\begin{bmatrix} \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}; \quad \lambda_2 = -8$$

$$\begin{bmatrix} 5 & 13 & 0 \\ 13 & 5 & 0 \\ 0 & 0 & -8 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \end{bmatrix} = \begin{bmatrix} -\frac{8\sqrt{6}}{6} \\ \frac{8\sqrt{6}}{6} \\ \frac{8\sqrt{6}}{3} \end{bmatrix} = (-8)\begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \end{bmatrix}; \quad \lambda_3 = -8$$

(**c**) $\mathbf{K}_1^T\mathbf{K}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix} = \frac{\sqrt{6}}{6} - \frac{\sqrt{6}}{6} = 0; \quad \mathbf{K}_1^T\mathbf{K}_3 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \end{bmatrix} = \frac{\sqrt{12}}{12} - \frac{\sqrt{12}}{12} = 0$

$$\mathbf{K}_2^T\mathbf{K}_3 = \begin{bmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \end{bmatrix} = \frac{\sqrt{18}}{18} + \frac{\sqrt{18}}{18} - \frac{\sqrt{18}}{9} = 0$$

4. (**a**)–(**b**) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}; \quad \lambda_1 = 0$

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}; \quad \lambda_2 = 3$$

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 12 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}; \quad \lambda_3 = 6$$

(**c**) $\mathbf{K}_1^T\mathbf{K}_2 = \begin{bmatrix} -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = -2 + 4 - 2 = 0; \quad \mathbf{K}_1^T\mathbf{K}_3 = \begin{bmatrix} -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = -4 + 2 + 2 = 0$

$$\mathbf{K}_2^T\mathbf{K}_3 = \begin{bmatrix} 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 2 + 2 - 4 = 0$$

5. Orthogonal. Columns form an orthonormal set.

6. Not orthogonal. Columns one and three are not unit vectors.

7. Orthogonal. Columns form an orthonormal set.

8. Not orthogonal. The matrix is singular.

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9. Not orthogonal. Columns are not unit vectors.

10. Orthogonal. Columns form an orthogonal set.

11. $\lambda_1 = -8,\ \lambda_2 = 10,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

12. $\lambda_1 = 7,\ \lambda_2 = 4,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

13. $\lambda_1 = 0,\ \lambda_2 = 10,\ \mathbf{K}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$

14. $\lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{2},\ \lambda_2 = \frac{1}{2} - \frac{\sqrt{5}}{2},\ \mathbf{K}_1 = \begin{bmatrix} 1+\sqrt{5} \\ 2 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1-\sqrt{5} \\ 2 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} \frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & \frac{1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}} \\ \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{2}{\sqrt{10-2\sqrt{5}}} \end{bmatrix}$

15. $\lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = 1,\ \mathbf{K}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$

16. $\lambda_1 = -1,\ \lambda_2 = 1-\sqrt{2},\ \lambda_3 = 1+\sqrt{2},\ \mathbf{K}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

17. $\lambda_1 = -11,\ \lambda_2 = 0,\ \lambda_3 = 6,\ \mathbf{K}_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} -\frac{3}{\sqrt{11}} & \frac{1}{\sqrt{66}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{11}} & -\frac{4}{\sqrt{66}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{11}} & \frac{7}{\sqrt{66}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

18. $\lambda_1 = -18,\ \lambda_2 = 0,\ \lambda_3 = 9,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

19. $\begin{bmatrix} \frac{3}{5} & a \\ \frac{4}{5} & b \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ implies $\frac{9}{25} + a^2 = 1$ and $\frac{16}{25} + b^2 = 1$. These equations give $a = \pm\frac{4}{5}$, $b = \pm\frac{3}{5}$. But $\frac{12}{25} + ab = 0$ indicates a and b must have opposite signs. Therefore choose $a = -\frac{4}{5}$, $b = \frac{3}{5}$.

The matrix $\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ is orthogonal.

20. $\begin{bmatrix} \frac{1}{\sqrt{5}} & b \\ a & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & a \\ b & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ implies $\frac{1}{5} + b^2 = 1$ and $a^2 + \frac{1}{5} = 1$. These equations give $a = \pm\frac{2}{\sqrt{5}}$, $b = \pm\frac{2}{\sqrt{5}}$. But $\frac{a}{\sqrt{5}} + \frac{b}{\sqrt{5}} = 0$ indicates a and b must have opposite signs. Therefore choose $a = -\frac{2}{\sqrt{5}}$, $b = \frac{2}{\sqrt{2}}$.

The matrix $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ is orthogonal.

21. **(a)** $\mathbf{AK}_1 = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 36 \\ 9 \\ -9 \end{bmatrix} = 9 \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} = \lambda_1 \mathbf{K}_1$

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$$\mathbf{AK}_2 = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ -36 \end{bmatrix} = -9 \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \lambda_2 \mathbf{K}_2$$

$$\mathbf{AK}_3 = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 36 \\ 0 \end{bmatrix} = -9 \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} = \lambda_3 \mathbf{K}_3$$

(b) For $\lambda_2 = \lambda_3 = -9$ we have

$$\left[\begin{array}{ccc|c} 16 & 4 & -4 & 0 \\ 4 & 1 & -1 & 0 \\ -4 & -1 & 1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $k_1 = -\frac{1}{4}k_2 + \frac{1}{4}k_3$. The choices $k_2 = 0$, $k_3 = 4$ and $k_2 = -4$, $k_3 = 0$ give the eigenvectors in part (a). These eigenvectors are not orthogonal. However, by choosing $k_2 = k_3 = 1$ and $k_2 = -2$, $k_3 = 2$ we obtain, respectively,

$$\mathbf{K}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{K}_3 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

Mutually orthogonal eigenvectors are

$$\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

22. The eigenvalues and eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = \lambda_2 = -1, \quad \lambda_3 = \lambda_4 = 3, \quad \mathbf{K}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{K}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{K}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Since $\|\mathbf{K}_1\| = \|\mathbf{K}_2\| = \|\mathbf{K}_3\| = \|\mathbf{K}_4\| = \sqrt{2}$, an orthogonal matrix is

$$\mathbf{P} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

23. Suppose $\mathbf{A}$ and $\mathbf{B}$ are orthogonal matrices. Then $\mathbf{A}^{-1} = \mathbf{A}^T$ and $\mathbf{B}^{-1} = \mathbf{B}^T$ and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{B}^T\mathbf{A}^T = (\mathbf{AB})^T.$$

Thus $\mathbf{AB}$ is an orthogonal matrix.

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Exercises 8.11

1. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and computing $\mathbf{X}_i = \mathbf{A}\mathbf{X}_{i-1}$ for $i = 1, 2, 3, 4$ we obtain

$$\mathbf{X}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 8 \\ 8 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} 16 \\ 16 \end{bmatrix}.$$

We conclude that a dominant eigenvector is $\mathbf{K} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with corresponding eigenvalue $\lambda = \dfrac{\mathbf{AK} \cdot \mathbf{K}}{\mathbf{K} \cdot \mathbf{K}} = \dfrac{4}{2} = 2$.

2. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and computing $\mathbf{X}_i = \mathbf{A}\mathbf{X}_{i-1}$ for $i = 1, 2, 3, 4, 5$ we obtain

$$\mathbf{X}_1 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 49 \\ -47 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} -437 \\ 439 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} 3937 \\ -3935 \end{bmatrix}, \quad \mathbf{X}_5 = \begin{bmatrix} -35429 \\ 35431 \end{bmatrix}.$$

We conclude that a dominant eigenvector is $\mathbf{K} = \dfrac{1}{35439}\begin{bmatrix} -35429 \\ 35431 \end{bmatrix} \approx \begin{bmatrix} -0.99994 \\ 1 \end{bmatrix}$ with corresponding eigenvalue $\lambda = \dfrac{\mathbf{AK} \cdot \mathbf{K}}{\mathbf{K} \cdot \mathbf{K}} = -8.9998$.

3. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and computing $\mathbf{A}\mathbf{X}_0 = \begin{bmatrix} 6 \\ 16 \end{bmatrix}$, we define $\mathbf{X}_1 = \dfrac{1}{16}\begin{bmatrix} 6 \\ 16 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 1 \end{bmatrix}$. Continuing in this manner we obtain

$$\mathbf{X}_2 = \begin{bmatrix} 0.3363 \\ 1 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 0.3335 \\ 1 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} 0.3333 \\ 1 \end{bmatrix}.$$

We conclude that a dominant eigenvector is $\mathbf{K} = \begin{bmatrix} 0.3333 \\ 1 \end{bmatrix}$ with corresponding eigenvalue $\lambda = 14$.

4. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and computing $\mathbf{A}\mathbf{X}_0 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, we define $\mathbf{X}_1 = \dfrac{1}{5}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$. Continuing in this manner we obtain

$$\mathbf{X}_2 = \begin{bmatrix} 0.2727 \\ 1 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 0.2676 \\ 1 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} 0.2680 \\ 1 \end{bmatrix}, \quad \mathbf{X}_5 = \begin{bmatrix} 0.2679 \\ 1 \end{bmatrix}.$$

We conclude that a dominant eigenvector is $\mathbf{K} = \begin{bmatrix} 0.2679 \\ 1 \end{bmatrix}$ with corresponding eigenvalue $\lambda = 6.4641$.

5. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and computing $\mathbf{A}\mathbf{X}_0 = \begin{bmatrix} 11 \\ 11 \\ 6 \end{bmatrix}$, we define $\mathbf{X}_1 = \dfrac{1}{11}\begin{bmatrix} 11 \\ 11 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0.5455 \end{bmatrix}$. Continuing in this manner we obtain

$$\mathbf{X}_2 = \begin{bmatrix} 1 \\ 1 \\ 0.5045 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 1 \\ 1 \\ 0.5005 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} 1 \\ 1 \\ 0.5 \end{bmatrix}.$$

We conclude that a dominant eigenvector is $\mathbf{K} = \begin{bmatrix} 1 \\ 1 \\ 0.5 \end{bmatrix}$ with corresponding eigenvalue $\lambda = 10$.

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6. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and computing $\mathbf{AX}_0 = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$, we define $\mathbf{X}_1 = \frac{1}{5}\begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.4 \\ 0.4 \end{bmatrix}$. Continuing in this manner we obtain

$$\mathbf{X}_2 = \begin{bmatrix} 1 \\ 0.2105 \\ 0.2105 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 1 \\ 0.1231 \\ 0.1231 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} 1 \\ 0.0758 \\ 0.0758 \end{bmatrix}, \quad \mathbf{X}_5 = \begin{bmatrix} 1 \\ 0.0481 \\ 0.0481 \end{bmatrix}.$$

At this point if we restart with $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ we see that $\mathbf{K} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a dominant eigenvector with corresponding eigenvalue $\lambda = 3$.

7. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and using scaling we obtain

$$\mathbf{X}_1 = \begin{bmatrix} 0.625 \\ 1 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 0.5345 \\ 1 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 0.5098 \\ 1 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} 0.5028 \\ 1 \end{bmatrix}, \quad \mathbf{X}_5 = \begin{bmatrix} 0.5008 \\ 1 \end{bmatrix}.$$

Taking $\mathbf{K} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ as the dominant eigenvector we find $\lambda_1 = 7$. Now the normalized eigenvector is $\mathbf{K}_1 = \begin{bmatrix} 0.4472 \\ 0.8944 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1.6 & -0.8 \\ -0.8 & 0.4 \end{bmatrix}$. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and using scaling again we obtain $\mathbf{X}_1 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$, $\mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$. Taking $\mathbf{K} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$ we find $\lambda_2 = 2$. The eigenvalues are 7 and 2.

8. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and using scaling we obtain $\mathbf{X}_1 = \begin{bmatrix} 0.3333 \\ 1 \end{bmatrix}$, $\mathbf{X}_2 = \begin{bmatrix} 0.3333 \\ 1 \end{bmatrix}$. Taking $\mathbf{K} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$ as the dominant eigenvector we find $\lambda_1 = 10$. Now the normalized eigenvector is $\mathbf{K}_1 = \begin{bmatrix} 0.3162 \\ 0.9486 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.An eigenvector for the zero matrix is $\lambda_2 = 0$. The eigenvalues are 10 and 0.

9. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and using scaling we obtain

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.6667 \\ 1 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 1 \\ -0.9091 \\ 1 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} 1 \\ -0.9767 \\ 1 \end{bmatrix}, \quad \mathbf{X}_5 = \begin{bmatrix} 1 \\ -0.9942 \\ 1 \end{bmatrix}.$$

Taking $\mathbf{K} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ as the dominant eigenvector we find $\lambda_1 = 4$. Now the normalized eigenvector is

$$\mathbf{K}_1 = \begin{bmatrix} 0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix} \text{and} \quad \mathbf{B} = \begin{bmatrix} 1.6667 & 0.3333 & -1.3333 \\ 0.3333 & 0.6667 & 0.3333 \\ -1.3333 & 0.3333 & 1.6667 \end{bmatrix}. \text{ If } \mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ is now chosen only one more}$$

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eigenvalue is found. Thus, try $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Using scaling we obtain

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 1 \\ 0.2 \\ -0.8 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 1 \\ 0.0714 \\ -0.9286 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} 1 \\ 0.0244 \\ -0.9756 \end{bmatrix}, \quad \mathbf{X}_5 = \begin{bmatrix} 1 \\ 0.0082 \\ -0.9918 \end{bmatrix}.$$

Taking $\mathbf{K} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ as the eigenvector we find $\lambda_2 = 3$. The normalized eigenvector in this case is $\mathbf{K}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 0.1667 & 0.3333 & 0.1667 \\ 0.3333 & 0.6667 & 0.3333 \\ 0.1667 & 0.3333 & 0.1667 \end{bmatrix}$. If $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is chosen, and scaling is used we obtain $\mathbf{X}_1 = \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix}$, $\mathbf{X}_2 = \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix}$. Taking $\mathbf{K} = \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix}$ we find $\lambda_3 = 1$. The eigenvalues are 4, 3, and 1. The difficulty in choosing $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to find the second eigenvector results from the fact that this vector is a linear combination of the eigenvectors corresponding to the other two eigenvalues, with 0 contribution from the second eigenvector. When this occurs the development of the power method, shown in the text, breaks down.

10. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and using scaling we obtain

$$\mathbf{X}_1 = \begin{bmatrix} -0.3636 \\ -0.3636 \\ 1 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} -0.2431 \\ 0.0884 \\ 1 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} -0.2504 \\ -0.0221 \\ 1 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} -0.2499 \\ -0.0055 \\ 1 \end{bmatrix}.$$

Taking $\mathbf{K} = \begin{bmatrix} -0.25 \\ 0 \\ 1 \end{bmatrix}$ as the dominant eigenvector we find $\lambda_1 = 16$. The normalized eigenvector is $\mathbf{K}_1 = \begin{bmatrix} -0.2425 \\ 0 \\ 0.9701 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -0.9412 & 0 & -0.2353 \\ 0 & -4 & 0 \\ -0.2353 & 0 & -0.0588 \end{bmatrix}$. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and using scaling we obtain

$$\mathbf{X}_1 = \begin{bmatrix} -0.2941 \\ -1 \\ -0.0735 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 0.0735 \\ 1 \\ 0.0184 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} -0.0184 \\ -1 \\ -0.0046 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} 0.0046 \\ 1 \\ 0.0011 \end{bmatrix}.$$

Taking $\mathbf{K} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as the eigenvector we find $\lambda_2 = -4$. The normalized eigenvector in this case is $\mathbf{K}_2 = \mathbf{K} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -0.9412 & 0 & -0.2353 \\ 0 & 0 & 0 \\ -0.2353 & 0 & -0.0588 \end{bmatrix}$. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and using scaling we obtain $\mathbf{X}_1 = \begin{bmatrix} -1 \\ 0 \\ -0.25 \end{bmatrix}$,

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$\mathbf{X}_2 = \begin{bmatrix} 1 \\ 0 \\ 0.25 \end{bmatrix}$. Using $\mathbf{K} = \begin{bmatrix} 1 \\ 0 \\ 0.25 \end{bmatrix}$ we find $\lambda_3 = -1$. The eigenvalues are 16, -4, and -1.

11. The inverse matrix is $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and using scaling we obtain

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ -0.6667 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 1 \\ -0.7857 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 1 \\ -0.7910 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} 1 \\ -0.7913 \end{bmatrix}.$$

Using $\mathbf{K} = \begin{bmatrix} 1 \\ -0.7913 \end{bmatrix}$ we find $\lambda = 4.7913$. The minimum eigenvalue of $\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$ is $1/4.7913 \approx 0.2087$.

12. The inverse matrix is $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$. Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and using scaling we obtain

$$\mathbf{X}_1 = \begin{bmatrix} 0.6667 \\ 1 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 0.7857 \\ 1 \end{bmatrix}, \ldots, \ \mathbf{X}_{10} = \begin{bmatrix} 0.75 \\ 1 \end{bmatrix}.$$

Using $\mathbf{K} = \begin{bmatrix} 0.75 \\ 1 \end{bmatrix}$ we find $\lambda = 5$. The minimum eigenvalue of $\begin{bmatrix} -0.2 & 0.3 \\ 0.4 & -0.1 \end{bmatrix}$ is $1/5 = 0.2$

13. **(a)** Replacing the second derivative with the difference expression we obtain

$$EI\,\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + Py_i = 0 \quad \text{or} \quad EI(y_{i+1} - 2y_i + y_{i-1}) + Ph^2 y_i = 0.$$

(b) Expanding the difference equation for $i = 1$, 2, 3 and using $h = L/4$, $y_0 = 0$, and $y_4 = 0$ we obtain

$$\begin{aligned} EI(y_2 - 2y_1 + y_0) + \frac{PL^2}{16}\,y_1 = 0 \qquad & \qquad 2y_1 - y_2 = \frac{PL^2}{16EI}\,y_1 \\ EI(y_3 - 2y_2 + y_1) + \frac{PL^2}{16}\,y_2 = 0 \qquad & \text{or} \qquad -y_1 + 2y_2 - y_3 = \frac{PL^2}{16EI}\,y_2 \\ EI(y_4 - 2y_3 + y_2) + \frac{PL^2}{16}\,y_3 = 0 \qquad & \qquad -y_2 + 2y_3 = \frac{PL^2}{16EI}\,y_3. \end{aligned}$$

In matrix form this becomes

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{PL^2}{16EI} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

(c) $\mathbf{A}^{-1} = \begin{bmatrix} 0.75 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.75 \end{bmatrix}$

(d) Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and using scaling we obtain

$$\mathbf{X}_1 = \begin{bmatrix} 0.75 \\ 1 \\ 0.75 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 0.7143 \\ 1 \\ 0.7143 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 0.7083 \\ 1 \\ 0.7083 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} 0.7073 \\ 1 \\ 0.7073 \end{bmatrix}, \quad \mathbf{X}_5 = \begin{bmatrix} 0.7071 \\ 1 \\ 0.7071 \end{bmatrix}.$$

Using $\mathbf{K} = \begin{bmatrix} 0.7071 \\ 1 \\ 0.7071 \end{bmatrix}$ we find $\lambda = 1.7071$. Then $1/\lambda = 0.5859$ is the minimum eigenvalue of $\mathbf{A}$.

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(e) Solving $\dfrac{PL^2}{16EI} = 0.5859$ for P we obtain $P = 9.3726\,\dfrac{EI}{L^2}$. In Example 3 of Section 3.9 we saw

$$P = \pi^2\,\frac{EI}{L^2} \approx 9.8696\,\frac{EI}{L^2}\,.$$

14. (a) The difference equation is

$$EI_i(y_{i+1} - 2y_i + y_{i-1}) + Ph^2y_i = 0, \quad i = 1,\ 2,\ 3,$$

where $I_0 = 0.00200$, $I_1 = 0.00175$, $I_2 = 0.00150$, $I_3 = 0.00125$, and $I_4 = 0.00100$. The system of equations is

$$\begin{aligned} 0.00175E(y_2 - 2y_1 + y_0) + \frac{PL^2}{16}\,y_1 &= 0 \\ 0.00150E(y_3 - 2y_2 + y_1) + \frac{PL^2}{16}\,y_2 &= 0 \\ 0.00125E(y_4 - 2y_3 + y_2) + \frac{PL^2}{16}\,y_2 &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 0.0035y_1 - 0.00175y_2 &= \frac{PL^2}{16E}\,y_1 \\ -0.0015y_1 + 0.003y_2 - 0.0015y_3 &= \frac{PL^2}{16E}\,y_2 \\ -0.00125y_2 + 0.0025y_3 &= \frac{PL^2}{16E}\,y_3. \end{aligned}$$

In matrix form this becomes

$$\begin{bmatrix} 0.0035 & -0.00175 & 0 \\ -0.0015 & 0.003 & -0.0015 \\ 0 & -0.00125 & 0.0025 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{PL^2}{16E} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

(b) The inverse of $\mathbf{A}$ is

$$\mathbf{A}^{-1} = \begin{bmatrix} 428.571 & 333.333 & 200 \\ 285.714 & 666.667 & 400 \\ 142.857 & 333.333 & 600 \end{bmatrix}.$$

Taking $\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and using scaling we obtain

$$\mathbf{X}_1 = \begin{bmatrix} 0.7113 \\ 1 \\ 0.7958 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 0.6710 \\ 1 \\ 0.7679 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 0.6645 \\ 1 \\ 0.7635 \end{bmatrix}, \quad \mathbf{X}_4 = \begin{bmatrix} 0.6634 \\ 1 \\ 0.7628 \end{bmatrix}, \quad \mathbf{X}_5 = \begin{bmatrix} 0.6632 \\ 1 \\ 0.7627 \end{bmatrix}.$$

This yields the eigenvalue $\lambda = 1161.23$. The smallest eigenvalue of $\mathbf{A}$ is then $1/\lambda = 0.0008612$. The lowest critical load is

$$P = \frac{16E}{L2}(0.0008612) - 0.01378\,\frac{E}{L^2}\,.$$

15. (a) $\mathbf{A}^{10} = \begin{bmatrix} 67{,}745{,}349 & -43{,}691{,}832 & 8{,}258{,}598 \\ -43{,}691{,}832 & 28{,}182{,}816 & -5{,}328{,}720 \\ 8{,}258{,}598 & -5{,}328{,}720 & 1{,}008{,}180 \end{bmatrix}$

(b) $\mathbf{X}_{10} = \mathbf{A}^{10} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 67{,}745{,}349 \\ -43{,}691{,}832 \\ 8{,}258{,}598 \end{bmatrix} \approx 67{,}745{,}349 \begin{bmatrix} 1 \\ -0.644942 \\ 0.121906 \end{bmatrix}$

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$$\mathbf{X}_{12} = \mathbf{A}^{12}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2{,}680{,}201{,}629\\-1{,}728{,}645{,}624\\326{,}775{,}222\end{bmatrix} \approx 2{,}680{,}201{,}629\begin{bmatrix}1\\-0.644968\\0.121922\end{bmatrix}.$$

The vectors appear to be approaching scalar multiples of $\mathbf{K} = (1, -0.644968, 0.121922)$, which approximates the dominant eigenvector.

(c) The dominant eigenvalue is $\lambda_1 = (\mathbf{AK}\cdot\mathbf{K})/(\mathbf{K}\cdot\mathbf{K}) = 6.28995$.

Exercises 8.12

1. Distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = 5$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix}-3 & 1\\1 & 1\end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix}1 & 0\\0 & 5\end{bmatrix}$$

2. Distinct eigenvalues $\lambda_1 = 0$, $\lambda_2 = 6$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix}-5 & -1\\4 & 2\end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix}0 & 0\\0 & 6\end{bmatrix}$$

3. For $\lambda_1 = \lambda_2 = 1$ we obtain the single eigenvector $\mathbf{K}_1 = \begin{bmatrix}1\\1\end{bmatrix}$. Hence $\mathbf{A}$ is not diagonalizable.

4. Distinct eigenvalues $\lambda_1 = \sqrt{5}$, $\lambda_2 = -\sqrt{5}$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix}\sqrt{5} & -\sqrt{5}\\1 & 1\end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix}\sqrt{5} & 0\\0 & -\sqrt{5}\end{bmatrix}$$

5. Distinct eigenvalues $\lambda_1 = -7$, $\lambda_2 = 4$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix}13 & 1\\2 & 1\end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix}-7 & 0\\0 & 4\end{bmatrix}$$

6. Distinct eigenvalues $\lambda_1 = -4$, $\lambda_2 = 10$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix}-3 & 1\\1 & -5\end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix}-4 & 0\\0 & 10\end{bmatrix}$$

7. Distinct eigenvalues $\lambda_1 = \dfrac{1}{3}$, $\lambda_2 = \dfrac{2}{3}$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix}\frac{1}{3} & 0\\0 & \frac{2}{3}\end{bmatrix}$$

8. For $\lambda_1 = \lambda_2 = -3$ we obtain the single eigenvector $\mathbf{K}_1 = \begin{bmatrix}1\\1\end{bmatrix}$. Hence $\mathbf{A}$ is not diagonalizable.

9. Distinct eigenvalues $\lambda_1 = -i$, $\lambda_2 = i$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix}1 & 1\\-i & i\end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix}-i & 0\\0 & i\end{bmatrix}$$

10. Distinct eigenvalues $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix}2 & 2\\i & -i\end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix}1+i & 0\\0 & 1-i\end{bmatrix}$$

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11. Distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

12. Distinct eigenvalues $\lambda_1 = 3$, $\lambda_2 = 4$, $\lambda_3 = 5$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

13. Distinct eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

14. Distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = -3i$, $\lambda_3 = 3i$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix} 0 & -3i & 3i \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3i & 0 \\ 0 & 0 & 3i \end{bmatrix}$$

15. The eigenvalues are $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$. For $\lambda_1 = \lambda_2 = 1$ we obtain the single eigenvector $\mathbf{K}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Hence $\mathbf{A}$ is not diagonalizable.

16. Distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

17. Distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = \sqrt{5}$, $\lambda_3 = -\sqrt{5}$ imply $\mathbf{A}$ is diagonalizable.

$$\mathbf{P} = \begin{bmatrix} 0 & 1+\sqrt{5} & 1-\sqrt{5} \\ 0 & 2 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

18. For $\lambda_1 = \lambda_2 = \lambda_3 = 1$ we obtain the single eigenvector $\mathbf{K}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Hence $\mathbf{A}$ is not diagonalizable.

19. For the eigenvalues $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 1$, $\lambda_4 = -1$ we obtain four linearly independent eigenvectors. Hence $\mathbf{A}$ is diagonalizable and

$$\mathbf{P} = \begin{bmatrix} -3 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

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20. The eigenvalues are $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = \lambda_4 = 3$. For $\lambda_3 = \lambda_4 = 3$ we obtain the single eigenvector $\mathbf{K}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

Hence $\mathbf{A}$ is not diagonalizable.

21. $\lambda_1 = 0,\ \lambda_2 = 2,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},\ \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

22. $\lambda_1 = -1,\ \lambda_2 = 4,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix},\ \mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

23. $\lambda_1 = 3,\ \lambda_2 = 10,\ \mathbf{K}_1 = \begin{bmatrix} -\sqrt{10} \\ 2 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} \sqrt{10} \\ 5 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} -\frac{\sqrt{10}}{\sqrt{14}} & \frac{\sqrt{10}}{\sqrt{35}} \\ \frac{2}{\sqrt{14}} & \frac{15}{\sqrt{35}} \end{bmatrix},\ \mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix}$

24. $\lambda_1 = -1,\ \lambda_2 = 3,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix},\ \mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

25. $\lambda_1 = -1,\ \lambda_2 = \lambda_3 = 1,\ \mathbf{K}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

26. $\lambda_1 = \lambda_2 = -1,\ \lambda_3 = 5,\ \mathbf{K}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix},$

$\mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

27. $\lambda_1 = 3,\ \lambda_2 = 6,\ \lambda_3 = 9,\ \mathbf{K}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix},\ \mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

28. $\lambda_1 = 1,\ \lambda_2 = 2 - \sqrt{2},\ \lambda_3 = 2 + \sqrt{2},\ \mathbf{K}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 - \sqrt{2} \\ 0 \\ 1 \end{bmatrix},\ \mathbf{K_3} = \begin{bmatrix} 1 + \sqrt{2} \\ 0 \\ 1 \end{bmatrix},$

$\mathbf{P} = \begin{bmatrix} 0 & -\frac{\sqrt{2-\sqrt{2}}}{2} & \frac{\sqrt{2+\sqrt{2}}}{2} \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2+\sqrt{2}}}{2} & \frac{\sqrt{2-\sqrt{2}}}{2} \end{bmatrix},\ \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 - 2\sqrt{2} & 0 \\ 0 & 0 & 2 + \sqrt{2} \end{bmatrix}$

29. $\lambda_1 = 1,\ \lambda_2 = -6,\ \lambda_3 = 8,\ \mathbf{K}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},\ \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

30. $\lambda_1 = \lambda_2 = 0,\ \lambda_3 = -2,\ \lambda_4 = 2,\ \mathbf{K}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix},\ \mathbf{K}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

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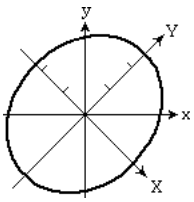
$$\mathbf{P} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

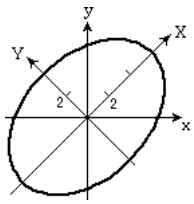
31. The given equation can be written as $\mathbf{X}^T\mathbf{A}\mathbf{X} = 24$: $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 24$. Using $\lambda_1 = 6,\ \lambda_2 = 4,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{X} = \mathbf{P}\mathbf{X}'$ we find



$$\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 24 \quad \text{or} \quad 6X^2 + 4Y^2 = 24.$$

The conic section is an ellipse. Now from $\mathbf{X}' = \mathbf{P}^T\mathbf{X}$ we see that the XY-coordinates of $(1, -1)$ and $(1, 1)$ are $(\sqrt{2}\,, 0)$ and $(0, \sqrt{2}\,)$, respectively. From this we conclude that the X-axis and Y-axis are as shown in the accompanying figure.

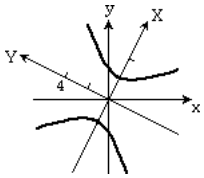
32. The given equation can be written as $\mathbf{X}^T\mathbf{A}\mathbf{X} = 288$: $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 288$. Using $\lambda_1 = 8,\ \lambda_2 = 18,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{X} = \mathbf{P}\mathbf{X}'$ we find



$$\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 18 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 288 \quad \text{or} \quad 8X^2 + 18Y^2 = 288.$$

The conic section is an ellipse. Now from $\mathbf{X}' = \mathbf{P}^T\mathbf{X}$ we see that the XY-coordinates of $(1, 1)$ and $(1, -1)$ are $(\sqrt{2}\,, 0)$ and $(0, -\sqrt{2}\,)$, respectively. From this we conclude that the X-axis and Y-axis are as shown in the accompanying figure.

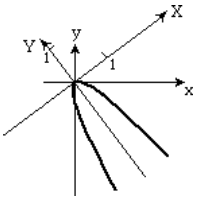
33. The given equation can be written as $\mathbf{X}^T\mathbf{A}\mathbf{X} = 20$: $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 20$. Using $\lambda_1 = 5,\ \lambda_2 = -5,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ and $\mathbf{X} = \mathbf{P}\mathbf{X}'$ we find



$$\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 20 \quad \text{or} \quad 5X^2 - 5Y^2 = 20.$$

The conic section is a hyperbola. Now from $\mathbf{X}' = \mathbf{P}^T\mathbf{X}$ we see that the XY-coordinates of $(1, 2)$ and $(-2, 1)$ are $(\sqrt{5}\,, 0)$ and $(0, \sqrt{5}\,)$, respectively. From this we conclude that the X-axis and Y-axis are as shown in the accompanying figure.

34. The given equation can be written as $\mathbf{X}^T\mathbf{A}\mathbf{X} = 288$:



$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Using $\lambda_1 = 25,\ \lambda_2 = 0,\ \mathbf{K}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{bmatrix}$ and $\mathbf{X} = \mathbf{P}\mathbf{X}'$ we find

$$\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 0 & 5 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0 \quad \text{or} \quad 25X^2 + 5Y = 0.$$

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The conic section is a parabola. Now from $\mathbf{X}' = \mathbf{P}^T\mathbf{X}$ we see that the XY-coordinates of $(4, 3)$ and $(3, -4)$ are $(5, 0)$ and $(0, -5)$, respectively. From this we conclude that the X-axis and Y-axis are as shown in the accompanying figure.

35. Since $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ we have $\mathbf{A} = \mathbf{PDP}^{-1}$. Hence

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}.$$

36. Since eigenvectors are mutually orthogonal we use an orthogonal matrix $\mathbf{P}$ and $\mathbf{A} = \mathbf{PDP}^T$.

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{8}{3} & \frac{4}{3} & -\frac{1}{3} \\ \frac{4}{3} & \frac{11}{3} & \frac{4}{3} \\ -\frac{1}{3} & \frac{4}{3} & \frac{8}{3} \end{bmatrix}$$

37. Since $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ we have

$$\mathbf{A} = \mathbf{PDP}^{-1}$$
$$\mathbf{A}^2 = \mathbf{PDP}^{-1}\mathbf{PDP}^{-1} = \mathbf{PDDP}^{-1} = \mathbf{PD}^2\mathbf{P}^{-1}$$
$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \mathbf{PD}^2\mathbf{P}^{-1}\mathbf{PDP}^{-1} = \mathbf{PD}^2\mathbf{DP}^{-1} = \mathbf{PD}^3\mathbf{P}^{-1}$$

and so on.

38. $$\begin{bmatrix} 2^4 & 0 & 0 & 0 \\ 0 & 3^4 & 0 & 0 \\ 0 & 0 & (-1)^4 & 0 \\ 0 & 0 & 0 & (5)^4 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 81 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 625 \end{bmatrix}$$

39. $\lambda_1 = 2,\ \lambda_2 = -1,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix},\ \mathbf{P}^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$

$$\mathbf{A}^5 = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 21 & 11 \\ 22 & 10 \end{bmatrix}$$

40. $\lambda_1 = 0,\ \lambda_2 = 1,\ \mathbf{K}_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix},\ \mathbf{P}^{-1} = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$

$$\mathbf{A}^{10} = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ 3 & -5 \end{bmatrix}$$

Exercises 8.13

1. **(a)** The message is $\mathbf{M} = \begin{bmatrix} 19 & 5 & 14 & 4 & 0 \\ 8 & 5 & 12 & 16 & 0 \end{bmatrix}$. The encoded message is

$$\mathbf{B} = \mathbf{AM} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 19 & 5 & 14 & 4 & 0 \\ 8 & 5 & 12 & 16 & 0 \end{bmatrix} = \begin{bmatrix} 35 & 15 & 38 & 36 & 0 \\ 27 & 10 & 26 & 20 & 0 \end{bmatrix}.$$

(b) The decoded message is

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 35 & 15 & 38 & 36 & 0 \\ 27 & 10 & 26 & 20 & 0 \end{bmatrix} = \begin{bmatrix} 19 & 5 & 14 & 4 & 0 \\ 8 & 5 & 12 & 16 & 0 \end{bmatrix}.$$

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2. **(a)** The message is $\mathbf{M} = \begin{bmatrix} 20 & 8 & 5 & 0 & 13 & 15 & 14 & 5 & 25 \\ 0 & 9 & 19 & 0 & 8 & 5 & 18 & 5 & 0 \end{bmatrix}$. The encoded message is

$$\mathbf{B} = \mathbf{AM} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 20 & 8 & 5 & 0 & 13 & 15 & 14 & 5 & 25 \\ 0 & 9 & 19 & 0 & 8 & 5 & 18 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 60 & 69 & 110 & 0 & 79 & 70 & 132 & 40 & 75 \\ 20 & 26 & 43 & 0 & 29 & 25 & 50 & 15 & 25 \end{bmatrix}.$$

(b) The decoded message is

$$\begin{aligned} \mathbf{M} = \mathbf{A}^{-1}\mathbf{B} &= \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 60 & 69 & 110 & 0 & 79 & 70 & 132 & 40 & 75 \\ 20 & 26 & 43 & 0 & 29 & 25 & 50 & 15 & 25 \end{bmatrix} \\ &= \begin{bmatrix} 20 & 8 & 5 & 0 & 13 & 15 & 14 & 5 & 25 \\ 0 & 9 & 19 & 0 & 8 & 5 & 18 & 5 & 0 \end{bmatrix}. \end{aligned}$$

3. **(a)** The message is $\mathbf{M} = \begin{bmatrix} 16 & 8 & 15 & 14 & 5 \\ 0 & 8 & 15 & 13 & 5 \end{bmatrix}$. The encoded message is

$$\mathbf{B} = \mathbf{AM} = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 16 & 8 & 15 & 14 & 5 \\ 0 & 8 & 15 & 13 & 5 \end{bmatrix} = \begin{bmatrix} 48 & 64 & 120 & 107 & 40 \\ 32 & 40 & 75 & 67 & 25 \end{bmatrix}.$$

(b) The decoded message is

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 48 & 64 & 120 & 107 & 40 \\ 32 & 40 & 75 & 67 & 25 \end{bmatrix} = \begin{bmatrix} 16 & 8 & 15 & 14 & 5 \\ 0 & 8 & 15 & 13 & 5 \end{bmatrix}.$$

4. **(a)** The message is $\mathbf{M} = \begin{bmatrix} 7 & 15 & 0 & 14 & 15 & 18 & 20 \\ 8 & 0 & 15 & 14 & 0 & 13 & 1 \\ 9 & 14 & 0 & 19 & 20 & 0 & 0 \end{bmatrix}$. The encoded message is

$$\mathbf{B} = \mathbf{AM} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 15 & 0 & 14 & 15 & 18 & 20 \\ 8 & 0 & 15 & 14 & 0 & 13 & 1 \\ 9 & 14 & 0 & 19 & 20 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 50 & 57 & 30 & 99 & 75 & 44 & 22 \\ 33 & 43 & 15 & 66 & 55 & 31 & 21 \\ 26 & 28 & 15 & 52 & 40 & 13 & 1 \end{bmatrix}.$$

(b) The decoded message is

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 50 & 57 & 30 & 99 & 75 & 44 & 22 \\ 33 & 43 & 15 & 66 & 55 & 31 & 21 \\ 26 & 28 & 15 & 52 & 40 & 13 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 15 & 0 & 14 & 15 & 18 & 20 \\ 8 & 0 & 15 & 14 & 0 & 13 & 1 \\ 9 & 14 & 0 & 19 & 20 & 0 & 0 \end{bmatrix}.$$

5. **(a)** The message is $\mathbf{M} = \begin{bmatrix} 7 & 15 & 0 & 14 & 15 & 18 & 20 \\ 8 & 0 & 15 & 14 & 0 & 13 & 1 \\ 9 & 14 & 0 & 19 & 20 & 0 & 0 \end{bmatrix}$. The encoded message is

$$\mathbf{B} = \mathbf{AM} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 7 & 15 & 0 & 14 & 15 & 18 & 20 \\ 8 & 0 & 15 & 14 & 0 & 13 & 1 \\ 9 & 14 & 0 & 19 & 20 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 31 & 44 & 15 & 61 & 50 & 49 & 41 \\ 24 & 29 & 15 & 47 & 35 & 31 & 21 \\ 1 & -15 & 15 & 0 & -15 & -5 & -19 \end{bmatrix}.$$

(b) The decoded message is

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 31 & 44 & 15 & 61 & 50 & 49 & 41 \\ 24 & 29 & 15 & 47 & 35 & 31 & 21 \\ 1 & -15 & 15 & 0 & -15 & -5 & -19 \end{bmatrix} = \begin{bmatrix} 7 & 15 & 0 & 14 & 15 & 18 & 20 \\ 8 & 0 & 15 & 14 & 0 & 13 & 1 \\ 9 & 14 & 0 & 19 & 20 & 0 & 0 \end{bmatrix}.$$

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6. **(a)** The message is $\mathbf{M} = \begin{bmatrix} 4 & 18 & 0 & 10 & 15 & 8 \\ 14 & 0 & 9 & 19 & 0 & 20 \\ 8 & 5 & 0 & 19 & 16 & 25 \end{bmatrix}$. The encoded message is

$$\mathbf{B} = \mathbf{AM} = \begin{bmatrix} 5 & 3 & 0 \\ 4 & 3 & -1 \\ 5 & 2 & 2 \end{bmatrix} \begin{bmatrix} 4 & 18 & 0 & 10 & 15 & 8 \\ 14 & 0 & 9 & 19 & 0 & 20 \\ 8 & 5 & 0 & 19 & 16 & 25 \end{bmatrix} = \begin{bmatrix} 62 & 90 & 27 & 107 & 75 & 100 \\ 50 & 67 & 27 & 78 & 44 & 67 \\ 64 & 100 & 18 & 126 & 107 & 130 \end{bmatrix}.$$

(b) The decoded message is

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 8 & -6 & -3 \\ -13 & 10 & 5 \\ -7 & 5 & 3 \end{bmatrix} \begin{bmatrix} 62 & 90 & 27 & 107 & 75 & 100 \\ 50 & 67 & 27 & 78 & 44 & 67 \\ 64 & 100 & 18 & 126 & 107 & 130 \end{bmatrix} = \begin{bmatrix} 4 & 18 & 0 & 10 & 15 & 8 \\ 14 & 0 & 9 & 19 & 0 & 20 \\ 8 & 5 & 0 & 19 & 16 & 25 \end{bmatrix}.$$

7. **(a)** The decoded message is

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix} \begin{bmatrix} 152 & 184 & 171 & 86 & 212 \\ 95 & 116 & 107 & 56 & 133 \end{bmatrix} = \begin{bmatrix} 19 & 20 & 21 & 4 & 25 \\ 0 & 8 & 1 & 18 & 4 \end{bmatrix}.$$

From correspondence (1) we obtain: STUDY_HARD.

8. The decoded message is

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 46 & -7 & -13 & 22 & -18 & 1 & 10 \\ 23 & -15 & -14 & 2 & -18 & -12 & 5 \end{bmatrix} = \begin{bmatrix} 23 & 8 & 1 & 20 & 0 & 13 & 5 \\ 0 & 23 & 15 & 18 & 18 & 25 & 0 \end{bmatrix}.$$

From correspondence (1) we obtain: WHAT_ME_WORRY_.

9. The decoded message is

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 31 & 21 & 21 & 22 & 20 & 9 \\ 19 & 0 & 9 & 13 & 16 & 15 \\ 13 & 1 & 20 & 8 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 13 & 1 & 20 & 8 & 0 & 9 \\ 19 & 0 & 9 & 13 & 16 & 15 \\ 18 & 20 & 1 & 14 & 20 & 0 \end{bmatrix}.$$

From correspondence (1) we obtain: MATH_IS_IMPORTANT.

10. The decoded message is

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 36 & 32 & 28 & 61 & 26 & 56 & 10 & 12 \\ -9 & -2 & -18 & -1 & -18 & -25 & 0 & 0 \\ 23 & 27 & 23 & 41 & 26 & 43 & 5 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 5 & 5 & 20 & 0 & 13 & 5 & 0 \\ 1 & 20 & 0 & 20 & 8 & 5 & 0 & 12 \\ 9 & 2 & 18 & 1 & 18 & 25 & 0 & 0 \end{bmatrix}.$$

From correspondence (1) we obtain: MEET_ME_AT_THE_LIBRARY__.

11. Let $\mathbf{A}^{-1} = \begin{bmatrix} u & v \\ x & y \end{bmatrix}$. Then

$$\mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} u & v \\ x & y \end{bmatrix} \begin{bmatrix} 17 & 16 & 18 & 5 & 34 & 0 & 34 & 20 & 9 & 5 & 25 \\ -30 & -31 & -32 & -10 & -59 & 0 & -54 & -35 & -13 & -6 & -50 \end{bmatrix},$$

so $17u - 30v = 4$, $16u - 31v = 1$ and $5x - 6y = 1$, $25x - 50y = 25$. Then $\mathbf{A}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ and

$$\mathbf{A}^{-1}\mathbf{B} \begin{bmatrix} 4 & 1 & 4 & 0 & 9 & 0 & 14 & 5 & 5 & 4 & 0 \\ 13 & 15 & 14 & 5 & 25 & 0 & 20 & 15 & 4 & 1 & 25 \end{bmatrix}.$$

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From correspondence (1) we obtain: DAD_I_NEED_MONEY_TODAY.

12. **(a)** $\mathbf{M}^T = \begin{bmatrix} 22 & 8 & 19 & 27 & 21 & 3 & 3 & 27 & 21 & 18 & 21 \\ 13 & 3 & 21 & 22 & 3 & 25 & 27 & 6 & 7 & 14 & 23 \\ 2 & 27 & 21 & 7 & 27 & 5 & 21 & 17 & 2 & 25 & 7 \end{bmatrix}$

(b) $\mathbf{B}^T = \mathbf{M} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 37 & 38 & 61 & 56 & 51 & 33 & 51 & 50 & 30 & 57 & 51 \\ 24 & 35 & 40 & 34 & 48 & 8 & 24 & 44 & 23 & 43 & 28 \\ 11 & -24 & 0 & 15 & -24 & 20 & 6 & -11 & 5 & -11 & 16 \end{bmatrix}$

(c) $\mathbf{BA}^{-1} = \mathbf{B} \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix} = \mathbf{M}$

13. **(a)** $\mathbf{B}' = \begin{bmatrix} 15 & 22 & 20 & 8 & 23 & 6 & 21 & 22 \\ 10 & 22 & 18 & 23 & 25 & 2 & 23 & 25 \\ 3 & 26 & 26 & 14 & 23 & 16 & 26 & 12 \end{bmatrix}$

(b) Using correspondence (1) the encoded message is: OVTHWFUVJVRWYBWYCZZNWPZL.

(c) $\mathbf{M}'\mathbf{A}^{-1}\mathbf{B}' = \begin{bmatrix} 1 & 4 & -3 \\ 2 & 3 & -2 \\ -2 & -4 & 3 \end{bmatrix} \mathbf{B}' = \begin{bmatrix} 46 & 32 & 14 & 58 & 54 & -34 & 35 & 86 \\ 54 & 58 & 42 & 57 & 75 & -14 & 59 & 95 \\ -61 & -54 & -34 & -66 & -77 & 28 & -56 & -108 \end{bmatrix}$

$\mathbf{M} = \mathbf{M}' \bmod 27 = \begin{bmatrix} 19 & 5 & 14 & 4 & 0 & 20 & 8 & 5 \\ 0 & 4 & 15 & 3 & 21 & 13 & 5 & 14 \\ 20 & 0 & 20 & 15 & 4 & 1 & 25 & 0 \end{bmatrix}$.

Using correspondence (1) the encoded message is: SEND_THE_DOCUMENT_TODAY.

Exercises 8.14

1. $[0 \quad 1 \quad 1 \quad 0]$

2. $[1 \quad 1 \quad 1 \quad 1]$

3. $[0 \quad 0 \quad 0 \quad 1 \quad 1]$

4. $[1 \quad 0 \quad 1 \quad 0 \quad 0]$

5. $[1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1]$

6. $[0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0]$

7. $[1 \quad 0 \quad 0]$

8. $[0 \quad 0 \quad 1]$

9. Parity error

10. $[1 \quad 0 \quad 1 \quad 0]$

11. $[1 \quad 0 \quad 0 \quad 1 \quad 1]$

12. Parity error

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In Problems 13-18 $\mathbf{D} = [\,c_1 \quad c_2 \quad c_3\,]$ and $\mathbf{P} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

13. $\mathbf{D}^T = \mathbf{P}\,[1 \quad 1 \quad 1 \quad 0]^T = [0 \quad 0 \quad 0]^T$; $\mathbf{C} = [0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0]$

14. $\mathbf{D}^T = \mathbf{P}\,[0 \quad 0 \quad 1 \quad 1]^T = [1 \quad 0 \quad 0]^T$; $\mathbf{C} = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1]$

15. $\mathbf{D}^T = \mathbf{P}\,[0 \quad 1 \quad 0 \quad 1]^T = [0 \quad 1 \quad 0]^T$; $\mathbf{C} = [0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1]$

16. $\mathbf{D}^T = \mathbf{P}\,[0 \quad 0 \quad 0 \quad 1]^T = [1 \quad 1 \quad 1]^T$; $\mathbf{C} = [1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1]$

17. $\mathbf{D}^T = \mathbf{P}\,[0 \quad 1 \quad 1 \quad 0]^T = [1 \quad 1 \quad 0]^T$; $\mathbf{C} = [1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0]$

18. $\mathbf{D}^T = \mathbf{P}\,[1 \quad 1 \quad 0 \quad 0]^T = [0 \quad 1 \quad 1]^T$; $\mathbf{C} = [0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0]$

In Problems 19-28 $\mathbf{W}$ represents the correctly decoded message.

19. $\mathbf{S} = \mathbf{H}\mathbf{R}^T = \mathbf{H}\,[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] = [0 \quad 0 \quad 0]^T$; a code word. $\mathbf{W} = [0 \quad 0 \quad 0 \quad 0]$

20. $\mathbf{S} = \mathbf{H}\mathbf{R}^T = \mathbf{H}\,[1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] = [0 \quad 1 \quad 1]^T$; not a code word. The error is in the third bit. $\mathbf{W} = [1 \quad 0 \quad 0 \quad 0]$

21. $\mathbf{S} = \mathbf{H}\mathbf{R}^T = \mathbf{H}\,[1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1] = [1 \quad 0 \quad 1]^T$; not a code word. The error is in the fifth bit. $\mathbf{W} = [0 \quad 0 \quad 0 \quad 1]$

22. $\mathbf{S} = \mathbf{H}\mathbf{R}^T = \mathbf{H}\,[0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0] = [0 \quad 0 \quad 0]^T$; a code word. $\mathbf{W} = [0 \quad 0 \quad 1 \quad 0]$

23. $\mathbf{S} = \mathbf{H}\mathbf{R}^T = \mathbf{H}\,[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1] = [0 \quad 0 \quad 0]^T$; a code word. $\mathbf{W} = [1 \quad 1 \quad 1 \quad 1]$

24. $\mathbf{S} = \mathbf{H}\mathbf{R}^T = \mathbf{H}\,[1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0] = [0 \quad 0 \quad 0]^T$; a code word. $\mathbf{W} = [0 \quad 1 \quad 1 \quad 0]$

25. $\mathbf{S} = \mathbf{H}\mathbf{R}^T = \mathbf{H}\,[0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1] = [0 \quad 1 \quad 0]^T$; not a code word. The error is in the second bit. $\mathbf{W} = [1 \quad 0 \quad 0 \quad 1]$

26. $\mathbf{S} = \mathbf{H}\mathbf{R}^T = \mathbf{H}\,[1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1] = [0 \quad 1 \quad 0]^T$; not a code word. The error is in the second bit. $\mathbf{W} = [0 \quad 0 \quad 0 \quad 1]$

27. $\mathbf{S} = \mathbf{H}\mathbf{R}^T = \mathbf{H}\,[1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1] = [1 \quad 1 \quad 1]^T$; not a code word. The error is in the seventh bit. $\mathbf{W} = [1 \quad 0 \quad 1 \quad 0]$

28. $\mathbf{S} = \mathbf{H}\mathbf{R}^T = \mathbf{H}\,[0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1] = [0 \quad 1 \quad 0]^T$; not a code word. The error is in the second bit. $\mathbf{W} = [1 \quad 0 \quad 1 \quad 1]$

29. **(a)** $2^7 = 128$

(b) $2^4 = 16$

(c) $[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$, $[1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1]$, $[0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0]$, $[1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1]$,
$[1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0]$, $[0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1]$, $[1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0]$, $[0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1]$,
$[1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]$, $[0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1]$, $[1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0]$, $[0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1]$,
$[0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0]$, $[1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1]$, $[0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0]$, $[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$

30. **(a)** $c_4 = 0$, $c_3 = 1$, $c_2 = 1$, $c_1 = 0$; $[0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0]$

(b) $\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ **(c)** $\mathbf{S} = \mathbf{H}\mathbf{R}^T = \mathbf{H}\,[0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0]^T = [0 \quad 0 \quad 0 \quad 0]^T$

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Exercises 8.15

1. We have $\mathbf{Y}^T = [\,1 \quad 2 \quad 3 \quad 2\,]$ and $\mathbf{A}^T = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

Now $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 54 & 14 \\ 14 & 4 \end{bmatrix}$ and $(\mathbf{A}^T\mathbf{A})^{-1} = \dfrac{1}{20}\begin{bmatrix} 4 & -14 \\ -14 & 54 \end{bmatrix}$

so $\mathbf{X} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y} = \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \end{bmatrix}$ and the least squares line is $y = 0.4x + 0.6$.

2. We have $\mathbf{Y}^T = [\,-1 \quad 3 \quad 5 \quad 7\,]$ and $\mathbf{A}^T = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

Now $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix}$ and $(\mathbf{A}^T\mathbf{A})^{-1} = \dfrac{1}{20}\begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix}$

so $\mathbf{X} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y} = \begin{bmatrix} \frac{13}{5} \\ -\frac{2}{5} \end{bmatrix}$ and the least squares line is $y = 2.6x - 0.4$.

3. We have $\mathbf{Y}^T = [\,1 \quad 1.5 \quad 3 \quad 4.5 \quad 5\,]$ and $\mathbf{A}^T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

Now $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix}$ and $(\mathbf{A}^T\mathbf{A})^{-1} = \dfrac{1}{50}\begin{bmatrix} 5 & -15 \\ -15 & 55 \end{bmatrix}$

so $\mathbf{X} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y} = \begin{bmatrix} 1.1 \\ -0.3 \end{bmatrix}$ and the least squares line is $y = 1.1x - 0.3$.

4. We have $\mathbf{Y}^T = [\,0 \quad 1.5 \quad 3 \quad 4.5 \quad 5\,]$ and $\mathbf{A}^T = \begin{bmatrix} 0 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

Now $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 54 & 14 \\ 14 & 5 \end{bmatrix}$ and $(\mathbf{A}^T\mathbf{A})^{-1} = \dfrac{1}{74}\begin{bmatrix} 5 & -14 \\ -14 & 54 \end{bmatrix}$

so $\mathbf{X} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y} = \begin{bmatrix} 1.06757 \\ -0.189189 \end{bmatrix}$ and the least squares line is $y = 1.06757x - 0.189189$.

5. We have $\mathbf{Y}^T = [\,2 \quad 3 \quad 5 \quad 5 \quad 9 \quad 8 \quad 10\,]$ and $\mathbf{A}^T = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

Now $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 91 & 21 \\ 21 & 7 \end{bmatrix}$ and $(\mathbf{A}^T\mathbf{A})^{-1} = \dfrac{1}{196}\begin{bmatrix} 7 & -21 \\ -21 & 91 \end{bmatrix}$

so $\mathbf{X} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y} = \begin{bmatrix} \frac{19}{14} \\ \frac{27}{14} \end{bmatrix}$ and the least squares line is $y = 1.35714x + 1.92857$.

6. We have $\mathbf{Y}^T = [\,2 \quad 2.5 \quad 1 \quad 1.5 \quad 2 \quad 3.2 \quad 5\,]$ and $\mathbf{A}^T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

Now $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 140 & 28 \\ 28 & 7 \end{bmatrix}$ and $(\mathbf{A}^T\mathbf{A})^{-1} = \dfrac{1}{196}\begin{bmatrix} 7 & -28 \\ -28 & 140 \end{bmatrix}$

so $\mathbf{X} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y} = \begin{bmatrix} 0.407143 \\ 0.828571 \end{bmatrix}$ and the least squares line is $y = 0.407143x + 0.828571$.

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7. We have $\mathbf{Y}^T = [\,220 \quad 200 \quad 180 \quad 170 \quad 150 \quad 135\,]$ and $\mathbf{A}^T = \begin{bmatrix} 20 & 40 & 60 & 80 & 100 & 120 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

Now $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 36400 & 420 \\ 420 & 6 \end{bmatrix}$ and $(\mathbf{A}^T\mathbf{A})^{-1} = \dfrac{1}{42000}\begin{bmatrix} 6 & -420 \\ -420 & 36400 \end{bmatrix}$

so $\mathbf{X} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y} = \begin{bmatrix} -\frac{117}{140} \\ \frac{703}{3} \end{bmatrix}$ and the least squares line is $v = -0.835714T + 234.333$. At $T = 140$, $v \approx 117.333$ and at $T = 160$, $v \approx 100.619$.

8. We have $\mathbf{Y}^T = [\,0.47 \quad 0.90 \quad 2.0 \quad 3.7 \quad 7.5 \quad 15\,]$ and $\mathbf{A}^T = \begin{bmatrix} 400 & 450 & 500 & 550 & 600 & 650 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

Now $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1697500 & 3150 \\ 3150 & 6 \end{bmatrix}$ and $(\mathbf{A}^T\mathbf{A})^{-1} = \dfrac{1}{262500}\begin{bmatrix} 6 & -3150 \\ -3150 & 1697500 \end{bmatrix}$

so $\mathbf{X} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y} = \begin{bmatrix} 0.0538 \\ -23.3167 \end{bmatrix}$ and the least squares line is $R = 0.0538T - 23.3167$. At $T = 700$, $R \approx 14.3433$.

Exercises 8.16

In Problems 1-5 we use the fact that the element τ_{ij} in the transfer matrix $\mathbf{T}$ is the rate of transfer from compartment j to compartment i, and the fact that the sum of each column in $\mathbf{T}$ is 1.

1. **(a)** The initial state and the transfer matrix are

$$\mathbf{X}_0 = \begin{bmatrix} 90 \\ 60 \end{bmatrix} \quad \text{and} \quad \mathbf{T} = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}.$$

(b) We have

$$\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0 = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}\begin{bmatrix} 90 \\ 60 \end{bmatrix} = \begin{bmatrix} 96 \\ 54 \end{bmatrix}$$

and

$$\mathbf{X}_2 = \mathbf{T}\mathbf{X}_1 = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}\begin{bmatrix} 96 \\ 54 \end{bmatrix} = \begin{bmatrix} 98.4 \\ 51.6 \end{bmatrix}.$$

(c) From $\mathbf{T}\hat{\mathbf{X}} - \hat{\mathbf{X}} = (\mathbf{T} - \mathbf{I})\hat{\mathbf{X}} = \mathbf{0}$ and the fact that the system is closed we obtain

$$\begin{aligned} -0.2x_1 + 0.4x_2 &= 0 \\ x_1 + \quad x_2 &= 150. \end{aligned}$$

The solution is $x_1 = 100$, $x_2 = 50$, so the equilibrium state is $\hat{\mathbf{X}} = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$.

2. **(a)** The initial state and the transfer matrix are

$$\mathbf{X}_0 = \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix} \quad \text{and} \quad \mathbf{T} = \begin{bmatrix} 0.7 & 0 & 0.5 \\ 0.3 & 0.8 & 0 \\ 0 & 0.2 & 0.5 \end{bmatrix}.$$

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(b) We have

$$\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0 = \begin{bmatrix} 145 \\ 190 \\ 115 \end{bmatrix} \quad \text{and} \quad \mathbf{X}_2 = \mathbf{T}\mathbf{X}_1 = \begin{bmatrix} 159 \\ 195.5 \\ 95.5 \end{bmatrix}.$$

(c) From $\mathbf{T}\hat{\mathbf{X}} - \hat{\mathbf{X}} = (\mathbf{T} - \mathbf{I})\hat{\mathbf{X}} = \mathbf{0}$ and the fact that the system is closed we obtain

$$\begin{aligned} -0.8x_1 + 0.5x_2 \qquad &= 0 \\ 0.3x_1 - 0.9x_2 \qquad &= 0 \\ x_1 + \quad x_2 + x_3 &= 450. \end{aligned}$$

The solution is $x_1 = 145.161$, $x_2 = 217.742$, $x_3 = 87.0968$, so the equilibrium state is $\hat{\mathbf{X}} = \begin{bmatrix} 145.161 \\ 217.742 \\ 87.097 \end{bmatrix}$.

3. (a) The initial state and the transfer matrix are

$$\mathbf{X}_0 = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{T} = \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0.3 & 0.1 & 0 \\ 0.5 & 0.4 & 1 \end{bmatrix}.$$

(b) We have

$$\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0 = \begin{bmatrix} 20 \\ 30 \\ 50 \end{bmatrix} \quad \text{and} \quad \mathbf{X}_2 = \mathbf{T}\mathbf{X}_1 = \begin{bmatrix} 19 \\ 9 \\ 72 \end{bmatrix}.$$

(c) From $\mathbf{T}\hat{\mathbf{X}} - \hat{\mathbf{X}} = (\mathbf{T} - \mathbf{I})\hat{\mathbf{X}} = \mathbf{0}$ and the fact that the system is closed we obtain

$$\begin{aligned} -0.8x_1 + 0.5x_2 \qquad &= 0 \\ 0.3x_1 - 0.9x_2 \qquad &= 0 \\ x_1 + \quad x_2 + x_3 &= 100. \end{aligned}$$

The solution is $x_1 = x_2 = 0$, $x_3 = 100$, so the equilibrium state is $\hat{\mathbf{X}} = \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}$.

4. (a) The transfer matrix is

$$\mathbf{T} = \begin{bmatrix} 0.7 & 0.05 & 0.15 \\ 0.3 & 0.75 & 0 \\ 0 & 0.2 & 0.85 \end{bmatrix}.$$

(b)

Year	Bare Space	Grasses	Small Shrubs
0	10.00	0.00	0.00
1	7.00	3.00	0.00
2	5.05	4.35	0.60
3	3.84	4.78	1.38
4	3.14	4.74	2.13
5	2.75	4.49	2.76
6	2.56	4.19	3.24

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5. **(a)** The initial state and the transfer matrix are

$$\mathbf{X}_0 = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{T} = \begin{bmatrix} 0.88 & 0.02 & 0 \\ 0.06 & 0.97 & 0.05 \\ 0.06 & 0.01 & 0.95 \end{bmatrix}.$$

(b)

Year	Phytoplankton	Water	Zooplankton
0	0.00	100.00	0.00
1	2.00	97.00	1.00
2	3.70	94.26	2.04
3	5.14	91.76	3.10
4	6.36	89.47	4.17
5	7.39	87.37	5.24
6	8.25	85.46	6.30
7	8.97	83.70	7.33
8	9.56	82.10	8.34
9	10.06	80.62	9.32
10	10.46	79.28	10.26
11	10.79	78.04	11.17
12	11.06	76.90	12.04

6. From $\mathbf{T}\hat{\mathbf{X}} = 1\hat{\mathbf{X}}$ we see that the equilibrium state vector $\hat{\mathbf{X}}$ is the eigenvector of the transfer matrix $\mathbf{T}$ corresponding to the eigenvalue 1. It has the properties that its components add up to the sum of the components of the initial state vector.

Chapter 8 Review Exercises

1. $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}$

2. 4×3

3. $\mathbf{AB} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$; $\mathbf{BA} = [11]$

4. $\mathbf{A}^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

5. False; consider $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

6. True

7. $\det\left(\frac{1}{2}\mathbf{A}\right) = \left(\frac{1}{2}\right)^3(5) = \frac{5}{8}$; $\det(-\mathbf{A}^T) = (-1)^3(5) = -5$

8. $\det \mathbf{AB}^{-1} = \det \mathbf{A}/\det \mathbf{B} = 6/2 = 3$

9. 0

10. $\det \mathbf{C} = (-1)^3/\det \mathbf{B} = -1/10^3(2) = -1/2000$

11. False; an eigenvalue can be 0.

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12. True

13. True

14. True, since complex roots of real polynomials occur in conjugate pairs.

15. False; if the characteristic equation of an $n \times n$ matrix has repeated roots, there may not be n linearly independent eigenvectors.

16. True

17. True

18. True

19. False; $\mathbf{A}$ is singular and thus not orthogonal.

20. True

21. $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$ where $\frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ is symmetric and $\frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$ is skew-symmetric.

22. Since $\det \mathbf{A}^2 = (\det \mathbf{A})^2 \geq 0$ and $\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$, there is no $\mathbf{A}$ such that $\mathbf{A}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

23. **(a)** $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ is nilpotent.

(b) Since $\det \mathbf{A}^n = (\det \mathbf{A})^n = 0$ we see that $\det \mathbf{A} = 0$ and $\mathbf{A}$ is singular.

24. **(a)** $\sigma_x\sigma_y = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = -\sigma_y\sigma_x; \quad \sigma_x\sigma_z = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\sigma_z\sigma_x; \quad \sigma_y\sigma_z = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = -\sigma_z\sigma_y$

(b) We first note that for anticommuting matrices $\mathbf{AB} = -\mathbf{BA}$, so $\mathbf{C} = 2\mathbf{AB}$. Then $\mathbf{C}_{xy} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}$, $\mathbf{C}_{yz} = \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix}$, and $\mathbf{C}_{zx} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

25. $\left[\begin{array}{ccc|c} 5 & -1 & 1 & -9 \\ 2 & 4 & 0 & 27 \\ 1 & 1 & 5 & 9 \end{array}\right] \xrightarrow{R_{13}} \left[\begin{array}{ccc|c} 1 & 1 & 5 & 9 \\ 2 & 4 & 0 & 27 \\ 5 & -1 & 1 & -9 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 1 & 5 & 9 \\ 0 & 1 & -5 & \frac{9}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right].$

The solution is $\mathbf{X} = [\, -\frac{1}{2} \quad 7 \quad \frac{1}{2} \,]^T$.

26. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -2 & 3 & 2 \\ 2 & 0 & -3 & 3 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -\frac{2}{3} & \frac{4}{3} \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right].$

The solution is $x_1 = 3$, $x_2 = 2$, $x_3 = 1$.

27. Multiplying the second row by abc we obtain the third row. Thus the determinant is 0.

28. Expanding along the first row we see that the result is an expression of the form $ay + bx^2 + cx + d = 0$, which is a parabola since, in this case $a \neq 0$ and $b \neq 0$. Letting $x = 1$ and $y = 2$ we note that the first and second rows are the same. Similarly, when $x = 2$ and $y = 3$, the first and third rows are the same; and when $x = 3$ and $y = 5$, the first and fourth rows are the same. In each case the determinant is 0 and the points lie on the parabola.

29. $4(-2)(3)(-1)(2)(5) = 240$

30. $(-3)(6)(9)(1) = -162$

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31. Since $\begin{vmatrix} 1 & -1 & 1 \\ 5 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 18 = 0$, the system has only the trivial solution.

32. Since $\begin{vmatrix} 1 & -1 & -1 \\ 5 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 0$, the system has infinitely many solutions.

33. From $x_1\mathrm{I}_2 + x_2\mathrm{HNO}_3 \rightarrow x_3\mathrm{HIO}_3 + x_4\mathrm{NO}_2 + x_5\mathrm{H}_2\mathrm{O}$ we obtain the system $2x_1 = x_3$, $x_2 = x_3 + 2x_5$, $x_2 = x_4$, $3x_2 = 3x_3 + 2x_4 + x_5$. Letting $x_4 = x_2$ in the fourth equation we obtain $x_2 = 3x_3 + x_5$. Taking $x_1 = t$ we see that $x_3 = 2t$, $x_2 = 2t + 2x_5$, and $x_2 = 6t + x_5$. From the latter two equations we get $x_5 = 4t$. Taking $t = 1$ we have $x_1 = 1$, $x_2 = 10$, $x_3 = 2$, $x_4 = 10$, and $x_5 = 4$. The balanced equation is $\mathrm{I}_2 + 10\mathrm{HNO}_3 \rightarrow 2\mathrm{HIO}_3 + 10\mathrm{NO}_2 + 4\mathrm{H}_2\mathrm{O}$.

34. From $x_1\mathrm{Ca} + x_2\mathrm{H}_3\mathrm{PO}_4 \rightarrow x_3\mathrm{Ca}_3\mathrm{P}_2\mathrm{O}_8 + x_4\mathrm{H}_2$ we obtain the system $x_1 = 3x_3$, $3x_2 = 2x_4$, $x_2 = 2x_3$, $4x_2 = 8x_3$. Letting $x_3 = t$ we see that $x_1 = 3t$, $x_2 = 2t$, and $x_4 = 3t$. Taking $t = 1$ we obtain the balanced equation $3\mathrm{Ca} + 2\mathrm{H}_3\mathrm{PO}_4 \rightarrow \mathrm{Ca}_3\mathrm{P}_2\mathrm{O}_8 + 3\mathrm{H}_2$.

35. $\det \mathbf{A} = -84$, $\det \mathbf{A}_1 = 42$, $\det \mathbf{A}_2 = -21$, $\det \mathbf{A}_3 = -56$; $x_1 = \dfrac{42}{-84} = -\dfrac{1}{2}$, $x_2 = \dfrac{-21}{-84} = \dfrac{1}{4}$, $x_3 = \dfrac{-56}{-84} = \dfrac{2}{3}$

36. $\det = 4$, $\det \mathbf{A}_1 = 16$, $\det \mathbf{A}_2 = -4$, $\det \mathbf{A}_3 = 0$; $x_1 = \dfrac{16}{4} = 4$, $x_2 = \dfrac{-4}{4} = -1$, $x_3 = \dfrac{0}{4} = 0$

37. $\det \mathbf{A} = \cos^2\theta + \sin^2\theta, \det \mathbf{A}_1 = X\cos\theta - Y\sin\theta, \det \mathbf{A}_2 = Y\cos\theta + X\sin\theta$;
$x_1 = X\cos\theta - Y\sin\theta$, $y = Y\cos\theta + X\sin\theta$

38. **(a)** $i_1 - i_2 - i_3 - i_4 = 0$, $i_2R_1 = E$, $i_2R_1 - i_3R_2 = 0$, $i_3R_2 - i_4R_3 = 0$

(b) $\det \mathbf{A} = \begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & R_1 & 0 & 0 \\ 0 & R_1 & -R_2 & 0 \\ 0 & 0 & R_2 & -R_3 \end{vmatrix} = R_1R_2R_3$;

$$\det \mathbf{A}_1 = \begin{vmatrix} 0 & -1 & -1 & -1 \\ E & R_1 & 0 & 0 \\ 0 & R_1 & -R_2 & 0 \\ 0 & 0 & R_2 & -R_3 \end{vmatrix} = -E[-R_2R_3 - R_1(R_3 + R_2)] = E(R_2R_3 + R_1R_3 + R_1R_2);$$

$$i_1 = \frac{\det \mathbf{A}_1}{\det \mathbf{A}} = \frac{E(R_2R_3 + R_1R_3 + R_1R_2)}{R_1R_2R_3} = E\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

39. $\mathbf{AX} = \mathbf{B}$ is $\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}$. Since $\mathbf{A}^{-1} = -\dfrac{1}{3}\begin{bmatrix} -2 & -3 & -2 \\ -1 & 0 & -1 \\ -4 & -6 & -7 \end{bmatrix}$, we have

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 7 \\ 5 \\ 23 \end{bmatrix}.$$

40. **(a)** $\mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{4} & -\frac{9}{4} \\ -1 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

(b) $\mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{4} & -\frac{9}{4} \\ -1 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \\ -2 \end{bmatrix}$

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41. From the characteristic equation $\lambda^2 - 4\lambda - 5 = 0$ we see that the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 5$. For $\lambda_1 = -1$ we have $2k_1 + 2k_2 = 0$, $4k_1 4k_2 = 0$ and $\mathbf{K}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. For $\lambda_2 = 5$ we have $-4k_1 + 2k_2 = 0$, $4k_1 - 2k_2 = 0$ and $\mathbf{K}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

42. From the characteristic equation $\lambda^2 = 0$ we see that the eigenvalues are $\lambda_1 = \lambda_2 = 0$. For $\lambda_1 = \lambda_2 = 0$ we have $4k_1 = 0$ and $\mathbf{K}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a single eigenvector.

43. From the characteristic equation $-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = -(\lambda + 1)^2(\lambda - 8) = 0$ we see that the eigenvalues are $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 8$. For $\lambda_1 = \lambda_2 = -1$ we have

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $\mathbf{K}_1 = [\,1 \quad -2 \quad 0\,]^T$ and $\mathbf{K}_2 = [\,1 \quad 0 \quad -1\,]^T$. For $\lambda_3 = 8$ we have

$$\left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $\mathbf{K}_3 = [\,2 \quad 1 \quad 2\,]^T$.

44. From the characteristic equation $-\lambda^3 + 18\lambda^2 - 99\lambda + 162 = -(\lambda - 9)(\lambda - 6)(\lambda - 3) = 0$ we see that the eigenvalues are $\lambda_1 = 9$, $\lambda_2 = 6$, and $\lambda_3 = 3$. For $\lambda_1 = 9$ we have

$$\left[\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ -2 & -3 & 2 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $\mathbf{K}_1 = [\,-2 \quad 2 \quad 1\,]^T$. For $\lambda_2 = 6$ we have

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $\mathbf{K}_2 = [\,2 \quad 1 \quad 2\,]^T$. For $\lambda_3 = 3$ we have

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & 0 \\ -2 & 3 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $\mathbf{K}_3 = [\,1 \quad 2 \quad -2\,]^T$.

45. From the characteristic equation $-\lambda^3 - \lambda^2 + 21\lambda + 45 = -(\lambda + 3)^2(\lambda - 5) = 0$ we see that the eigenvalues are $\lambda_1 = \lambda_2 = -3$ and $\lambda_3 = 5$. For $\lambda_1 = \lambda_2 = -3$ we have

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $\mathbf{K}_1 = [\,-2 \quad 1 \quad 0\,]^T$ and $\mathbf{K}_2 = [\,3 \quad 0 \quad 1\,]^T$. For $\lambda_3 = 5$ we have

$$\left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & -\frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

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Thus $\mathbf{K}_3 = [\,-1 \quad -2 \quad 1\,]^T$.

46. From the characteristic equation $-\lambda^3 + \lambda^2 + 2\lambda = -\lambda(\lambda+1)(\lambda-2) = 0$ we see that the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -1$, and $\lambda_3 = 2$. For $\lambda_1 = 0$ we have $k_3 = 0$, $2k_1 + 2k_2 + k_3 = 0$ and $\mathbf{K}_1 = [\,1 \quad -1 \quad 0\,]^T$. For $\lambda_2 = -1$ we have

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $\mathbf{K}_2 = [\,0 \quad 1 \quad -1\,]^T$. For $\lambda_3 = 2$ we have

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 2 & 2 & -1 & 0 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $\mathbf{K}_3 = [\,0 \quad 1 \quad 2\,]^T$.

47. Let $\mathbf{X}_1 = [\,a \quad b \quad c\,]^T$ be the first column of the matrix. Then $\mathbf{X}_1^T\,[\,-\frac{1}{\sqrt{2}} \quad 0 \quad \frac{1}{\sqrt{2}}\,]^T = \frac{1}{\sqrt{2}}(c-a) = 0$ and $\mathbf{X}_1^T\,[\,\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}}\,]^T = \frac{1}{\sqrt{3}}(a+b+c) = 0$. Also $\mathbf{X}_1^T\mathbf{X}_1 = a^2 + b^2 + c^2 = 1$. We see that $c = a$ and $b = -2a$ from the first two equations. Then $a^2 + 4a^2 + a^2 = 6a^2 = 1$ and $a = \frac{1}{\sqrt{6}}$. Thus $\mathbf{X}_1 = [\,\frac{1}{\sqrt{6}} \quad -\frac{2}{\sqrt{6}} \quad \frac{1}{\sqrt{6}}\,]^T$.

48. **(a)** Eigenvalues are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 5$ with corresponding eigenvectors $\mathbf{K}_1 = [\,0 \quad 1 \quad 0\,]^T$, $\mathbf{K}_2 = [\,2 \quad 0 \quad 1\,]^T$, and $\mathbf{K}_3 = [\,-1 \quad 0 \quad 2\,]^T$. Since $\|\mathbf{K}_1\| = 1$, $\|\mathbf{K}_2\| = \sqrt{5}$, and $\|\mathbf{K}_3\| = \sqrt{5}$, we have

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \quad \text{and} \quad \mathbf{P}^{-1} = \mathbf{P}^T = \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}.$$

(b) $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

49. We identify $\mathbf{A} = \begin{bmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$. Eigenvalues are $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = \frac{5}{2}$ so $\mathbf{D} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{5}{2} \end{bmatrix}$ and the equation becomes $[\,X \quad Y\,]\,\mathbf{D}\begin{bmatrix} X \\ Y \end{bmatrix} = -\frac{1}{2}X^2 + \frac{5}{2}Y^2 = 1$. The graph is a hyperbola.

50. We measure years in units of 10, with 0 corresponding to 1890. Then $\mathbf{Y} = [\,63 \quad 76 \quad 92 \quad 106 \quad 123\,]^T$ and $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$, so $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 30 & 10 \\ 10 & 5 \end{bmatrix}$. Thus

$$\mathbf{X} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y} = \frac{1}{50}\begin{bmatrix} 5 & -10 \\ -10 & 30 \end{bmatrix}\mathbf{A}^T\mathbf{Y} = \begin{bmatrix} 15 \\ 62 \end{bmatrix},$$

and the least squares line is $y = 15t + 62$. At $t = 5$ (corresponding to 1940) we have $y = 137$. The error in the predicted population is 5 million or 3.7%.

51. The encoded message is

$$\mathbf{B} = \mathbf{A}\mathbf{M} = \begin{bmatrix} 10 & 1 \\ 9 & 1 \end{bmatrix}\begin{bmatrix} 19 & 1 & 20 & 5 & 12 & 12 & 9 & 20 & 5 & 0 & 12 & 1 & 21 \\ 14 & 3 & 8 & 5 & 4 & 0 & 15 & 14 & 0 & 6 & 18 & 9 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 204 & 13 & 208 & 55 & 124 & 120 & 105 & 214 & 50 & 6 & 138 & 19 & 210 \\ 185 & 12 & 188 & 50 & 112 & 108 & 96 & 194 & 45 & 6 & 126 & 18 & 189 \end{bmatrix}.$$

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52. The encoded message is

$$\mathbf{B} = \mathbf{AM} = \begin{bmatrix} 10 & 1 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 19 & 5 & 3 & 0 & 1 & 7 & 14 & 20 & 0 & 1 & 18 \\ 18 & 22 & 19 & 0 & 20 & 21 & 5 & 19 & 0 & 1 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 208 & 72 & 49 & 0 & 30 & 91 & 145 & 219 & 0 & 11 & 193 \\ 189 & 67 & 46 & 0 & 29 & 84 & 131 & 199 & 0 & 10 & 175 \end{bmatrix}.$$

53. The decoded message is

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -3 & 2 & -1 \\ 1 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 19 & 0 & 15 & 14 & 0 & 20 \\ 35 & 10 & 27 & 53 & 1 & 54 \\ 5 & 15 & -3 & 48 & 2 & 39 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 12 & 16 & 0 & 9 \\ 19 & 0 & 15 & 14 & 0 & 20 \\ 8 & 5 & 0 & 23 & 1 & 25 \end{bmatrix}.$$

From correspondence (1) we obtain: HELP_IS_ON_THE_WAY.

54. The decoded message is

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -3 & 2 & -1 \\ 1 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & 21 \\ 27 & 17 & 40 \\ 21 & 13 & -2 \end{bmatrix} = \begin{bmatrix} 18 & 15 & 19 \\ 5 & 2 & 21 \\ 4 & 0 & 0 \end{bmatrix}.$$

From correspondence (1) we obtain: ROSEBUD__.

55. **(a)** The parity is even so the decoded message is $[1 \quad 1 \quad 0 \quad 0 \quad 1]$

(b) The parity is odd; there is a parity error.

56. From $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ we obtain the codeword $[0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1]$.

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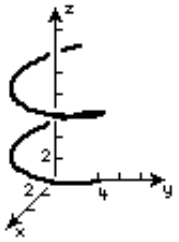
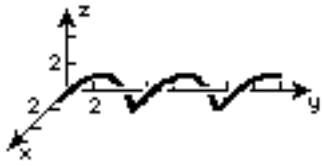
9 Vector Calculus

Exercises 9.1

1.



2.



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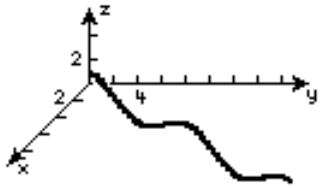

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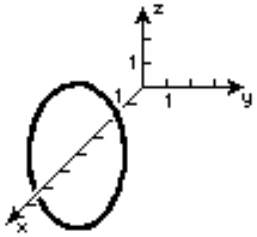
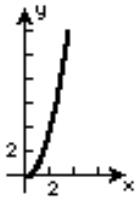
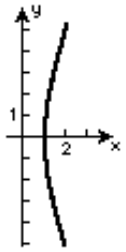


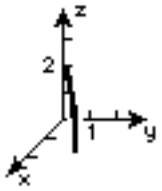
5.



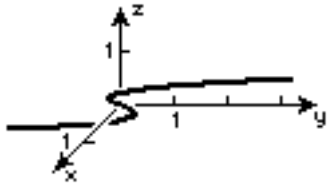
6.



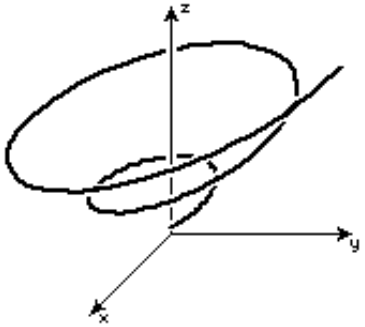
7.



8.

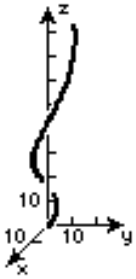


9.



Note: the scale is distorted in this graph. For $t = 0$, the graph starts at $(1, 0, 1)$. The upper loop shown intersects the xz-plane at about $(286751, 0, 286751)$.

10.



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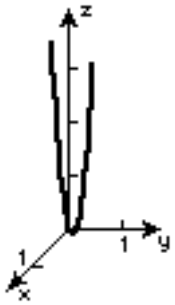
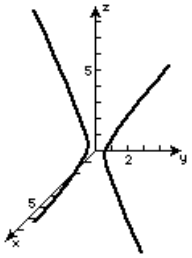
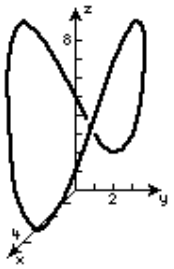
11. $x = t,\ y = t,\ z = t^2 + t^2 = 2t^2;\ \ \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}$



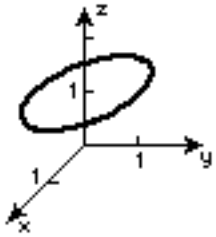
12. $x = t,\ y = 2t,\ z = \pm\sqrt{t^2 + 4t^2 + 1} = \pm\sqrt{5t^2 - 1}\,;\ \ \mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} \pm \sqrt{5t^2 - 1}\,\mathbf{k}$



13. $x = 3\cos t,\ z = 9 - 9\cos^2 t = 9\sin^2 t,\ y = 3\sin t;\ \ \mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 9\sin^2 t\mathbf{k}$



14. $x = \sin t,\ z = 1,\ y = \cos t;\ \ \mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$



15. $\mathbf{r}(t) = \dfrac{\sin 2t}{t}\,\mathbf{i} + (t-2)^5\mathbf{j} + \dfrac{\ln t}{1/t}\,\mathbf{k}$. Using L'Hôpital's Rule,

$$\lim_{t\to 0^+} \mathbf{r}(t) = \left[\frac{2\cos 2t}{1}\,\mathbf{i} + (t-2)^5\mathbf{j} + \frac{1/t}{-1/t^2}\,\mathbf{k}\right] = 2\mathbf{i} - 32\mathbf{j}.$$

16. **(a)** $\lim_{t\to\alpha}[-4\mathbf{r}_1(t) + 3\mathbf{r}_2(t)] = -4(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + 3(2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}) = 2\mathbf{i} + 23\mathbf{j} + 17\mathbf{k}$

(b) $\lim_{t\to\alpha} \mathbf{r}_1(t) \cdot \mathbf{r}_2(t) = (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}) = -1$

17. $\mathbf{r}'(t) = \dfrac{1}{t}\mathbf{i} - \dfrac{1}{t^2}\,\mathbf{j};\ \ \mathbf{r}''(t) = -\dfrac{1}{t^2}\mathbf{i} + \dfrac{2}{t^3}\,\mathbf{j}$

18. $\mathbf{r}'(t) = \langle -t\sin t, 1 - \sin t\rangle;\ \ \mathbf{r}''(t) = \langle -t\cos t - \sin t, -\cos t\rangle$

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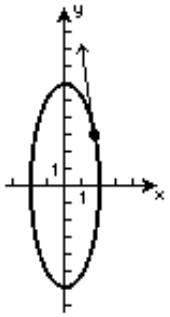
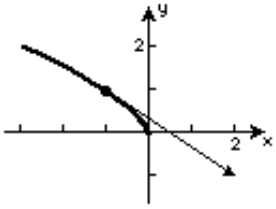
19. $\mathbf{r}'(t) = \langle 2te^{2t} + e^{2t}, 3t^2, 8t - 1\rangle$; $\mathbf{r}''(t) = \langle 4te^{2t} + 4e^{2t}, 6t, 8\rangle$

20. $\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j} + \dfrac{1}{1+t^2}\,\mathbf{k}$; $\mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j} - \dfrac{2t}{(1+t^2)^2}\,\mathbf{k}$

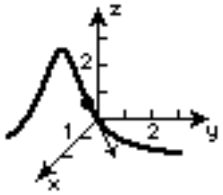
21. $\mathbf{r}'(t) = -2\sin t\mathbf{i} + 6\cos t\mathbf{j}$
$\mathbf{r}'(\pi/6) = -\mathbf{i} + 3\sqrt{3}\,\mathbf{j}$



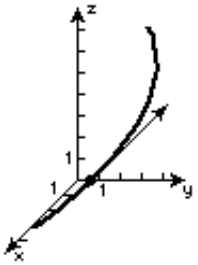
22. $\mathbf{r}'(t) = 3t^2\mathbf{i} + 2t\mathbf{j}$
$\mathbf{r}'(-1) = 3\mathbf{i} - 2\mathbf{j}$



23. $\mathbf{r}'(t) = \mathbf{j} - \dfrac{8t}{(1+t^2)^2}\,\mathbf{k}$
$\mathbf{r}'(1) = \mathbf{j} - 2\mathbf{k}$



24. $\mathbf{r}'(t) = -3\sin t\mathbf{i} + 3\cos t\mathbf{j} + 2\mathbf{k}$
$\mathbf{r}'(\pi/4) = \dfrac{-3\sqrt{2}}{2}\,\mathbf{i} + \dfrac{3\sqrt{2}}{2}\,\mathbf{j} + 2\mathbf{k}$



25. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{2}t^2\mathbf{j} + \dfrac{1}{3}t^3\mathbf{k}$; $\mathbf{r}(2) = 2\mathbf{i} + 2\mathbf{j} + \dfrac{8}{3}\mathbf{k}$; $\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$; $\mathbf{r}'(2) = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

Using the point $(2, 2, 8/3)$ and the direction vector $\mathbf{r}'(2)$, we have $x = 2 + t$, $y = 2 + 2t$, $z = 8/3 + 4t$.

26. $\mathbf{r}(t) = (t^3 - t)\mathbf{i} + \dfrac{6t}{t+1}\,\mathbf{j} + (2t+1)^2\mathbf{k}$; $\mathbf{r}(1) = 3\mathbf{j} + 9\mathbf{k}$; $\mathbf{r}'(t) = (3t^2 - 1)\mathbf{i} + \dfrac{6}{(t+1)^2}\mathbf{j} + (8t+4)\mathbf{k}$; $\mathbf{r}'(1) = 2\mathbf{i} + \dfrac{3}{2}\mathbf{j} + 12\mathbf{k}$.

Using the point $(0, 3, 9)$ and the direction vector $\mathbf{r}'(1)$, we have $x = 2t$, $y = 3 + \frac{3}{2}t$, $z = 9 + 12t$.

27. $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t) + \mathbf{r}'(t) \times \mathbf{r}'(t) = \mathbf{r}(t) \times \mathbf{r}''(t)$

28. $\dfrac{d}{dt}[\mathbf{r}(t) \cdot (t\mathbf{r}(t))] = \mathbf{r}(t) \cdot \dfrac{d}{dt}(t\mathbf{r}(t)) + \mathbf{r}'(t) \cdot (t\mathbf{r}(t)) = \mathbf{r}(t) \cdot (t\mathbf{r}'(t) + \mathbf{r}(t)) + \mathbf{r}'(t) \cdot (t\mathbf{r}(t))$
$= \mathbf{r}(t) \cdot (t\mathbf{r}'(t)) + \mathbf{r}(t) \cdot \mathbf{r}(t) + \mathbf{r}'(t) \cdot (t\mathbf{r}(t)) = 2t(\mathbf{r}(t) \cdot \mathbf{r}'(t)) + \mathbf{r}(t) \cdot \mathbf{r}(t)$

29. $\dfrac{d}{dt}[\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))] = \mathbf{r}(t) \cdot \dfrac{d}{dt}(\mathbf{r}'(t) \times \mathbf{r}''(t)) + \mathbf{r}'(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))$
$= \mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}'''(t) + \mathbf{r}''(t) \times \mathbf{r}''(t)) + \mathbf{r}'(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))$
$= \mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}'''(t))$

30. $\dfrac{d}{dt}[\mathbf{r}_1(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3(t))] = \mathbf{r}_1(t) \times \dfrac{d}{dt}(\mathbf{r}_2(t) \times \mathbf{r}_3(t)) + \mathbf{r}_1'(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3(t))$
$= \mathbf{r}_1(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3'(t) + \mathbf{r}_2'(t) \times \mathbf{r}_3(t)) + \mathbf{r}_1'(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3(t))$
$= \mathbf{r}_1(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3'(t)) + \mathbf{r}_1(t) \times (\mathbf{r}_2'(t) \times \mathbf{r}_3(t)) + \mathbf{r}_1(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3(t))$

31. $\dfrac{d}{dt}\Big[\mathbf{r}_1(2t) + \mathbf{r}_2\Big(\dfrac{1}{t}\Big)\Big] = 2\mathbf{r}_1'(2t) - \dfrac{1}{t^2}\mathbf{r}_2'\Big(\dfrac{1}{t}\Big)$

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32. $\dfrac{d}{dt}[t^3\mathbf{r}(t^2)] = t^3(2t)\mathbf{r}'(t^2) + 3t^2\mathbf{r}(t^2) = 2t^4\mathbf{r}'(t^2) + 3t^2\mathbf{r}(t^2)$

33. $\displaystyle\int_{-1}^{2} \mathbf{r}(t)\,dt = \left[\int_{-1}^{2} t\,dt\right]\mathbf{i} + \left[\int_{-1}^{2} 3t^2\,dt\right]\mathbf{j} + \left[\int_{-1}^{2} 4t^3\,dt\right]\mathbf{k} = \frac{1}{2}t^2\,\Big|_{-1}^{2}\,\mathbf{i} + t^3\,\Big|_{-1}^{2}\,\mathbf{j} + t^4\,\Big|_{-1}^{2}\,\mathbf{k} = \frac{3}{2}\mathbf{i} + 9\mathbf{j} + 15\mathbf{k}$

34. $\displaystyle\int_0^4 \mathbf{r}(t)\,dt = \left[\int_0^4 \sqrt{2t+1}\,dt\right]\mathbf{i} + \left[\int_0^4 -\sqrt{t}\;dt\right]\mathbf{j} + \left[\int_0^4 \sin \pi t\,dt\right]\mathbf{k}$

$$= \frac{1}{3}(2t+1)^{3/2}\,\Big|_0^4\,\mathbf{i} - \frac{2}{3}t^{3/2}\,\Big|_0^4\,\mathbf{j} - \frac{1}{\pi}\cos \pi t\,\Big|_0^4\,\mathbf{k} = \frac{26}{3}\mathbf{i} - \frac{16}{3}\mathbf{j}$$

35. $\displaystyle\int \mathbf{r}(t)\,dt = \left[\int te^t\,dt\right]\mathbf{i} + \left[\int -e^{-2t}\,dt\right]\mathbf{j} + \left[\int te^{t^2}\,dt\right]\mathbf{k}$

$$= [te^t - e^t + c_1]\mathbf{i} + \left[\frac{1}{2}e^{-2t} + c_2\right]\mathbf{j} + \left[\frac{1}{2}e^{t^2} + c_3\right]\mathbf{k} = e^t(t-1)\mathbf{i} + \frac{1}{2}e^{-2t}\mathbf{j} + \frac{1}{2}e^{t^2}\mathbf{k} + \mathbf{c},$$

where $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

36. $\displaystyle\int \mathbf{r}(t)\,dt = \left[\int \frac{1}{1+t^2}\,dt\right]\mathbf{i} + \left[\int \frac{t}{1+t^2}\,dt\right]\mathbf{j} + \left[\int \frac{t^2}{1+t^2}\,dt\right]\mathbf{k}$

$$= [\tan^{-1} t + c_1]\mathbf{i} + \left[\frac{1}{2}\ln(1+t^2) + c_2\right]\mathbf{j} + \left[\int \left(1 - \frac{1}{1+t^2}\right)dt\right]\mathbf{k}$$

$$= [\tan^{-1} t + c_1]\mathbf{i} + \left[\frac{1}{2}\ln(1+t^2) + c_2\right]\mathbf{j} + [t - \tan^{-1} t + c_3]\mathbf{k}$$

$$= \tan^{-1} t\,\mathbf{i} + \frac{1}{2}\ln(1+t^2)\mathbf{j} + (t - \tan^{-1} t)\mathbf{k} + \mathbf{c},$$

where $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

37. $\mathbf{r}(t) = \displaystyle\int \mathbf{r}'(t)\,dt = \left[\int 6\,dt\right]\mathbf{i} + \left[\int 6t\,dt\right]\mathbf{j} + \left[\int 3t^2\,dt\right]\mathbf{k} = [6t + c_1]\mathbf{i} + [3t^2 + c_2]\mathbf{j} + [t^3 + c_3]\mathbf{k}$

Since $\mathbf{r}(0) = \mathbf{i} - 2\mathbf{j} + \mathbf{k} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, $c_1 = 1$, $c_2 = -2$, and $c_3 = 1$. Thus,

$$\mathbf{r}(t) = (6t+1)\mathbf{i} + (3t^2 - 2)\mathbf{j} + (t^3 + 1)\mathbf{k}.$$

38. $\mathbf{r}(t) = \displaystyle\int \mathbf{r}'(t)\,dt = \left[\int t\sin t^2\,dt\right]\mathbf{i} + \left[\int -\cos 2t\,dt\right]\mathbf{j} = -[\frac{1}{2}\cos t^2 + c_1]\mathbf{i} + [-\frac{1}{2}\sin 2t + c_2]\mathbf{j}$

Since $\mathbf{r}(0) = \frac{3}{2}\mathbf{i} = (-\frac{1}{2} + c_1)\mathbf{i} + c_2\mathbf{j}$, $c_1 = 2$ and $c_2 = 0$. Thus,

$$\mathbf{r}(t) = \left(-\frac{1}{2}\cos t^2 + 2\right)\mathbf{i} - \frac{1}{2}\sin 2t\mathbf{j}.$$

39. $\mathbf{r}'(t) = \displaystyle\int \mathbf{r}''(t)\,dt = \left[\int 12t\,dt\right]\mathbf{i} + \left[\int -3t^{-1/2}\,dt\right]\mathbf{j} + \left[\int 2\,dt\right]\mathbf{k} = [6t^2 + c_1]\mathbf{i} + [-6t^{1/2} + c_2]\mathbf{j} + [2t + c_3]\mathbf{k}$

Since $\mathbf{r}'(1) = \mathbf{j} = (6 + c_1)\mathbf{i} + (-6 + c_2)\mathbf{j} + (2 + c_3)\mathbf{k}$, $c_1 = -6$, $c_2 = 7$, and $c_3 = -2$. Thus,

$$\mathbf{r}'(t) = (6t^2 - 6)\mathbf{i} + (-6t^{1/2} + 7)\mathbf{j} + (2t - 2)\mathbf{k}.$$

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \left[\int (6t^2 - 6)\,dt\right]\mathbf{i} + \left[\int (-6t^{1/2} + 7)\,dt\right]\mathbf{j} + \left[\int (2t - 2)\,dt\right]\mathbf{k}$$

$$= [2t^3 - 6t + c_4]\mathbf{i} + [-4t^{3/2} + 7t + c_5]\mathbf{j} + [t^2 - 2t + c_6]\mathbf{k}.$$

Since

$$\mathbf{r}(1) = 2\mathbf{i} - \mathbf{k} = (-4 + c_4)\mathbf{i} + (3 + c_5)\mathbf{j} + (-1 + c_6)\mathbf{k},$$

$c_4 = 6$, $c_5 = -3$, and $c_6 = 0$. Thus,

$$\mathbf{r}(t) = (2t^3 - 6t + 6)\mathbf{i} + (-4t^{3/2} + 7t - 3)\mathbf{j} + (t^2 - 2t)\mathbf{k}.$$

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40. $\mathbf{r}'(t) = \displaystyle\int \mathbf{r}''(t)\,dt = \left[\int \sec^2 t\,dt\right]\mathbf{i} + \left[\int \cos t\,dt\right]\mathbf{j} + \left[\int -\sin t\,dt\right]\mathbf{k}$

$= [\tan t + c_1]\mathbf{i} + [\sin t + c_2]\mathbf{j} + [\cos t + c_3]\mathbf{k}$

Since $\mathbf{r}'(0) = \mathbf{i} + \mathbf{j} + \mathbf{k} = c_1\mathbf{i} + c_2\mathbf{j} + (1 + c_3)\mathbf{k}$, $c_1 = 1$, $c_2 = 1$, and $c_3 = 0$. Thus,

$$\mathbf{r}'(t) = (\tan t + 1)\mathbf{i} + (\sin t + 1)\mathbf{j} + \cos t\mathbf{k}.$$

$\mathbf{r}(t) = \displaystyle\int \mathbf{r}'(t)\,dt = \left[\int (\tan t + 1)\,dt\right]\mathbf{i} + \left[\int (\sin t + 1)\,dt\right]\mathbf{j} + \left[\int \cos t\,dt\right]\mathbf{k}$

$= [\ln|\sec t| + t + c_4]\mathbf{i} + [-\cos t + t + c_5]\mathbf{j} + [\sin t + c_6]\mathbf{k}.$

Since $\mathbf{r}(0) = -\mathbf{j} + 5\mathbf{k} = c_4\mathbf{i} + (-1 + c_5)\mathbf{j} + c_6\mathbf{k}$, $c_4 = 0$, $c_5 = 0$, and $c_6 = 5$. Thus,

$$\mathbf{r}(t) = (\ln|\sec t| + t)\mathbf{i} + (-\cos t + t)\mathbf{j} + (\sin t + 5)\mathbf{k}.$$

41. $\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + c\mathbf{k}$; $\|\mathbf{r}'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} = \sqrt{a^2 + c^2}$

$s = \displaystyle\int_0^{2\pi} \sqrt{a^2 + c^2}\,dt = \sqrt{a^2 + c^2}\,t\,\Big|_0^{2\pi} = 2\pi\sqrt{a^2 + c^2}$

42. $\mathbf{r}'(t) = \mathbf{i} + (\cos t - t\sin t)\mathbf{j} + (\sin t + t\cos t)\mathbf{k}$

$\|\mathbf{r}'(t)\| = \sqrt{1^2 + (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} = \sqrt{2 + t^2}$

$s = \displaystyle\int_0^{\pi} \sqrt{2 + t^2}\,dt = \left(\frac{t}{2}\sqrt{2 + t^2} + \ln\left|t + \sqrt{2 + t^2}\,\right|\right)\Big|_0^{\pi} = \frac{\pi}{2}\sqrt{2 + \pi^2} + \ln(\pi + \sqrt{2 + \pi^2}\,) - \ln\sqrt{2}$

43. $\mathbf{r}'(t) = (-2e^t\sin 2t + e^t\cos 2t)\mathbf{i} + (2e^t\cos 2t + e^t\sin 2t)\mathbf{j} + e^t\mathbf{k}$

$\|\mathbf{r}'(t)\| = \sqrt{5e^{2t}\cos^2 2t + 5e^{2t}\sin^2 2t + e^{2t}} = \sqrt{6e^{2t}} = \sqrt{6}\,e^t$

$s = \displaystyle\int_0^{3\pi} \sqrt{6}\,e^t\,dt = \sqrt{6}\,e^t\Big|_0^{3\pi} = \sqrt{6}\,(e^{3\pi} - 1)$

44. $\mathbf{r}'(t) = 3\mathbf{i} + 2\sqrt{3}\,t\mathbf{j} + 2t^2\mathbf{k}$; $\|\mathbf{r}'(t)\| = \sqrt{3^2 + (2\sqrt{3}\,t)^2 + (2t^2)^2} = \sqrt{9 + 12t^2 + 4t^4} = 3 + 2t^2$

$s = \displaystyle\int_0^1 (3 + 2t^2)\,dt = (3t + \frac{2}{3}t^3)\,\Big|_0^1 = 3 + \frac{2}{3} = \frac{11}{3}$

45. $\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j}$; $\|\mathbf{r}'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$, $a > 0$; $s = \displaystyle\int_0^t a\,du = at$

$\mathbf{r}(s) = a\cos(s/a)\mathbf{i} + a\sin(s/a)\mathbf{j}$; $\mathbf{r}'(s) = -\sin(s/a)\mathbf{i} + \cos(s/a)\mathbf{j}$

$\|\mathbf{r}'(s)\| = \sqrt{\sin^2(s/a) + \cos^2(s/a)} = 1$

46. $\mathbf{r}'(s) = -\dfrac{2}{\sqrt{5}}\sin(s/\sqrt{5}\,)\mathbf{i} + \dfrac{2}{\sqrt{5}}\cos(s/\sqrt{5}\,)\mathbf{j} + \dfrac{1}{\sqrt{5}}\mathbf{k}$

$\|\mathbf{r}'(s)\| = \sqrt{\dfrac{4}{5}\sin^2(s/\sqrt{5}\,) + \dfrac{4}{5}\cos^2(s/\sqrt{5}\,) + \dfrac{1}{5}} = \sqrt{\dfrac{4}{5} + \dfrac{1}{5}} = 1$

47. Since $\dfrac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \dfrac{d}{dt}\|\mathbf{r}\|^2 = \dfrac{d}{dt}\,c^2 = 0$ and $\dfrac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \mathbf{r}\cdot\mathbf{r}' + \mathbf{r}'\cdot\mathbf{r} = 2\mathbf{r}\cdot\mathbf{r}'$, we have $\mathbf{r}\cdot\mathbf{r}' = 0$. Thus, $\mathbf{r}'$ is perpendicular to $\mathbf{r}$.

48. Since $\|\mathbf{r}(t)\|$ is the length of $\mathbf{r}(t)$, $\|\mathbf{r}(t)\| = c$ represents a curve lying on a sphere of radius c centered at the origin.

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49. Let $\mathbf{r}_1(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then

$$\frac{d}{dt}[u(t)\mathbf{r}_1(t)] = \frac{d}{dt}[u(t)x(t)\mathbf{i} + u(t)y(t)\mathbf{j}] = [u(t)x'(t) + u'(t)x(t)]\mathbf{i} + [u(t)y'(t) + u'(t)y(t)]\mathbf{j}$$
$$= u(t)[x'(t)\mathbf{i} + y'(t)\mathbf{j}] + u'(t)[x(t)\mathbf{i} + y(t)\mathbf{j}] = u(t)\mathbf{r}_1'(t) + u'(t)\mathbf{r}_1(t).$$

50. Let $\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j}$ and $\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j}$. Then

$$\frac{d}{dt}[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)] = \frac{d}{dt}[x_1(t)x_2(t) + y_1(t)y_2(t)] = x_1(t)x_2'(t) + x_1'(t)x_2(t) + y_1(t)y_2'(t) + y_1'(t)y_2(t)$$
$$= [x_1(t)x_2'(t) + y_1(t)y_2'(t)] + [x_1'(t)x_2(t) + y_1'(t)y_2(t)] = \mathbf{r}_1(t)\cdot\mathbf{r}_2'(t) + \mathbf{r}_1'(t)\cdot\mathbf{r}_2(t).$$

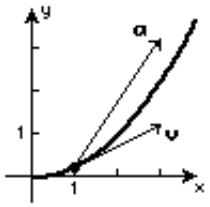
51.
$$\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)] = \lim_{h\to 0}\frac{\mathbf{r}_1(t+h)\times\mathbf{r}_2(t+h) - \mathbf{r}_1(t)\times\mathbf{r}_2(t)}{h}$$
$$= \lim_{h\to 0}\frac{\mathbf{r}_1(t+h)\times\mathbf{r}_2(t+h) - \mathbf{r}_1(t+h)\times\mathbf{r}_2(t) + \mathbf{r}_1(t+h)\times\mathbf{r}_2(t) - \mathbf{r}_1(t)\times\mathbf{r}_2(t)}{h}$$
$$= \lim_{h\to 0}\frac{\mathbf{r}_1(t+h)\times[\mathbf{r}_2(t+h) - \mathbf{r}_2(t)]}{h} + \lim_{h\to 0}\frac{[\mathbf{r}_1(t+h) - \mathbf{r}_1(t)]\times\mathbf{r}_2(t)}{h}$$
$$= \mathbf{r}_1(t)\times\left(\lim_{h\to 0}\frac{\mathbf{r}_2(t+h) - \mathbf{r}_2(t)}{h}\right) + \left(\lim_{h\to 0}\frac{\mathbf{r}_1(t+h) - \mathbf{r}_1(t)}{h}\right)\times\mathbf{r}_2(t)$$
$$= \mathbf{r}_1(t)\times\mathbf{r}_2'(t) + \mathbf{r}_1'(t)\times\mathbf{r}_2(t)$$

52. Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then

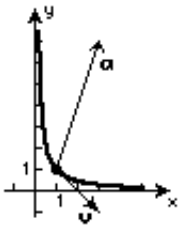
$$\int_a^b \mathbf{v}\cdot\mathbf{r}(t)\,dt = \int_a^b [ax(t) + by(t)]\,dt = a\int_a^b x(t)\,dt + b\int_a^b y(t)\,dt = \mathbf{v}\cdot\int_a^b \mathbf{r}(t)\,dt.$$

Exercises 9.2

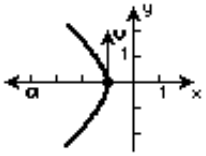
1. $\mathbf{v}(t) = 2t\mathbf{i} + t^3\mathbf{j}$; $\mathbf{v}(1) = 2\mathbf{i} + \mathbf{j}$; $\|\mathbf{v}(1)\| = \sqrt{4+1} = \sqrt{5}$;
$\mathbf{a}(t) = 2\mathbf{i} + 3t^2\mathbf{j}$; $\mathbf{a}(1) = 2\mathbf{i} + 3\mathbf{j}$



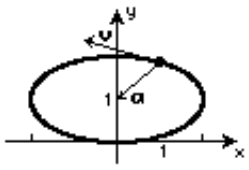
2. $\mathbf{v}(t) = 2t\mathbf{i} - \dfrac{2}{t^3}\mathbf{j}$; $\mathbf{v}(1) = 2\mathbf{i} - 2\mathbf{j}$; $\|\mathbf{v}(1)\| = \sqrt{4+4} = 2\sqrt{2}$;
$\mathbf{a}(t) = 2\mathbf{i} + \dfrac{6}{t^4}\mathbf{j}$; $\mathbf{a}(1) = 2\mathbf{i} + 6\mathbf{j}$



3. $\mathbf{v}(t) = -2\sinh 2t\mathbf{i} + 2\cosh 2t\mathbf{j}$; $\mathbf{v}(0) = 2\mathbf{j}$; $\|\mathbf{v}(0)\| = 2$;
$\mathbf{a}(t) = -4\cosh 2t\mathbf{i} + 4\sinh 2t\mathbf{j}$; $\mathbf{a}(0) = -4\mathbf{i}$



4. $\mathbf{v}(t) = -2\sin t\mathbf{i} + \cos t\mathbf{j}$; $\mathbf{v}(\pi/3) = -\sqrt{3}\,\mathbf{i} + \dfrac{1}{2}\mathbf{j}$; $\|\mathbf{v}(\pi/3)\| = \sqrt{3+1/4} = \sqrt{13}/2$;
$\mathbf{a}(t) = -2\cos t\mathbf{i} - \sin t\mathbf{j}$; $\mathbf{a}(\pi/3) = -\mathbf{i} - \dfrac{\sqrt{3}}{2}\,\mathbf{j}$



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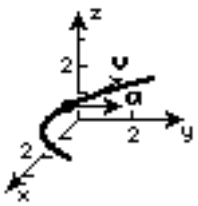
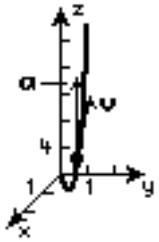
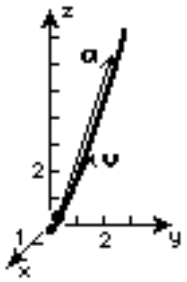
5. $\mathbf{v}(t) = (2t-2)\mathbf{j} + \mathbf{k}$; $\mathbf{v}(2) = 2\mathbf{j} + \mathbf{k}$ $\|\mathbf{v}(2)\| = \sqrt{4+1} = \sqrt{5}$;
$\mathbf{a}(t) = 2\mathbf{j}$; $\mathbf{a}(2) = 2\mathbf{j}$



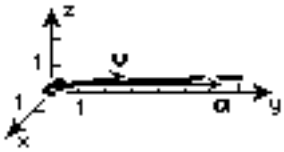
6. $\mathbf{v}(t) = \mathbf{i}+\mathbf{j}+3t^2\mathbf{k}$; $\mathbf{v}(2) = \mathbf{i}+\mathbf{j}+12\mathbf{k}$; $\|\mathbf{v}(2)\| = \sqrt{1+1+144} = \sqrt{146}$; $\mathbf{a}(t) = 6t\mathbf{k}$;
$\mathbf{a}(2) = 12\mathbf{k}$



7. $\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$; $\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$; $\|\mathbf{v}(1)\| = \sqrt{1+4+9} = \sqrt{14}$;
$\mathbf{a}(t) = 2\mathbf{j} + 6t\mathbf{k}$; $\mathbf{a}(1) = 2\mathbf{j} + 6\mathbf{k}$



8. $\mathbf{v}(t) = \mathbf{i} + 3t^2\mathbf{j} + \mathbf{k}$; $\mathbf{v}(1) = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$; $\|\mathbf{v}(1)\| = \sqrt{1+9+1} = \sqrt{11}$;
$\mathbf{a}(t) = 6t\mathbf{j}$; $\mathbf{a}(1) = 6\mathbf{j}$



9. The particle passes through the xy-plane when $z(t) = t^2 - 5t = 0$ or $t = 0,\ 5$ which gives us the points $(0, 0, 0)$ and $(25, 115, 0)$. $\mathbf{v}(t) = 2t\mathbf{i}+(3t^2-2)\mathbf{j}+(2t-5)\mathbf{k}$; $\mathbf{v}(0) = -2\mathbf{j}-5\mathbf{k}$, $\mathbf{v}(5) = 10\mathbf{i}+73\mathbf{j}+5\mathbf{k}$; $\mathbf{a}(t) = 2\mathbf{i}+6t\mathbf{j}+2\mathbf{k}$; $\mathbf{a}(0) = 2\mathbf{i} + 2\mathbf{k}$, $\mathbf{a}(5) = 2\mathbf{i} + 30\mathbf{j} + 2\mathbf{k}$

10. If $\mathbf{a}(t) = \mathbf{0}$, then $\mathbf{v}(t) = \mathbf{c}_1$ and $\mathbf{r}(t) = \mathbf{c}_1 t + \mathbf{c}_2$. The graph of this equation is a straight line.

11. Initially we are given $\mathbf{s}_0 = \mathbf{0}$ and $\mathbf{v}_0 = (480\cos 30°)\mathbf{i} + (480\sin 30°)\mathbf{j} = 240\sqrt{3}\,\mathbf{i} + 240\mathbf{j}$. Using $\mathbf{a}(t) = -32\mathbf{j}$ we find

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = -32t\mathbf{j} + \mathbf{c}$$

$$240\sqrt{3}\,\mathbf{i} + 240\mathbf{j} = \mathbf{v}(0) = \mathbf{c}$$

$$\mathbf{v}(t) = -32t\mathbf{j} + 240\sqrt{3}\,\mathbf{i} + 240\mathbf{j} = 240\sqrt{3}\,\mathbf{i} + (240 - 32t)\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = 240\sqrt{3}\,t\mathbf{i} + (240t - 16t^2)\mathbf{j} + \mathbf{b}$$

$$\mathbf{0} = \mathbf{r}(0) = \mathbf{b}.$$

(a) The shell's trajectory is given by $\mathbf{r}(t) = 240\sqrt{3}\,t\mathbf{i} + (240t - 16t^2)\mathbf{j}$ or $x = 240\sqrt{3}\,t$, $y = 240t - 16t^2$.

(b) Solving $dy/dt = 240 - 32t = 0$, we see that y is maximum when $t = 15/2$. The maximum altitude is $y(15/2) = 900$ ft.

(c) Solving $y(t) = 240t - 16t^2 = 16t(15 - t) = 0$, we see that the shell is at ground level when $t = 0$ and $t = 15$. The range of the shell is $x(15) = 3600\sqrt{3} \approx 6235$ ft.

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(d) From **(c)**, impact is when $t = 15$. The speed at impact is

$$\|\mathbf{v}(15)\| = |240\sqrt{3}\,\mathbf{i} + (240 - 32 \cdot 15)\mathbf{j}| = \sqrt{240^2 \cdot 3 + (-240)^2} = 480 \text{ ft/s}.$$

12. Initially we are given $\mathbf{s}_0 = 1600\mathbf{j}$ and $\mathbf{v}_0 = (480\cos 30°)\mathbf{i} + (480\sin 30°)\mathbf{j} = 240\sqrt{3}\,\mathbf{i} + 240\mathbf{j}$. Using $\mathbf{a}(t) = -32\mathbf{j}$ we find

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = -32t\mathbf{j} + \mathbf{c}$$

$$240\sqrt{3}\,\mathbf{i} + 240\mathbf{j} = \mathbf{v}(0) = \mathbf{c}$$

$$\mathbf{v}(t) = -32t\mathbf{j} + 240\sqrt{3}\,\mathbf{i} + 240\mathbf{j} = 240\sqrt{3}\,\mathbf{i} + (240 - 32t)\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = 240\sqrt{3}\,t\mathbf{i} + (240t - 16t^2)\mathbf{j} + \mathbf{b}$$

$$1600\mathbf{j} = \mathbf{r}(0) = \mathbf{b}.$$

(a) The shell's trajectory is given by $\mathbf{r}(t) = 240\sqrt{3}\,t\mathbf{i} + (240t - 16t^2 + 1600)\mathbf{j}$ or $x = 240\sqrt{3}\,t$, $y = 240t - 16t^2 + 1600$.

(b) Solving $dy/dt = 240 - 32t = 0$, we see that y is maximum when $t = 15/2$. The maximum altitude is $y(15/2) = 2500$ ft.

(c) Solving $y(t) = -16t^2 + 240t + 1600 = -16(t - 20)(t + 5) = 0$, we see that the shell hits the ground when $t = 20$. The range of the shell is $x(20) = 4800\sqrt{3} \approx 8314$ ft.

(d) From **(c)**, impact is when $t = 20$. The speed at impact is

$$|\mathbf{v}(20)\| = |240\sqrt{3}\,\mathbf{i} + (240 - 32 \cdot 20)\mathbf{j}| = \sqrt{240^2 \cdot 3 + (-400)^2} = 160\sqrt{13} \approx 577 \text{ ft/s}.$$

13. We are given $\mathbf{s}_0 = 81\mathbf{j}$ and $\mathbf{v}_0 = 4\mathbf{i}$. Using $\mathbf{a}(t) = -32\mathbf{j}$, we have

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = -32t\mathbf{j} + \mathbf{c}$$

$$4\mathbf{i} = \mathbf{v}(0) = \mathbf{c}$$

$$\mathbf{v}(t) = 4\mathbf{i} - 32t\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = 4t\mathbf{i} - 16t^2\mathbf{j} + \mathbf{b}$$

$$81\mathbf{j} = \mathbf{r}(0) = \mathbf{b}$$

$$\mathbf{r}(t) = 4t\mathbf{i} + (81 - 16t^2)\mathbf{j}.$$

Solving $y(t) = 81 - 16t^2 = 0$, we see that the car hits the water when $t = 9/4$. Then

$$\|\mathbf{v}(9/4)\| = |4\mathbf{i} - 32(9/4)\mathbf{j}| = \sqrt{4^2 + 72^2} = 20\sqrt{13} \approx 72.11 \text{ ft/s}.$$

14. Let θ be the angle of elevation. Then $\mathbf{v}(0) = 98\cos\theta\mathbf{i} + 98\sin\theta\mathbf{j}$. Using $\mathbf{a}(t) = -9.8\mathbf{j}$, we have

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = -9.8t\mathbf{j} + \mathbf{c}$$

$$98\cos\theta\mathbf{i} + 98\sin\theta\mathbf{j} = \mathbf{v}(0) = \mathbf{c}$$

$$\mathbf{v}(t) = 98\cos\theta\mathbf{i} + (98\sin\theta - 9.8t)\mathbf{j}$$

$$\mathbf{r}(t) = 98t\cos\theta\mathbf{i} + (98t\sin\theta - 4.9t^2)\mathbf{j} + \mathbf{b}.$$

Since $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ and $\mathbf{r}(t) = 98t\cos\theta\mathbf{i} + (98t\sin\theta - 4.9t^2)\mathbf{j}$. Setting $y(t) = 98t\sin\theta - 4.9t^2 = t(98\sin\theta - 4.9t) = 0$, we see that the projectile hits the ground when $t = 20\sin\theta$. Thus, using $x(t) = 98t\cos\theta$, $490 = x(t) = 98(20\sin\theta)\cos\theta$ or $\sin 2\theta = 0.5$. Then $2\theta = 30°$ or $150°$. The angles of elevation are $15°$ and $75°$.

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15. Let s be the initial speed. Then $\mathbf{v}(0) = s\cos 45^\circ \mathbf{i} + s\sin 45^\circ \mathbf{j} = \dfrac{s\sqrt{2}}{2}\,\mathbf{i} + \dfrac{s\sqrt{2}}{2}\,\mathbf{j}$. Using $\mathbf{a}(t) = -32\mathbf{j}$, we have

$$\begin{aligned}
\mathbf{v}(t) &= \int \mathbf{a}(t)\,dt = -32t\mathbf{j} + \mathbf{c} \\
\frac{s\sqrt{2}}{2}\,\mathbf{i} + \frac{s\sqrt{2}}{2}\,\mathbf{j} &= \mathbf{v}(0) = \mathbf{c} \\
\mathbf{v}(t) &= \frac{s\sqrt{2}}{2}\,\mathbf{i} + \left(\frac{s\sqrt{2}}{2} - 32t\right)\mathbf{j} \\
\mathbf{r}(t) &= \frac{s\sqrt{2}}{2}\,t\mathbf{i} + \left(\frac{s\sqrt{2}}{2}\,t - 16t^2\right)\mathbf{j} + \mathbf{b}.
\end{aligned}$$

Since $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ and

$$\mathbf{r}(t) = \frac{s\sqrt{2}}{2}\,t\mathbf{i} + \left(\frac{s\sqrt{2}}{2}\,t - 16t^2\right)\mathbf{j}.$$

Setting $y(t) = s\sqrt{2}\,t/2 - 16t^2 = t(s\sqrt{2}/2 - 16t) = 0$ we see that the ball hits the ground when $t = \sqrt{2}\,s/32$. Thus, using $x(t) = s\sqrt{2}\,t/2$ and the fact that 100 yd = 300 ft, $300 = x(t) = \dfrac{s\sqrt{2}}{2}(\sqrt{2}\,s/32) = \dfrac{s^2}{32}$ and $s = \sqrt{9600} \approx 97.98$ ft/s.

16. Let s be the initial speed and θ the initial angle. Then $\mathbf{v}(0) = s\cos\theta\mathbf{i} + s\sin\theta\mathbf{j}$. Using $\mathbf{a}(t) = -32\mathbf{j}$, we have

$$\begin{aligned}
\mathbf{v}(t) &= \int \mathbf{a}(t)\,dt = -32t\mathbf{j} + \mathbf{c} \\
s\cos\theta\mathbf{i} + s\sin\theta\mathbf{j} &= \mathbf{v}(0) = \mathbf{c} \\
\mathbf{v}(t) &= s\cos\theta\mathbf{i} + (s\sin\theta - 32t)\mathbf{j} \\
\mathbf{r}(t) &= st\cos\theta\mathbf{i} + (st\sin\theta - 16t^2)\mathbf{j} + \mathbf{b}.
\end{aligned}$$

Since $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ and $\mathbf{r}(t) = st\cos\theta\mathbf{i} + (st\sin\theta - 16t^2)\mathbf{j}$. Setting $y(t) = st\sin\theta - 16t^2 = t(s\sin\theta - 16t) = 0$, we see that the ball hits the ground when $t = (s\sin\theta)/16$. Using $x(t) = st\cos\theta\mathbf{i}$, we see that the range of the ball is

$$x\left(\frac{s\sin\theta}{16}\right) = \frac{s^2\sin\theta\cos\theta}{16} = \frac{s^2\sin 2\theta}{32}.$$

For $\theta = 30^\circ$, the range is $s^2\sin 60^\circ/32 = \sqrt{3}\,s^2/64$ and for $\theta = 60^\circ$ the range is $s^2\sin 120^\circ/32 = \sqrt{3}\,s^2/64$. In general, when the angle is $90^\circ - \theta$ the range is

$$[s^2\sin 2(90^\circ - \theta)]/32 = s^2[\sin(180^\circ - 2\theta)]/32 = s^2(\sin 2\theta)/32.$$

Thus, for angles θ and $90^\circ - \theta$, the range is the same.

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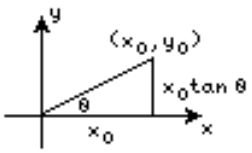
17. Let the initial speed of the projectile be s and let the target be at (x_0, y_0). Then $\mathbf{v}_p(0) = s\cos\theta\mathbf{i} + s\sin\theta\mathbf{j}$ and $\mathbf{v}_t(0) = \mathbf{0}$. Using $\mathbf{a}(t) = -32\mathbf{j}$, we have



$$\mathbf{v}_p(t) = \int \mathbf{a}(t)\,dt = -32t\mathbf{j} + \mathbf{c}$$
$$s\cos\theta\mathbf{i} + s\sin\theta\mathbf{j} = \mathbf{v}_p(0) = \mathbf{c}$$
$$\mathbf{v}_p(t) = s\cos\theta\mathbf{i} + (s\sin\theta - 32t)\mathbf{j}$$
$$\mathbf{r}_p(t) = st\cos\theta\mathbf{i} + (st\sin\theta - 16t^2)\mathbf{j} + \mathbf{b}.$$

Since $\mathbf{r}_p(0) = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ and $\mathbf{r}_p(t) = st\cos\theta\mathbf{i} + (st\sin\theta - 16t^2)\mathbf{j}$. Also, $\mathbf{v}_t(t) = -32t\mathbf{j} + \mathbf{c}$ and since $\mathbf{v}_t(0) = \mathbf{0}$, $\mathbf{c} = \mathbf{0}$ and $\mathbf{v}_t(t) = -32t\mathbf{j}$. Then $\mathbf{r}_t(t) = -16t^2\mathbf{j} + \mathbf{b}$. Since $\mathbf{r}_t(0) = x_0\mathbf{i} + y_0\mathbf{j}$, $\mathbf{b} = x_0\mathbf{i} + y_0\mathbf{j}$ and $\mathbf{r}_t(t) = x_0\mathbf{i} + (y_0 - 16t^2)\mathbf{j}$. Now, the horizontal component of $\mathbf{r}_p(t)$ will be x_0 when $t = x_0/s\cos\theta$ at which time the vertical component of $\mathbf{r}_p(t)$ will be

$$(sx_0/s\cos\theta)\sin\theta - 16(x_0/s\cos\theta)^2 = x_0\tan\theta - 16(x_0/s\cos\theta)^2 = y_0 - 16(x_0/s\cos\theta)^2.$$

Thus, $\mathbf{r}_p(x_0/s\cos\theta) = \mathbf{r}_t(x_0/s\cos\theta)$ and the projectile will strike the target as it falls.

18. The initial angle is $\theta = 0$, the initial height is 1024 ft, and the initial speed is $s = 180(5280)/3600 = 264$ ft/s. Then $x(t) = 264t$ and $y(t) = -16t^2 + 1024$. Solving $y(t) = 0$ we see that the pack hits the ground at $t = 8$ seconds The horizontal distance travelled is $x(8) = 2112$ feet. From the figure in the text, $\tan\alpha = 1024/2112 = 16/33$ and $\alpha \approx 0.45$ radian or $25.87°$.

19. $\mathbf{r}'(t) = \mathbf{v}(t) = -r_0\omega\sin\omega t\mathbf{i} + r_0\omega\cos\omega t\mathbf{j}$; $v = \|\mathbf{v}(t)\| = \sqrt{r_0^2\omega^2\sin^2\omega t + r_0^2\omega^2\cos^2\omega t} = r_0\omega$
$\omega = v/r_0$; $\mathbf{a}(t) = \mathbf{r}''(t) = -r_0\omega^2\cos\omega t\mathbf{i} - r_0\omega^2\sin\omega t\mathbf{j}$
$a = \|\mathbf{a}(t)\| = \sqrt{r_0^2\omega^4\cos^2\omega t + r_0^2\omega^4\sin^2\omega t} = r_0\omega^2 = r_0(v/r_0)^2 = v^2/r_0$.

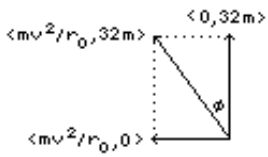
20. **(a)** $\mathbf{v}(t) = -b\sin t\mathbf{i} + b\cos t\mathbf{j} + c\mathbf{k}$; $\|\mathbf{v}(t)\| = \sqrt{b^2\sin^2 t + b^2\cos^2 t + c^2} = \sqrt{b^2 + c^2}$

(b) $s = \displaystyle\int_0^t \|\mathbf{v}(u)\|\,du \int_0^t \sqrt{b^2 + c^2}\,du = t\sqrt{b^2 + c^2}$; $\dfrac{ds}{dt} = \sqrt{b^2 + c^2}$

(c) $\dfrac{d^2s}{dt^2} = 0$; $\mathbf{a}(t) = -b\cos t\mathbf{i} - b\sin t\mathbf{j}$; $\|\mathbf{a}(t)\| = \sqrt{b^2\cos^2 t + b^2\sin^2 t} = |b|$. Thus, $d^2s/dt^2 \neq \|\mathbf{a}(t)\|$.

21. By Problem 19, $a = v^2/r_0 = 1530^2/(4000 \cdot 5280) \approx 0.1108$. We are given $mg = 192$, so $m = 192/32$ and $w_e = 192 - (192/32)(0.1108) \approx 191.33$ lb.

22. By Problem 19, the centripetal acceleration is v^2/r_0. Then the horizontal force is mv^2/r_0. The vertical force is $32m$. The resultant force is $\mathbf{U} = (mv^2/r_0)\mathbf{i} + 32m\mathbf{j}$. From the figure, we see that $\tan\phi = (mv^2/r_0)/32m = v^2/32r_0$. Using $r_0 = 60$ and $v = 44$ we obtain $\tan\phi = 44^2/32(60) \approx 1.0083$ and $\phi \approx 45.24°$.



23. Solving $x(t) = (v_0\cos\theta)t$ for t and substituting into $y(t) = -\frac{1}{2}gt^2 + (v_0\sin\theta)t + s_0$ we obtain

$$y = -\frac{1}{2}\,g\left(\frac{x}{v_0\cos\theta}\right)^2 + (v_0\sin\theta)\frac{x}{v_0\cos\theta} + s_0 = -\frac{g}{2v_0^2\cos^2\theta}\,x^2 + (\tan\theta)x + s_0,$$

which is the equation of a parabola.

24. Since the projectile is launched from ground level, $s_0 = 0$. To find the maximum height we maximize $y(t) = -\frac{1}{2}gt^2 + (v_0\sin\theta)t$. Solving $y'(t) = -gt + v_0\sin\theta = 0$, we see that $t = (v_0/g)\sin\theta$ is a critical point.

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Since $y''(t) = -g < 0$,

$$H = y\left(\frac{v_0 \sin\theta}{g}\right) = -\frac{1}{2}\, g\, \frac{v_0^2 \sin^2\theta}{g^2} + v_0 \sin\theta \frac{v_0 \sin\theta}{g} = \frac{v_0^2 \sin^2\theta}{2g}$$

is the maximum height. To find the range we solve $y(t) = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t = t(v_0 \sin\theta - \frac{1}{2}gt) = 0$. The positive solution of this equation is $t = (2v_0 \sin\theta)/g$. The range is thus

$$x(t) = (v_0 \cos\theta)\, \frac{2v_0 \sin\theta}{g} = \frac{v_0^2 \sin 2\theta}{g}\,.$$

25. Letting $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, the equation $d\mathbf{r}/dt = \mathbf{v}$ is equivalent to $dx/dt = 6t^2x$, $dy/dt = -4ty^2$, $dz/dt = 2t(z+1)$. Separating variables and integrating, we obtain $dx/x = 6t^2\,dt$, $dy/y^2 = -4t\,dt$, $dz/(z+1) = 2t\,dt$, and $\ln x = 2t^3 + c_1$, $-1/y = -2t^2 + c_2$, $\ln(z+1) = t^2 + c_3$. Thus,

$$\mathbf{r}(t) = k_1 e^{2t^3}\mathbf{i} + \frac{1}{2t^2 + k_2}\,\mathbf{j} + (k_3 e^{t^2} - 1)\mathbf{k}.$$

26. We require the fact that $d\mathbf{r}/dt = \mathbf{v}$. Then

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p} = \boldsymbol{\tau} + \mathbf{v} \times \mathbf{p} = \boldsymbol{\tau} + \mathbf{v} \times m\mathbf{v} = \boldsymbol{\tau} + m(\mathbf{v} \times \mathbf{v}) = \boldsymbol{\tau} + \mathbf{0} = \boldsymbol{\tau}.$$

27. **(a)** Since $\mathbf{F}$ is directed along $\mathbf{r}$ we have $\mathbf{F} = c\mathbf{r}$ for some constant c. Then

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (c\mathbf{r}) = c(\mathbf{r} \times \mathbf{r}) = \mathbf{0}.$$

(b) If $\boldsymbol{\tau} = \mathbf{0}$ then $d\mathbf{L}/dt = \mathbf{0}$ and $\mathbf{L}$ is constant.

28. **(a)** Using Problem 27, $\mathbf{F} = -k(Mm/r^2)\mathbf{u} = m\mathbf{a}$. Then $\mathbf{a} = d^2\mathbf{r}/dt = -k(M/r^2)\mathbf{u}$.

(b) Using $\mathbf{u} = \mathbf{r}/r$ we have

$$\mathbf{r} \times \mathbf{r}'' = \mathbf{r} \times \left(-k\,\frac{M}{r^2}\,\mathbf{u}\right) = -\frac{kM}{r^2}\left[\mathbf{r} \times (\frac{1}{r}\,\mathbf{r})\right] = -\frac{kM}{r^3}(\mathbf{r} \times \mathbf{r}) = \mathbf{0}.$$

(c) From Theorem 9.4 (iv) we have

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{r} \times \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{v} = \mathbf{r} \times \mathbf{r}'' + \mathbf{v} \times \mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

(d) Since $\mathbf{r} = r\mathbf{u}$ we have $\mathbf{c} = \mathbf{r} \times \mathbf{v} = r\mathbf{u} \times r\mathbf{u}' = r^2(\mathbf{u} \times \mathbf{u}')$.

(e) Since $\mathbf{u} = (1/r)\mathbf{r}$ is a unit vector, $\mathbf{u} \cdot \mathbf{u} = 1$ and

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{u}) = \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{u}}{dt} \cdot \mathbf{u} = 2\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = \frac{d}{dt}(1) = 0.$$

Thus, $\mathbf{u} \cdot \mathbf{u}' = 0$.

(f) $$\begin{aligned}\frac{d}{dt}(\mathbf{v} \times \mathbf{c}) &= \mathbf{v} \times \frac{d\mathbf{c}}{dt} + \frac{d\mathbf{v}}{dt} \times \mathbf{c} = \mathbf{v} \times \mathbf{0} + \mathbf{a} \times \mathbf{c} = -\frac{kM}{r^2}\,\mathbf{u} \times \mathbf{c} = -\frac{kM}{r^2}\,\mathbf{u} \times [r^2(\mathbf{u} \times \mathbf{u}')] \\ &= -kM[\mathbf{u} \times (\mathbf{u} \times \mathbf{u}')] = -kM = -kM[(\mathbf{u} \cdot \mathbf{u}')\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u}'] \qquad \boxed{\text{by (10) of 7.4}} \\ &= -kM[\mathbf{0} - \mathbf{u}'] = kM\mathbf{u}' = kM\,\frac{d\mathbf{u}}{dt}\end{aligned}$$

(g) Since

$$\begin{aligned}\mathbf{r} \cdot (\mathbf{v} \times \mathbf{c}) &= (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{c} \qquad \boxed{\text{by Problem 61 in 7.4}} \\ &= \mathbf{c} \cdot \mathbf{c} = c^2 \qquad \boxed{\text{where } c = \|\mathbf{c}\|}\end{aligned}$$

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and

$$(kM\mathbf{u}+\mathbf{d})\cdot\mathbf{r} = (kM\mathbf{u}+\mathbf{d})\cdot r\mathbf{u} = kMr\mathbf{u}\cdot\mathbf{u} + rd\mathbf{u}\cdot\mathbf{u}$$
$$= kMr + rd\cos\theta \qquad \boxed{\text{where } d=\|\mathbf{d}\|}$$

we have $c^2 = kMr + rd\cos\theta$ or $r = \dfrac{c^2}{kM + d\cos\theta} = \dfrac{c^2/kM}{1+(d/kM)\cos\theta}$.

(h) First note that $c>0$ (otherwise there is no orbit) and $d>0$ (since the orbit is not a circle). We recognize the equation in **(g)** to be that of a conic section with eccentricity $e = d/kM$. Since the orbit of the planet is closed it must be an ellipse.

(i) At perihelion $c = \|\mathbf{c}\| = \|\mathbf{r}\times\mathbf{v}\| = r_0 v_0 \sin(\pi/r) = r_0 v_0$. Since r is minimum at this point, we want the denominator in the equation $r_0 = [c^2/kM]/[1+(d/kM)\cos\theta]$ to be maximum. This occurs when $\theta = 0$. In this case

$$r_0 = \frac{r_0^2 v_0^2/kM}{1+d/kM} \qquad \text{and} \qquad d = r_0 v_0^2 - kM.$$

Exercises 9.3

1. $\mathbf{r}'(t) = -t\sin t\mathbf{i} + t\cos t\mathbf{j} + 2t\mathbf{k}$; $|\mathbf{r}'(t)| = \sqrt{t^2\sin^2 t + t^2\cos^2 t + 4t^2} = \sqrt{5}\,t$;
$\mathbf{T}(t) = -\dfrac{\sin t}{\sqrt{5}}\,\mathbf{i} + \dfrac{\cos t}{\sqrt{5}}\,\mathbf{j} + \dfrac{2}{\sqrt{5}}\,\mathbf{k}$

2. $\mathbf{r}'(t) = e^t(-\sin t+\cos t)\mathbf{i} + e^t(\cos t+\sin t)\mathbf{j} + \sqrt{2}\,e^t\mathbf{k}$,
$|\mathbf{r}'(t)| = [e^{2t}(\sin^2 t - 2\sin t\cos t + \cos^2 t) + e^{2t}(\cos^2 t + 2\sin t\cos t + \sin^2 t) + 2e^{2t}]^{1/2} = \sqrt{4e^{2t}} = 2e^t$;
$\mathbf{T}(t) = \dfrac{1}{2}(-\sin t+\cos t)\mathbf{i} + \dfrac{1}{2}(\cos t+\sin t)\mathbf{j} + \dfrac{\sqrt{2}}{2}\,\mathbf{k}$

3. We assume $a>0$. $\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + c\mathbf{k}$; $|\mathbf{r}'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + c^2} = \sqrt{a^2+c^2}$;
$\mathbf{T}(t) = -\dfrac{a\sin t}{\sqrt{a^2+c^2}}\,\mathbf{i} + \dfrac{a\cos t}{\sqrt{a^2+c^2}}\,\mathbf{j} + \dfrac{c}{\sqrt{a^2+c^2}}\,\mathbf{k}$; $\dfrac{d\mathbf{T}}{dt} = -\dfrac{a\cos t}{\sqrt{a^2+c^2}}\mathbf{i} - \dfrac{a\sin t}{\sqrt{a^2+c^2}}\,\mathbf{j}$,
$\left|\dfrac{d\mathbf{T}}{dt}\right| = \sqrt{\dfrac{a^2\cos^2 t}{a^2+c^2} + \dfrac{a^2\sin^2 t}{a^2+c^2}} = \dfrac{a}{\sqrt{a^2+c^2}}$; $\mathbf{N} = -\cos t\mathbf{i} - \sin t\mathbf{j}$;

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\dfrac{a\sin t}{\sqrt{a^2+c^2}} & \dfrac{a\cos t}{\sqrt{a^2+c^2}} & \dfrac{c}{\sqrt{a^2+c^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{c\sin t}{\sqrt{a^2+c^2}}\,\mathbf{i} - \frac{c\cos t}{\sqrt{a^2+c^2}}\,\mathbf{j} + \frac{a}{\sqrt{a^2+c^2}}\,\mathbf{k};$$

$\kappa = \dfrac{|d\mathbf{T}/dt|}{|\mathbf{r}'(t)|} = \dfrac{a/\sqrt{a^2+c^2}}{\sqrt{a^2+c^2}} = \dfrac{a}{a^2+c^2}$

4. $\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$, $\mathbf{r}'(1) = \mathbf{i}+\mathbf{j}+\mathbf{k}$; $|\mathbf{r}'(t)| = \sqrt{1+t^2+t^4}$, $|\mathbf{r}'(1)| = \sqrt{3}$;
$\mathbf{T}(t) = (1+t^2+t^4)^{-1/2}(\mathbf{i}+t\mathbf{j}+t^2\mathbf{k})$, $\mathbf{T}(1) = \dfrac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$;

$$\frac{d\mathbf{T}}{dt} = -\frac{1}{2}(1+t^2+t^4)^{-3/2}(2t+4t^3)\mathbf{i} + [(1+t^2+t^4)^{-1/2} - \frac{t}{2}(1+t^2+t^4)^{-3/2}(2t+4t^3)]\mathbf{j}$$
$$+ [2t(1+t^2+t^4)^{-1/2} - \frac{t^2}{2}(1+t^2+t^4)^{-3/2}(2t+4t^3)]\mathbf{k};$$

$\dfrac{d}{dt}\,\mathbf{T}(1) = -\dfrac{1}{\sqrt{3}}\,\mathbf{i} + \dfrac{1}{\sqrt{3}}\,\mathbf{k}$, $\left|\dfrac{d}{dt}\,\mathbf{T}(1)\right| = \sqrt{\dfrac{1}{3}+\dfrac{1}{3}} = \dfrac{\sqrt{2}}{\sqrt{3}}$; $\mathbf{N}(1) = -\dfrac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{k})$;

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$$\mathbf{B}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{vmatrix} = \frac{1}{\sqrt{6}}(\mathbf{i} - 2\mathbf{j} + \mathbf{k});\ \ \kappa = \left|\frac{d}{dt}\,\mathbf{T}(1)\right| / |\mathbf{r}'(1)| = \frac{\sqrt{2}/\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{2}}{3}$$

5. From Example 2 in the text, a normal to the osculating plane is $\mathbf{B}(\pi/4) = \frac{1}{\sqrt{26}}(3\mathbf{i} - 3\mathbf{j} + 2\sqrt{2}\,\mathbf{k})$. The point on the curve when $t = \pi/4$ is $(\sqrt{2}\,, \sqrt{2}\,, 3\pi/4)$. An equation of the plane is $3(x - \sqrt{2}\,) - 3(y - \sqrt{2}\,) + 2\sqrt{2}(z - 3\pi/4) = 0$, $3x - 3y + 2\sqrt{2}\,z = 3\sqrt{2}\,\pi/2$, or $3\sqrt{2}\,x - 3\sqrt{2}\,y + 4z = 3\pi$.

6. From Problem 4, a normal to the osculating plane is $\mathbf{B}(1) = \frac{1}{\sqrt{6}}(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$. The point on the curve when $t = 1$ is $(1, 1/2, 1/3)$. An equation of the plane is $(x - 1) - 2(y - 1/2) + (z - 1/3) = 0$ or $x - 2y + z = 1/3$.

7. $\mathbf{v}(t) = \mathbf{j} + 2t\mathbf{k}$, $|\mathbf{v}(t)| = \sqrt{1 + 4t^2}$; $\mathbf{a}(t) = 2\mathbf{k}$; $\mathbf{v} \cdot \mathbf{a} = 4t$, $\mathbf{v} \times \mathbf{a} = 2\mathbf{i}$, $|\mathbf{v} \times \mathbf{a}| = 2$;
$a_T = \dfrac{4t}{\sqrt{1 + 4t^2}}$, $a_N = \dfrac{2}{\sqrt{1 + 4t^2}}$

8. $\mathbf{v}(t) = -3\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$,
$|\mathbf{v}(t)| = \sqrt{9\sin^2 t + 4\cos^2 t + 1} = \sqrt{5\sin^2 t + 4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}\,\sqrt{\sin^2 t + 1}\,;$
$\mathbf{a}(t) = -3\cos t\mathbf{i} - 2\sin t\mathbf{j}$; $\mathbf{v} \cdot \mathbf{a} = 9\sin t\cos t - 4\sin t\cos t = 5\sin t\cos t$,
$\mathbf{v} \times \mathbf{a} = 2\sin t\mathbf{i} - 3\cos t\mathbf{j} + 6\mathbf{k}$, $|\mathbf{v} \times \mathbf{a}| = \sqrt{4\sin^2 t + 9\cos^2 t + 36} = \sqrt{5}\,\sqrt{\cos^2 t + 8}\,;$
$a_T = \dfrac{\sqrt{5}\sin t\cos t}{\sqrt{\sin^2 t + 1}}$, $a_N = \sqrt{\dfrac{\cos^2 t + 8}{\sin^2 t + 1}}$

9. $\mathbf{v}(t) = 2t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}$, $|\mathbf{v}(t)| = 2\sqrt{6}\,t$, $t > 0$; $\mathbf{a}(t) = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$; $\mathbf{v} \cdot \mathbf{a} = 24t$, $\mathbf{v} \times \mathbf{a} = \mathbf{0}$;
$a_T = \dfrac{24t}{2\sqrt{6}\,t} = 2\sqrt{6}$, $a_N = 0$, $t > 0$

10. $\mathbf{v}(t) = 2t\mathbf{i} - 3t^2\mathbf{j} + 4t^3\mathbf{k}$, $|\mathbf{v}(t)| = t\sqrt{4 + 9t^2 + 16t^4}$, $t > 0$; $\mathbf{a}(t) = 2\mathbf{i} - 6t\mathbf{j} + 12t^2\mathbf{k}$;
$\mathbf{v} \cdot \mathbf{a} = 4t + 18t^3 + 48t^5$; $\mathbf{v} \times \mathbf{a} = -12t^4\mathbf{i} - 16t^3\mathbf{j} - 6t^2\mathbf{k}$, $|\mathbf{v} \times \mathbf{a}| = 2t^2\sqrt{36t^4 + 64t^2 + 9}\,;$
$a_T = \dfrac{4 + 18t^2 + 48t^4}{\sqrt{4 + 9t^2 + 16t^4}}$, $a_N = \dfrac{2t\sqrt{36t^4 + 64t^2 + 9}}{\sqrt{4 + 9t^2 + 16t^4}}$, $t > 0$

11. $\mathbf{v}(t) = 2\mathbf{i} + 2t\mathbf{j}$, $|\mathbf{v}(t)| = 2\sqrt{1 + t^2}\,;$ $\mathbf{a}(t) = 2\mathbf{j}$; $\mathbf{v} \cdot \mathbf{a} = 4t$; $\mathbf{v} \times \mathbf{a} = 4\mathbf{k}$, $|\mathbf{v} \times \mathbf{a}| = 4$;
$a_T = \dfrac{2t}{\sqrt{1 + t^2}}$, $a_N = \dfrac{2}{\sqrt{1 + t^2}}$

12. $\mathbf{v}(t) = \dfrac{1}{1 + t^2}\,\mathbf{i} + \dfrac{t}{1 + t^2}\,\mathbf{j}$, $|\mathbf{v}(t)| = \dfrac{\sqrt{1 + t^2}}{1 + t^2}\,;$ $\mathbf{a}(t) = -\dfrac{2t}{(1 + t^2)^2}\,\mathbf{i} + \dfrac{1 - t^2}{(1 + t^2)^2}\,\mathbf{j}\,;$
$\mathbf{v} \cdot \mathbf{a} = -\dfrac{2t}{(1 + t^2)^3} + \dfrac{t - t^3}{(1 + t^2)^3} = -\dfrac{t}{(1 + t^2)^2}\,;$ $\mathbf{v} \times \mathbf{a} = \dfrac{1}{(1 + t^2)^2}\,\mathbf{k}$, $|\mathbf{v} \times \mathbf{a}| = \dfrac{1}{(1 + t^2)^2}\,;$
$a_T = -\dfrac{t/(1 + t^2)^2}{\sqrt{1 + t^2}/(1 + t^2)} = -\dfrac{t}{(1 + t^2)^{3/2}}$, $a_N = \dfrac{1/(1 + t^2)^2}{\sqrt{1 + t^2}/(1 + t^2)} = \dfrac{1}{(1 + t^2)^{3/2}}$

13. $\mathbf{v}(t) = -5\sin t\mathbf{i} + 5\cos t\mathbf{j}$, $|\mathbf{v}(t)| = 5$; $\mathbf{a}(t) = -5\cos t\mathbf{i} - 5\sin t\mathbf{j}$; $\mathbf{v} \cdot \mathbf{a} = 0$, $\mathbf{v} \times \mathbf{a} = 25\mathbf{k}$, $|\mathbf{v} \times \mathbf{a}| = 25$;
$a_T = 0$, $a_N = 5$

14. $\mathbf{v}(t) = \sinh t\mathbf{i} + \cosh t\mathbf{j}$, $|\mathbf{v}(t)| = \sqrt{\sinh^2 t + \cosh^2 t}\,;$ $\mathbf{a}(t) = \cosh t\mathbf{i} + \sinh t\mathbf{j}$ $\mathbf{v} \cdot \mathbf{a} = 2\sinh t\cosh t$;
$\mathbf{v} \times \mathbf{a} = (\sinh^2 t - \cosh^2 t)\mathbf{k} = -\mathbf{k}$, $|\mathbf{v} \times \mathbf{a}| = 1$; $a_T = \dfrac{2\sinh t\cosh t}{\sqrt{\sinh^2 t + \cosh^2 t}}$, $a_N = \dfrac{1}{\sqrt{\sinh^2 t + \cosh^2 t}}$

15. $\mathbf{v}(t) = -e^{-t}(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $|\mathbf{v}(t)| = \sqrt{3}\,e^{-t}$; $\mathbf{a}(t) = e^{-t}(\mathbf{i} + \mathbf{j} + \mathbf{k})$; $\mathbf{v} \cdot \mathbf{a} = -3e^{-2t}$; $\mathbf{v} \times \mathbf{a} = \mathbf{0}$, $|\mathbf{v} \times \mathbf{a}| = 0$;
$a_T = -\sqrt{3}\,e^{-t}$, $a_N = 0$

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16. $\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, $|\mathbf{v}(t)| = \sqrt{21}$; $\mathbf{a}(t) = \mathbf{0}$; $\mathbf{v} \cdot \mathbf{a} = 0$, $\mathbf{v} \times \mathbf{a} = \mathbf{0}$, $|\mathbf{v} \times \mathbf{a}| = 0$; $a_T = 0$, $a_N = 0$

17. $\mathbf{v}(t) = -a\sin t\mathbf{i} + b\cos t\mathbf{j} + c\mathbf{k}$, $|\mathbf{v}(t)| = \sqrt{a^2\sin^2 t + b^2\cos^2 t + c^2}$; $\mathbf{a}(t) = -a\cos t\mathbf{i} - b\sin t\mathbf{j}$;
$\mathbf{v} \times \mathbf{a} = bc\sin t\mathbf{i} - ac\cos t\mathbf{j} + ab\mathbf{k}$, $|\mathbf{v} \times \mathbf{a}| = \sqrt{b^2c^2\sin^2 t + a^2c^2\cos^2 t + a^2b^2}$
$\kappa = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \dfrac{\sqrt{b^2c^2\sin^2 t + a^2c^2\cos^2 t + a^2b^2}}{(a^2\sin^2 t + b^2\cos^2 t + c^2)^{3/2}}$;

18. **(a)** $\mathbf{v}(t) = -a\sin t\mathbf{i} + b\cos t\mathbf{j}$, $|\mathbf{v}(t)| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$; $\mathbf{a}(t) = -a\cos t\mathbf{i} - b\sin t\mathbf{j}$;
$\mathbf{v} \times \mathbf{a} = ab\mathbf{k}$; $|\mathbf{v} \times \mathbf{a}| = ab$; $\kappa = \dfrac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$

(b) When $a = b$, $|\mathbf{v}(t)| = a$, $|\mathbf{v} \times \mathbf{a}| = a^2$, and $\kappa = a^2/a^3 = 1/a$.

19. The equation of a line is $\mathbf{r}(t) = \mathbf{b} + t\mathbf{c}$, where $\mathbf{b}$ and $\mathbf{c}$ are constant vectors.
$\mathbf{v}(t) = \mathbf{c}$, $|\mathbf{v}(t)| = |\mathbf{c}|$; $\mathbf{a}(t) = \mathbf{0}$; $\mathbf{v} \times \mathbf{a} = \mathbf{0}$, $|\mathbf{v} \times \mathbf{a}| = 0$; $\kappa = |\mathbf{v} \times \mathbf{a}|/|\mathbf{v}|^3 = 0$

20. $\mathbf{v}(t) = a(1 - \cos t)\mathbf{i} + a\sin t\mathbf{j}$; $\mathbf{v}(\pi) = 2a\mathbf{i}$, $|\mathbf{v}(\pi)| = 2a$; $\mathbf{a}(t) = a\sin t\mathbf{i} + a\cos t\mathbf{j}$, $\mathbf{a}(\pi) = -a\mathbf{j}$;
$|\mathbf{v} \times \mathbf{a}| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2a & 0 & 0 \\ 0 & -a & 0 \end{vmatrix} = -2a^2\mathbf{k}$; $|\mathbf{v} \times \mathbf{a}| = 2a^2$; $\kappa = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \dfrac{2a^2}{8a^3} = \dfrac{1}{4a}$

21. $\mathbf{v}(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$, $|\mathbf{v}(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$; $\mathbf{a}(t) = f''(t)\mathbf{i} + g''(t)\mathbf{j}$;
$\mathbf{v} \times \mathbf{a} = [f'(t)g''(t) - g'(t)f''(t)]\mathbf{k}$, $|\mathbf{v} \times \mathbf{a}| = |f'(t)g''(t) - g'(t)f''(t)|$;
$\kappa = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \dfrac{|f'(t)g''(t) - g'(t)f''(t)|}{([f'(t)]^2 + [g'(t)]^2)^{3/2}}$

22. For $y = F(x)$, $\mathbf{r}(x) = x\mathbf{i} + F(x)\mathbf{j}$. We identify $f(x) = x$ and $g(x) = F(x)$ in Problem 21. Then $f'(x) = 1$, $f''(x) = 0$, $g'(x) = F'(x)$, $g''(x) = F''(x)$, and $\kappa = |F''(x)|/(1 + [F'(x)]^2)^{3/2}$.

23. $F(x) = x^2$, $F(0) = 0$, $F(1) = 1$; $F'(x) = 2x$, $F'(0) = 0$, $F'(1) = 2$; $F''(x) = 2$, $F''(0) = 2$, $F''(1) = 2$;
$\kappa(0) = \dfrac{2}{(1 + 0^2)^{3/2}} = 2$; $\rho(0) = \dfrac{1}{2}$; $\kappa(1) = \dfrac{2}{(1 + 2^2)^{3/2}} = \dfrac{2}{5\sqrt{5}} \approx 0.18$;
$\rho(1) = \dfrac{5\sqrt{5}}{2} \approx 5.59$; Since $2 > 2/5\sqrt{5}$, the curve is "sharper" at $(0, 0)$.

24. $F(x) = x^3$, $F(-1) = -1$, $F(1/2) = 1/8$; $F'(x) = 3x^2$, $F'(-1) = 3$, $F'(1/2) = 3/4$; $F''(x) = 6x$,
$F''(-1) = -6$, $F''(1/2) = 3$; $\kappa(-1) = \dfrac{|-6|}{(1 + 3^2)^{3/2}} = \dfrac{6}{10\sqrt{10}} = \dfrac{3}{5\sqrt{10}} \approx 0.19$;
$\rho(-1) = \dfrac{5\sqrt{10}}{3} \approx 5.27$; $\kappa(\dfrac{1}{2}) = \dfrac{3}{[1 + (3/4)^2]^{3/2}} = \dfrac{3}{125/64} = \dfrac{192}{125} \approx 1.54$; $\rho(\dfrac{1}{2}) = \dfrac{125}{192} \approx 0.65$
Since $1.54 > 0.19$, the curve is "sharper" at $(1/2, 1/8)$.

25. At a point of inflection $(x_0, F(x_0))$, if $F''(x_0)$ exists then $F''(x_0) = 0$. Thus, assuming that $\lim_{x \to x_0} F''(x)$ exists, $F''(x)$ and hence κ is near 0 for x near x_0.

26. We use the fact that $\mathbf{T} \cdot \mathbf{N} = 0$ and $\mathbf{T} \cdot \mathbf{T} = \mathbf{N} \cdot \mathbf{N} = 1$. Then

$$|\mathbf{a}(t)|^2 = \mathbf{a} \cdot \mathbf{a} = (a_N\mathbf{N} + a_T\mathbf{T}) \cdot (a_N\mathbf{N} + a_T\mathbf{T}) = a_N^2\mathbf{N} \cdot \mathbf{N} + 2a_Na_T\mathbf{N} \cdot \mathbf{T} + a_T^2\mathbf{T} \cdot \mathbf{T} = a_N^2 + a_T^2.$$

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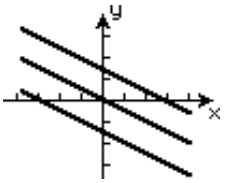
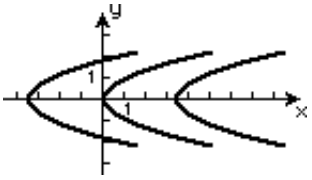
Exercises 9.4

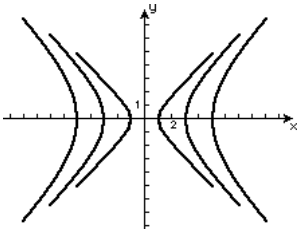
1. $y = -\frac{1}{2}x + C$



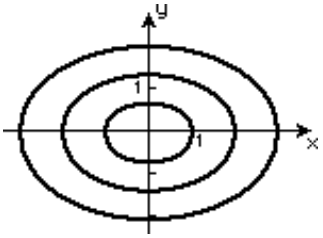
2. $x = y^2 - c$



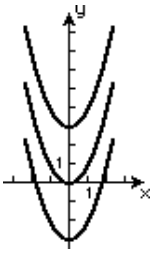
3. $x^2 - y^2 = 1 + c^2$



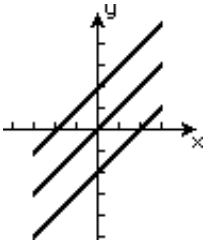
4. $4x^2 + 9y^2 = 36 - c^2$, $-6 \le c \le 6$



5. $y = x^2 + \ln c$, $c > 0$



6. $y = x + \tan c$, $-\pi/x < c < \pi/2$



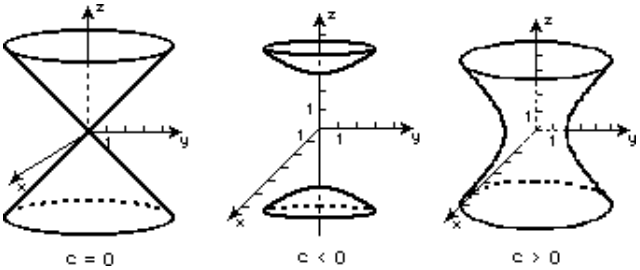
7. $x^2/9 + z^2/4 = c$; elliptical cylinder

8. $x^2 + y^2 + z^2 = c$; sphere

9. $x^2 + 3y^2 + 6z^2 = c$; ellipsoid

10. $4y - 2z + 1 = c$; plane

11.



12. Setting $x = -4$, $y = 2$, and $z = -3$ in $x^2/16 + y^2/4 + z^2/9 = c$ we obtain $c = 3$. The equation of the surface is $x^2/16 + y^2/4 + z^2/9 = 3$. Setting $y = z = 0$ we find the x-intercepts are $\pm 4\sqrt{3}$. Similarly, the y-intercepts are $\pm 2\sqrt{3}$ and the z-intercepts are $\pm 3\sqrt{3}$.

13. $z_x = 2x - y^2$; $z_y = -2xy + 20y^4$

14. $z_x = -3x^2 + 12xy^3$; $z_y = 18x^2y^2 + 10y$

15. $z_x = 20x^3y^3 - 2xy^6 + 30x^4$; $z_y = 15x^4y^2 - 6x^2y^5 - 4$

16. $z_x = 3x^2y^2 \sec^2(x^3y^2)$; $z_y = 2x^3y \sec^2(x^3y^2)$

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17. $z_x = \dfrac{2}{\sqrt{x}\,(3y^2+1)}$; $z_y = -\dfrac{24y\sqrt{x}}{(3y^2+1)^2}$

18. $z_x = 12x^2 - 10x + 8$; $z_y = 0$

19. $z_x = -(x^3-y^2)^{-2}(3x^2) = -3x^2(x^3-y^2)^{-2}$; $z_y = -(x^3-y^2)^{-2}(-2y) = 2y(x^3-y^2)^{-2}$

20. $z_x = 6(-x^4+7y^2+3y)^5(-4x^3) = -24x^3(-x^4+7y^2+3y)^5$; $z_y = 6(-x^4+7y^2+3y)^5(14y+3)$

21. $z_x = 2(\cos 5x)(-\sin 5x)(5) = -10\sin 5x\cos 5x$; $z_y = 2(\sin 5y)(\cos 5y)(5) = 10\sin 5y\cos 5y$

22. $z_x = (2x\tan^{-1}y^2)e^{x^2\tan^{-1}y^2}$; $z_y = \dfrac{2x^2y}{1+y^4}\,e^{x^2\tan^{-1}y^2}$

23. $f_x = x(3x^2ye^{x^3y} + e^{x^3y} = (3x^3y+1)e^{x^3y}$; $f_y = x^4e^{x^3y}$

24. $f_\theta = \phi^2\left(\cos\dfrac{\theta}{\phi}\right)\left(\dfrac{1}{\phi}\right) = \phi\cos\dfrac{\theta}{\phi}$; $f_\phi = \phi^2\left(\cos\dfrac{\theta}{\phi}\right)\left(-\dfrac{\theta}{\phi^2}\right) + 2\phi\sin\dfrac{\theta}{\phi} = -\theta\cos\dfrac{\theta}{\phi} + 2\phi\sin\dfrac{\theta}{\phi}$

25. $f_x = \dfrac{(x+2y)3-(3x-y)}{(x+2y)^2} = \dfrac{7y}{(x+2y)^2}$; $f_y = \dfrac{(x+2y)(-1)-(3x-y)(2)}{(x+2y)^2} = \dfrac{-7x}{(x+2y)^2}$

26. $f_x = \dfrac{(x^2-y^2)^2y - xy[2(x^2-y^2)2x]}{(x^2-y^2)^4} = \dfrac{-3x^2y-y^3}{(x^2-y^2)^3}$;

$f_y = \dfrac{(x^2-y^2)^2x - xy[2(x^2-y^2)(-2y)]}{(x^2-y^2)^4} = \dfrac{3xy^2+x^3}{(x^2-y^2)^3}$

27. $g_u = \dfrac{8u}{4u^2+5v^3}$; $g_v = \dfrac{15v^2}{4u^2+5v^3}$

28. $h_r = \dfrac{1}{2s\sqrt{r}} + \dfrac{\sqrt{s}}{r^2}$; $h_s = -\dfrac{\sqrt{r}}{s^2} - \dfrac{1}{2r\sqrt{s}}$

29. $w_x = \dfrac{y}{\sqrt{x}}$; $w_y = 2\sqrt{x} - y\left(\dfrac{1}{z}\,e^{y/z}\right) - e^{y/z} = 2\sqrt{x} - \left(\dfrac{y}{z}+1\right)e^{y/z}$; $w_z = -ye^{y/z}\left(-\dfrac{y}{z^2}\right) = \dfrac{y^2}{z^2}\,e^{y/z}$

30. $w_x = xy\left(\dfrac{1}{x}\right) + (\ln xz)y = y + y\ln xz$; $w_y = x\ln xz$; $w_z = \dfrac{xy}{z}$

31. $F_u = 2uw^2 - v^3 - vwt^2\sin(ut^2)$; $F_v = -3uv^2 + w\cos(ut^2)$;
$F_x = 4(2x^2t)^3(4xt) = 16xt(2x^2t)^3 = 128x^7t^4$; $F_t = -2uvwt\sin(ut^2) + 64x^8t^3$

32. $G_p = r^4s^5(p^2q^3)^{r^4s^5-1}(2pq^3) = 2pq^3r^4s^5(p^2q^3)^{r^4s^5-1}$;
$G_q = r^4s^5(p^2q^3)^{r^4s^5-1}(3p^2q^2) = 3p^2q^2r^4s^5(p^2q^3)^{r^4s^5-1}$; $G_r = (p^2q^3)^{r^4s^5}(4r^3s^5)\ln(p^2q^3)$;
$G_s = (p^2q^3)^{r^4s^5}(5r^4s^4)\ln(p^2q^3)$

33. $\dfrac{\partial z}{\partial x} = \dfrac{2x}{x^2+y^2}$, $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{(x^2+y^2)2-2x(2x)}{(x^2+y^2)^2} = \dfrac{2y^2-2x^2}{(x^2+y^2)^2}$; $\dfrac{\partial z}{\partial y} = \dfrac{2y}{x^2+y^2}$,
$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{(x^2+y^2)2-2y(2y)}{(x^2+y^2)^2} = \dfrac{2x^2-2y^2}{(x^2+y^2)^2}$; $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2} = 0$

34. $\dfrac{\partial z}{\partial x} = e^{x^2-y^2}(-2y\sin 2xy) + 2xe^{x^2-y^2}\cos 2xy$
$\dfrac{\partial^2 z}{\partial x^2} = e^{x^2-y^2}(-4y^2\cos 2xy - 8xy\sin 2xy + 4x^2\cos 2xy + 2\cos 2xy)$
$\dfrac{\partial z}{\partial y} = e^{x^2-y^2}(-2x\sin 2xy) - 2ye^{x^2-y^2}\cos 2xy$
$\dfrac{\partial^2 z}{\partial y^2} = e^{x^2-y^2}(-4x^2\cos 2xy + 8xy\sin 2xy + 4y^2\cos 2xy - 2\cos 2xy)$

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Adding the second partial derivatives gives

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = [-4(y^2 + x^2)\cos 2xy + 4(x^2 + y^2)\cos 2xy] = 0.$$

35. $\dfrac{\partial u}{\partial x} = \cos at \cos x, \ \dfrac{\partial^2 u}{\partial x^2} = -\cos at \sin x; \ \dfrac{\partial u}{\partial t} = -a \sin at \sin x, \ \dfrac{\partial^2 u}{\partial t^2} = -a^2 \cos at \sin x;$

$a^2 \dfrac{\partial^2 u}{\partial x^2} = a^2(-\cos at \sin x) = \dfrac{\partial^2 u}{\partial t^2}$

36. $\dfrac{\partial u}{\partial x} = -\sin(x + at) + \cos(x - at), \ \dfrac{\partial^2 u}{\partial x^2} = -\cos(x + at) - \sin(x - at);$

$\dfrac{\partial u}{\partial t} = -a \sin(x + at) - a\cos(x - at), \ \dfrac{\partial^2 u}{\partial t^2} = -a^2 \cos(x + at) - a^2 \sin(x - at);$

$a^2 \dfrac{\partial^2 u}{\partial x^2} = -a^2 \cos(x + at) - a^2 \sin(x - at) = \dfrac{\partial^2 u}{\partial t^2}$

37. $\dfrac{\partial C}{\partial x} = -\dfrac{2x}{kt} t^{-1/2} e^{-x^2/kt}, \ \dfrac{\partial^2 C}{\partial x^2} = \dfrac{4x^2}{k^2 t^2} t^{-1/2} e^{-x^2/kt} - \dfrac{2}{kt} t^{-1/2} e^{-x^2/kt};$

$\dfrac{\partial C}{\partial t} = t^{-1/2} \dfrac{x^2}{kt^2} e^{-x^2/kt} - \dfrac{t^{-3/2}}{2} e^{-x^2/kt}; \ \dfrac{k}{4} \dfrac{\partial^2 C}{\partial x^2} = \dfrac{x^2}{kt^2} t^{-1/2} e^{-x^2/kt} - \dfrac{t^{-1/2}}{2t} e^{-x^2/kt} = \dfrac{\partial C}{\partial t}$

38. **(a)** $P_v = -k(T/V^2)$

(b) $PV = kt, \ PV_T = k, \ V_T = k/P$

(c) $PV = kT, \ V = kT_p, \ T_p = V/k$

39. $z_x = v^2 e^{uv^2}(3x^2) + 2uve^{uv^2}(1) = 3x^2 v^2 e^{uv^2} + 2uve^{uv^2}; \ z_y = v^2 e^{uv^2}(0) + 2uve^{uv^2}(-2y) = -4yuve^{uv^2}$

40. $z_x = (2u\cos 4v)(2xy^3) - (4u^2 \sin 4v)(3x^2) = 4xy^3 u \cos 4v - 12x^2 u^2 \sin 4v$

$z_y = (2u\cos 4v)(3x^2 y^2) - (4v^2 \sin 4v)(3y^2) = 6x^2 y^2 u \cos 4v - 12y^2 u^2 \sin 4v$

41. $z_u = 4(4u^3) - 10y[2(2u - v)(2)] = 16u^3 - 40(2u - v)y$

$z_v = 4(-24v^2) - 10y[2(2u - v)(-1)] = -96v^2 + 20(2u - v)y$

42. $z_u = \dfrac{2y}{(x + y)^2}\left(\dfrac{1}{v}\right) + \dfrac{-2x}{(x + y)^2}\left(-\dfrac{v^2}{u^2}\right) = \dfrac{2y}{v(x + y)^2} + \dfrac{2xv^2}{u^2(x + y)^2}$

$z_v = \dfrac{2y}{(x + y)^2}\left(-\dfrac{u}{v^2}\right) + \dfrac{-2x}{(x + y)^2}\left(\dfrac{2v}{u}\right) = -\dfrac{2yu}{v^2(x + y)^2} - \dfrac{4xv}{u(x + y)^2}$

43. $w_t = \dfrac{3}{2}(u^2 + v^2)^{1/2}(2u)(-e^{-t}\sin\theta) + \dfrac{3}{2}(u^2 + v^2)^{1/2}(2v)(-e^{-t}\cos\theta)$

$= -3u(u^2 + v^2)^{1/2} e^{-t}\sin\theta - 3v(u^2 + v^2)^{1/2} e^{-t}\cos\theta$

$w_\theta = \dfrac{3}{2}(u^2 + v^2)^{1/2}(2u)e^{-t}\cos\theta + \dfrac{3}{2}(u^2 + v^2)^{1/2}(2v)(-e^{-t}\sin\theta)$

$= 3u(u^2 + v^2)^{1/2} e^{-t}\cos\theta - 3v(u^2 + v^2)^{1/2} e^{-t}\sin\theta$

44. $w_r = \dfrac{v/2\sqrt{uv}}{1 + uv}(2r) + \dfrac{u/2\sqrt{uv}}{1 + uv}(2rs^2) = \dfrac{rv}{\sqrt{uv}\,(1 + uv)} + \dfrac{rs^2 u}{\sqrt{uv}\,(1 + uv)}$

$w_s = \dfrac{v/2\sqrt{uv}}{1 + uv}(-2s) + \dfrac{u/2\sqrt{uv}}{1 + uv}(2r^2 s) = \dfrac{-sv}{\sqrt{uv}\,(1 + uv)} + \dfrac{r^2 su}{\sqrt{uv}\,(1 + uv)}$

45. $R_u = s^2 t^4(e^{v^2}) + 2rst^4(-2uve^{-u^2}) + 4rs^2 t^3(2uv^2 e^{u^2 v^2}) = s^2 t^4 e^{v^2} - 4uvrst^4 e^{-u^2} + 8uv^2 rs^2 t^3 e^{u^2 v^2}$

$R_v = s^2 t^4(2uve^{v^2}) + 2rst^4(e^{-u^2}) + 4rs^2 t^3(2u^2 ve^{u^2 v^2}) = 2s^2 t^4 uve^{v^2} + 2rst^4 e^{-u^2} + 8rs^2 t^3 u^2 ve^{u^2 v^2}$

46. $Q_x = \dfrac{1}{P}\left(\dfrac{t^2}{\sqrt{1 - x^2}}\right) + \dfrac{1}{q}\left(\dfrac{1}{t^2}\right) + \dfrac{1}{r}\left(\dfrac{1/t}{1 + (x/t)^2}\right) = \dfrac{t^2}{p\sqrt{1 - x^2}} + \dfrac{1}{qt^2} + \dfrac{t}{r(t^2 + x^2)}$

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$$Q_t = \frac{1}{p}(2t\sin^{-1}x) + \frac{1}{q}\left(-\frac{2x}{t^3}\right) + \frac{1}{r}\left(\frac{-x/t^2}{1+(x/t)^2}\right) = \frac{2t\sin^{-1}x}{p} - \frac{2x}{qt^3} - \frac{x}{r(t^2+x^2)}$$

47. $w_t = \dfrac{2x}{2\sqrt{x^2+y^2}}\,\dfrac{u}{rs+tu} + \dfrac{2y}{2\sqrt{x^2+y^2}}\,\dfrac{\cosh rs}{u} = \dfrac{xu}{\sqrt{x^2+y^2}\,(rs+tu)} + \dfrac{y\cosh rs}{u\sqrt{x^2+y^2}}$

$w_r = \dfrac{2x}{2\sqrt{x^2+y^2}}\,\dfrac{s}{rs+tu} + \dfrac{2y}{2\sqrt{x^2+y^2}}\,\dfrac{st\sinh rs}{u} = \dfrac{xs}{\sqrt{x^2+y^2}\,(rs+tu)} + \dfrac{yst\sinh rs}{u\sqrt{x^2+y^2}}$

$w_u = \dfrac{2x}{2\sqrt{x^2+y^2}}\,\dfrac{t}{rs+tu} + \dfrac{2y}{2\sqrt{x^2+y^2}}\,\dfrac{-t\cosh rs}{u^2} = \dfrac{xt}{\sqrt{x^2+y^2}\,(rs+tu)} - \dfrac{yt\cosh rs}{u^2\sqrt{x^2+y^2}}$

48. $s_\phi = 2pe^{3\theta} + 2q[-\sin(\phi+\theta)] - 2r\theta^2 + 4(2) = 2pe^{3\theta} - 2q\sin(\phi+\theta) - 2r\theta^2 + 8$

$s_\theta = 2p(3\phi e^{3\theta}) + 2q[-\sin(\phi+\theta)] - 2r(2\phi\theta) + 4(8) = 6p\phi e^{3\theta} - 2q\sin(\phi+\theta) - 4r\phi\theta + 32$

49. $\dfrac{dz}{dt} = \dfrac{2u}{u^2+v^2}(2t) + \dfrac{2v}{u^2+v^2}(-2t^{-3}) = \dfrac{4ut - 4vt^{-3}}{u^2+v^2}$

50. $\dfrac{dz}{dt} = (3u^2v - v^4)(-5e^{-5t}) + (u^3 - 4uv^3)(5\sec 5t\tan 5t) = -5(3u^2v - v^4)e^{-5t} + 5(u^3 - 4uv^3)\sec 5t\tan 5t$

51. $\dfrac{dw}{dt} = -3\sin(3u+4v)(2) - 4\sin(3u+4v)(-1);\;\; u(\pi) = 5\pi/2,\;\; v(\pi) = -5\pi/4$

$\left.\dfrac{dw}{dt}\right|_{\pi} = -6\sin\left(\dfrac{15\pi}{2} - 5\pi\right) + 4\sin\left(\dfrac{15\pi}{2} - 5\pi\right) = -2\sin\dfrac{5\pi}{2} = -2$

52. $\dfrac{dw}{dt} = ye^{xy}\left[\dfrac{-8}{(2t+1)^2}\right] + xe^{xy}(3);\;\; x(0) = 4,\;\; y(0) = 5;\;\; \left.\dfrac{dw}{dt}\right|_0 = 5e^{20}(-8) + 4e^{20}(3) = -28e^{20}$

53. With $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r}\cos\theta + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r}\sin\theta = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta}(-r\sin\theta) + \frac{\partial u}{\partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta}(r\cos\theta)$$

$$= -r\frac{\partial u}{\partial x}\cos\theta + r^2\frac{\partial^2 u}{\partial x^2}\sin^2\theta - r\frac{\partial u}{\partial y}\sin\theta + r^2\frac{\partial^2 u}{\partial y^2}\cos^2\theta.$$

Using $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, we have

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + \frac{1}{r}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)$$

$$+ \frac{1}{r^2}\left(-r\frac{\partial u}{\partial x}\cos\theta + r^2\frac{\partial^2 u}{\partial x^2}\sin^2\theta - r\frac{\partial u}{\partial y}\sin\theta + r^2\frac{\partial^2 u}{\partial y^2}\cos^2\theta\right)$$

$$= \frac{\partial^2 u}{\partial x^2}(\cos^2\theta + \sin^2\theta) + \frac{\partial^2 u}{\partial y^2}(\sin^2\theta + \cos^2\theta) + \frac{\partial u}{\partial x}\left(\frac{1}{r}\cos\theta - \frac{1}{r}\cos\theta\right)$$

$$+ \frac{\partial u}{\partial y}\left(\frac{1}{r}\sin\theta - \frac{1}{r}\sin\theta\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

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54. $\dfrac{dP}{dt} = \dfrac{(V-0.0427)(0.08)\,dT/dt}{(V-0.0427)^2} - \dfrac{0.08T(dV/dt)}{(V-0.0427)^2} + \dfrac{3.6}{V^3}\dfrac{dV}{dt}$

$= \dfrac{0.08}{V-0.0427}\dfrac{dT}{dt} + \left(\dfrac{3.6}{V^3} - \dfrac{0.08T}{(V-0.0427)^2}\right)\dfrac{dV}{dt}$

55. Since $dT/dT = 1$ and $\partial P/\partial T = 0$,

$$0 = F_T = \frac{\partial F}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial F}{\partial V}\frac{\partial V}{\partial T} + \frac{\partial F}{\partial T}\frac{dT}{dT} \implies \frac{\partial V}{\partial T} = -\frac{\partial F/\partial T}{\partial F/\partial V} = -\frac{1}{\partial T/\partial V}\,.$$

56. We are given $dE/dt = 2$ and $dR/dt = -1$. Then $\dfrac{dI}{dt} = \dfrac{\partial I}{\partial E}\dfrac{dE}{dt} + \dfrac{\partial I}{\partial R}\dfrac{dR}{dt} = \dfrac{1}{R}(2) - \dfrac{E}{R^2}(-1)$, and when $E = 60$ and $R = 50$, $\dfrac{dI}{dt} = \dfrac{2}{50} + \dfrac{60}{50^2} = \dfrac{1}{25} + \dfrac{3/5}{25} = \dfrac{8}{125}$ amp/min.

57. Since the height of the triangle is $x\sin\theta$, the area is given by $A = \frac{1}{2}xy\sin\theta$. Then

$$\frac{dA}{dt}\frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt} = \frac{1}{2}y\sin\theta\,\frac{dx}{dt} + \frac{1}{2}x\sin\theta\,\frac{dy}{dt} + \frac{1}{2}xy\cos\theta\,\frac{d\theta}{dt}\,.$$

When $x = 10$, $y = 8$, $\theta = \pi/6$, $dx/dt = 0.3$, $dy/dt = 0.5$, and $d\theta/dt = 0.1$,

$$\frac{dA}{dt} = \frac{1}{2}(8)\left(\frac{1}{2}\right)(0.3) + \frac{1}{2}(10)\left(\frac{1}{2}\right)(0.5) + \frac{1}{2}(10)(8)\left(\frac{\sqrt{3}}{2}\right)(0.1)$$

$$= 0.6 + 1.25 + 2\sqrt{3} = 1.85 + 2\sqrt{3} \approx 5.31 \text{ cm}^2/\text{s}.$$

58. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = \dfrac{x\,dx/dt + y\,dy/dt + z\,dz/dt}{\sqrt{x^2+y^2+z^2}} = \dfrac{-4x\sin t + 4y\cos t + 5z}{\sqrt{16\cos^2 t + 16\sin^2 t + 25t^2}}$

$= \dfrac{-16\sin t\cos t + 16\sin t\cos t + 25t}{\sqrt{16+25t^2}} = \dfrac{25t}{\sqrt{16+25t^2}}$

$\left.\dfrac{dw}{dt}\right|_{t=5\pi/2} = \dfrac{125\pi/2}{\sqrt{16+625\pi^2/4}} = \dfrac{125\pi}{\sqrt{64+625\pi^2}} \approx 4.9743$

Exercises 9.5

1. $\nabla f = (2x - 3x^2y^2)\mathbf{i} + (4y^3 - 2x^3y)\mathbf{j}$

2. $\nabla f = 4xye^{-2x^2y}\mathbf{i} + (1 + 2x^2e^{-2x^2y})\mathbf{j}$

3. $\nabla F = \dfrac{y^2}{z^3}\,\mathbf{i} + \dfrac{2xy}{z^3}\,\mathbf{j} - \dfrac{3xy^2}{z^4}\,\mathbf{k}$

4. $\nabla F = y\cos yz\mathbf{i} + (x\cos yz - xyz\sin yz)\mathbf{j} - xy^2\sin yz\mathbf{k}$

5. $\nabla f = 2x\mathbf{i} - 8y\mathbf{j}$; $\nabla f(2,4) = 4\mathbf{i} - 32\mathbf{j}$

6. $\nabla f = \dfrac{3x^2}{2\sqrt{x^3y - y^4}}\,\mathbf{i} + \dfrac{x^3 - 4y^3}{2\sqrt{x^3y - y^4}}\,\mathbf{j}$; $\nabla f(3,2) = \dfrac{27}{\sqrt{38}}\,\mathbf{i} - \dfrac{5}{2\sqrt{38}}\,\mathbf{j}$

7. $\nabla F = 2xz^2\sin 4y\mathbf{i} + 4x^2z^2\cos 4y\mathbf{j} + 2x^2z\sin 4y\mathbf{k}$

$\nabla F(-2, \pi/3, 1) = -4\sin\dfrac{4\pi}{3}\,\mathbf{i} + 16\cos\dfrac{4\pi}{3}\,\mathbf{j} + 8\sin\dfrac{4\pi}{3}\,\mathbf{k} = 2\sqrt{3}\,\mathbf{i} - 8\mathbf{j} - 4\sqrt{3}\,\mathbf{k}$

8. $\nabla F = \dfrac{2x}{x^2+y^2+z^2}\,\mathbf{i} + \dfrac{2y}{x^2+y^2+z^2}\,\mathbf{j} + \dfrac{2z}{x^2+y^2+z^2}\,\mathbf{k}$; $\nabla F(-4,3,5) = -\dfrac{4}{25}\,\mathbf{i} + \dfrac{3}{25}\,\mathbf{j} + \dfrac{1}{5}\,\mathbf{k}$

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9. $D_{\mathbf{u}}f(x,y) = \lim\limits_{h\to 0}\dfrac{f(x+h\sqrt{3}/2, y+h/2)-f(x,y)}{h} = \lim\limits_{h\to 0}\dfrac{(x+h\sqrt{3}/2)^2+(y+h/2)^2-x^2-y^2}{h}$

$= \lim\limits_{h\to 0}\dfrac{h\sqrt{3}\,x+3h^2/4+hy+h^2/4}{h} = \lim\limits_{h\to 0}(\sqrt{3}\,x+3h/4+y+h/4) = \sqrt{3}\,x+y$

10. $D_{\mathbf{u}}f(x,y) = \lim\limits_{h\to 0}\dfrac{f(x+h\sqrt{2}/2, y+h\sqrt{2}/2)-f(x,y)}{h} = \lim\limits_{h\to 0}\dfrac{3x+3h\sqrt{2}/2-(y+h\sqrt{2}/2)^2-3x+y^2}{h}$

$= \lim\limits_{h\to 0}\dfrac{3h\sqrt{2}/2-h\sqrt{2}\,y-h^2/2}{h} = \lim\limits_{h\to 0}(3\sqrt{2}/2-\sqrt{2}\,y-h/2) = 3\sqrt{2}/2-\sqrt{2}\,y$

11. $\mathbf{u} = \dfrac{\sqrt{3}}{2}\,\mathbf{i}+\dfrac{1}{2}\,\mathbf{j}$; $\nabla f = 15x^2y^6\mathbf{i}+30x^3y^5\mathbf{j}$; $\nabla f(-1,1) = 15\mathbf{i}-30\mathbf{j}$;

$D_{\mathbf{u}}f(-1,1) = \dfrac{15\sqrt{3}}{2}-15 = \dfrac{15}{2}(\sqrt{3}-2)$

12. $\mathbf{u} = \dfrac{\sqrt{2}}{2}\,\mathbf{i}+\dfrac{\sqrt{2}}{2}\,\mathbf{j}$; $\nabla f = (4+y^2)\mathbf{i}+(2xy-5)\mathbf{j}$; $\nabla f(3,-1) = 5\mathbf{i}-11\mathbf{j}$;

$D_{\mathbf{u}}f(3,-1) = \dfrac{5\sqrt{2}}{2}-\dfrac{11\sqrt{2}}{2} = -3\sqrt{2}$

13. $\mathbf{u} = \dfrac{\sqrt{10}}{10}\,\mathbf{i}-\dfrac{3\sqrt{10}}{10}\,\mathbf{j}$; $\nabla f = \dfrac{-y}{x^2+y^2}\,\mathbf{i}+\dfrac{x}{x^2+y^2}\,\mathbf{j}$; $\nabla f(2,-2) = \dfrac{1}{4}\,\mathbf{i}+\dfrac{1}{4}\,\mathbf{j}$

$D_{\mathbf{u}}f(2,-2) = \dfrac{\sqrt{10}}{40}-\dfrac{3\sqrt{10}}{40} = -\dfrac{\sqrt{10}}{20}$

14. $\mathbf{u} = \dfrac{6}{10}\,\mathbf{i}+\dfrac{8}{10}\,\mathbf{j} = \dfrac{3}{5}\mathbf{i}+\dfrac{4}{5}\,\mathbf{j}$; $\nabla f = \dfrac{y^2}{(x+y)^2}\,\mathbf{i}+\dfrac{x^2}{(x+y)^2}\,\mathbf{j}$; $\nabla f(2,-1) = \mathbf{i}+4\mathbf{j}$

$D_{\mathbf{u}}f(2,-1) = \dfrac{3}{5}+\dfrac{16}{5} = \dfrac{19}{5}$

15. $\mathbf{u} = (2\mathbf{i}+\mathbf{j})/\sqrt{5}$; $\nabla f = 2y(xy+1)\mathbf{i}+2x(xy+1)\mathbf{j}$; $\nabla f(3,2) = 28\mathbf{i}+42\mathbf{j}$

$D_{\mathbf{u}}f(3,2) = \dfrac{2(28)}{\sqrt{5}}+\dfrac{42}{\sqrt{5}} = \dfrac{98}{\sqrt{5}}$

16. $\mathbf{u} = -\mathbf{i}$; $\nabla f = 2x\tan y\mathbf{i}+x^2\sec^2 y\mathbf{j}$; $\nabla f(1/2,\pi/3) = \sqrt{3}\,\mathbf{i}+\mathbf{j}$; $D_{\mathbf{u}}f(1/2,\pi/3) = -\sqrt{3}$

17. $\mathbf{u} = \dfrac{1}{\sqrt{2}}\,\mathbf{j}+\dfrac{1}{\sqrt{2}}\,\mathbf{k}$; $\nabla F = 2xy^2(2z+1)^2\mathbf{i}+2x^2y(2z+1)^2\mathbf{j}+4x^2y^2(2z+1)\mathbf{k}$

$\nabla F(1,-1,1) = 18\mathbf{i}-18\mathbf{j}+12\mathbf{k}$; $D_{\mathbf{u}}F(1,-1,1) = -\dfrac{18}{\sqrt{2}}+\dfrac{12}{\sqrt{2}} = -\dfrac{6}{\sqrt{2}} = -3\sqrt{2}$

18. $\mathbf{u} = \dfrac{1}{\sqrt{6}}\,\mathbf{i}-\dfrac{2}{\sqrt{6}}\,\mathbf{j}+\dfrac{1}{\sqrt{6}}\,\mathbf{k}$; $\nabla F = \dfrac{2x}{z^2}\,\mathbf{i}-\dfrac{2y}{z^2}\,\mathbf{j}+\dfrac{2y^2-2x^2}{z^3}\,\mathbf{k}$; $\nabla F(2,4,-1) = 4\mathbf{i}-8\mathbf{j}-24\mathbf{k}$

$D_{\mathbf{u}}F(2,4,-1) = \dfrac{4}{\sqrt{6}}-\dfrac{16}{\sqrt{6}}-\dfrac{24}{\sqrt{6}} = -6\sqrt{6}$

19. $\mathbf{u} = -\mathbf{k}$; $\nabla F = \dfrac{xy}{\sqrt{x^2y+2y^2z}}\,\mathbf{i}+\dfrac{x^2+4z}{2\sqrt{x^2y+2y^2z}}\,\mathbf{j}+\dfrac{y^2}{\sqrt{x^2y+2y^2z}}\,\mathbf{k}$

$\nabla F(-2,2,1) = -\mathbf{i}+\mathbf{j}+\mathbf{k}$; $D_{\mathbf{u}}F(-2,2,1) = -1$

20. $\mathbf{u} = -(4\mathbf{i}-4\mathbf{j}+2\mathbf{k})/\sqrt{36} = -\dfrac{2}{3}\,\mathbf{i}+\dfrac{2}{3}\,\mathbf{j}-\dfrac{1}{3}\,\mathbf{k}$; $\nabla F = 2\mathbf{i}-2y\mathbf{j}+2z\mathbf{k}$; $\nabla F(4,-4,2) = 2\mathbf{i}+8\mathbf{j}+4\mathbf{k}$

$D_{\mathbf{u}}F(4,-4,2) = -\dfrac{4}{3}+\dfrac{16}{3}-\dfrac{4}{3} = \dfrac{8}{3}$

21. $\mathbf{u} = (-4\mathbf{i}-\mathbf{j})/\sqrt{17}$; $\nabla f = 2(x-y)\mathbf{i}-2(x-y)\mathbf{j}$; $\nabla f(4,2) = 4\mathbf{i}-4\mathbf{j}$; $\quad D_{\mathbf{u}}F(4,2) = -\dfrac{16}{\sqrt{17}}\dfrac{4}{\sqrt{17}} = -\dfrac{12}{\sqrt{17}}$

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22. $\mathbf{u} = (-2\mathbf{i} + 5\mathbf{j})/\sqrt{29}$; $\nabla f = (3x^2 - 5y)\mathbf{i} - (5x - 2y)\mathbf{j}$; $\nabla f(1,1) = -2\mathbf{i} - 3\mathbf{j}$;
$D_{\mathbf{u}} f(1,1) = \dfrac{4}{\sqrt{29}} - \dfrac{15}{\sqrt{29}} = -\dfrac{11}{\sqrt{29}}$

23. $\nabla f = 2e^{2x}\sin y\mathbf{i} + e^{2x}\cos y\mathbf{j}$; $\nabla f(0,\pi/4) = \sqrt{2}\,\mathbf{i} + \dfrac{\sqrt{2}}{2}\,\mathbf{j}$
The maximum $D_{\mathbf{u}}$ is $[(\sqrt{2}\,)^2 + (\sqrt{2}/2)^2]^{1/2} = \sqrt{5/2}$ in the direction $\sqrt{2}\,\mathbf{i} + (\sqrt{2}/2)\mathbf{j}$.

24. $\nabla f = (xye^{x-y} + ye^{x-y})\mathbf{i} + (-xye^{x-y} + xe^{x-y})\mathbf{j}$; $\nabla f(5,5) = 30\mathbf{i} - 20\mathbf{j}$
The maximum $D_{\mathbf{u}}$ is $[30^2 + (-20)^2]^{1/2} = 10\sqrt{13}$ in the direction $30\mathbf{i} - 20\mathbf{j}$.

25. $\nabla F = (2x + 4z)\mathbf{i} + 2z^2\mathbf{j} + (4x + 4yz)\mathbf{k}$; $\nabla F(1,2,-1) = -2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$
The maximum $D_{\mathbf{u}}$ is $[(-2)^2 + 2^2 + (-4)^2]^{1/2} = 2\sqrt{6}$ in the direction $-2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

26. $\nabla F = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; $\nabla F(3,1,-5) = -5\mathbf{i} - 15\mathbf{j} + 3\mathbf{k}$
The maximum $D_{\mathbf{u}}$ is $[(-5)^2 + (-15)^2 + 3^2]^{1/2} = \sqrt{259}$ in the direction $-5\mathbf{i} - 15\mathbf{j} + 3\mathbf{k}$.

27. $\nabla f = 2x\sec^2(x^2 + y^2)\mathbf{i} + 2y\sec^2(x^2 + y^2)\mathbf{j}$;
$\nabla f(\sqrt{\pi/6}\,, \sqrt{\pi/6}\,) = 2\sqrt{\pi/6}\sec^2(\pi/3)(\mathbf{i} + \mathbf{j}) = 8\sqrt{\pi/6}\,(\mathbf{i} + \mathbf{j})$
The minimum $D_{\mathbf{u}}$ is $-8\sqrt{\pi/6}\,(1^2 + 1^2)^{1/2} = -8\sqrt{\pi/3}$ in the direction $-(\mathbf{i} + \mathbf{j})$.

28. $\nabla f = 3x^2\mathbf{i} - 3y^2\mathbf{j}$; $\nabla f(2,-2) = 12\mathbf{i} - 12\mathbf{j} = 12(\mathbf{i} - \mathbf{j})$
The minimum $D_{\mathbf{u}}$ is $-12[1^2 + (-1)^2]^{1/2} = -12\sqrt{2}$ in the direction $-(\mathbf{i} - \mathbf{j}) = -\mathbf{i} + \mathbf{j}$.

29. $\nabla F = \dfrac{\sqrt{z}\,e^y}{2\sqrt{x}}\,\mathbf{i} + \sqrt{xz}\,e^y\mathbf{j} + \dfrac{\sqrt{x}}{2\sqrt{z}}\,\mathbf{k}$; $\nabla F(16,0,9) = \dfrac{3}{8}\,\mathbf{i} + 12\mathbf{j} + \dfrac{2}{3}\,\mathbf{k}$. The minimum $D_{\mathbf{u}}$ is
$-[(3/8)^2 + 12^2 + (2/3)^2]^{1/2} = -\sqrt{83{,}281}/24$ in the direction $-\dfrac{3}{8}\,\mathbf{i} - 12\mathbf{j} - \dfrac{2}{3}\,\mathbf{k}$.

30. $\nabla F = \dfrac{1}{x}\,\mathbf{i} + \dfrac{1}{y}\,\mathbf{j} - \dfrac{1}{z}\,\mathbf{k}$; $\nabla F(1/2, 1/6, 1/3) = 2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$
The minimum $D_{\mathbf{u}}$ is $-[2^2 + 6^2 + (-3)^2]^{1/2} = -7$ in the direction $-2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$.

31. Using implicit differentiation on $2x^2 + y^2 = 9$ we find $y' = -2x/y$. At $(2,1)$ the slope of the tangent line is $-2(2)/1 = -4$. Thus, $\mathbf{u} = \pm(\mathbf{i} - 4\mathbf{j})/\sqrt{17}$. Now, $\nabla f = \mathbf{i} + 2y\mathbf{j}$ and $\nabla f(3,4) = \mathbf{i} + 8\mathbf{j}$. Thus,
$D_{\mathbf{u}} = \pm(1/\sqrt{17} - 32\sqrt{17}\,) = \pm 31/\sqrt{17}$.

32. $\nabla f = (2x + y - 1)\mathbf{i} + (x + 2y)\mathbf{j}$; $D_{\mathbf{u}} f(x,y) = \dfrac{2x + y - 1}{\sqrt{2}} + \dfrac{x + 2y}{\sqrt{2}} = \dfrac{3x + 3y - 1}{\sqrt{2}}$

Solving $(3x + 3y - 1)/\sqrt{2} = 0$ we see that $D_{\mathbf{u}}$ is 0 for all points on the line $3x + 3y = 1$.

33. **(a)** Vectors perpendicular to $4\mathbf{i} + 3\mathbf{j}$ are $\pm(3\mathbf{i} - 4\mathbf{j})$. Take $\mathbf{u} = \pm\left(\dfrac{3}{5}\,\mathbf{i} - \dfrac{4}{5}\,\mathbf{j}\right)$.

(b) $\mathbf{u} = (4\mathbf{i} + 3\mathbf{j})/\sqrt{16 + 9} = \dfrac{4}{5}\,\mathbf{i} + \dfrac{3}{5}\,\mathbf{j}$

(c) $\mathbf{u} = -\dfrac{4}{5}\,\mathbf{i} - \dfrac{3}{5}\,\mathbf{j}$

34. $D_{-\mathbf{u}} f(a,b) = \nabla f(a,b)\cdot(-\mathbf{u}) = -\nabla f(a,b)\cdot\mathbf{u} = -D_{\mathbf{u}} f(a,b) = -6$

35. **(a)** $\nabla f = (3x^2 - 6xy^2)\mathbf{i} + (-6x^2y + 3y^2)\mathbf{j}$
$D_{\mathbf{u}} f(x,y) = \dfrac{3(3x^2 - 6xy^2)}{\sqrt{10}} + \dfrac{-6x^2y + 3y^2}{\sqrt{10}} = \dfrac{9x^2 - 18xy^2 - 6x^2y + 3y^2}{\sqrt{10}}$

(b) $F(x,y) = \dfrac{3}{\sqrt{10}}(3x^2 - 6xy^2 - 2x^2y + y^2)$; $\nabla F = \dfrac{3}{\sqrt{10}}[(6x - 6y^2 - 4xy)\mathbf{i} + (-12xy - 2x^2 + 2y)\mathbf{j}]$

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$$D_{\mathbf{u}}F(x,y) = \left(\frac{3}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right)(6x - 6y^2 - 4xy) + \left(\frac{1}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right)(-12xy - 2x^2 + 2y)$$
$$= \frac{9}{5}(3x - 3y^2 - 2xy) + \frac{3}{5}(-6xy - x^2 + y) = \frac{1}{5}(27x - 27y^2 - 36xy - 3x^2 + 3y)$$

36. $\nabla U = \dfrac{Gmx}{(x^2+y^2)^{3/2}}\,\mathbf{i} + \dfrac{Gmy}{(x^2+y^2)^{3/2}}\,\mathbf{j} = \dfrac{Gm}{(x^2+y^2)^{3/2}}(x\mathbf{i} + y\mathbf{j})$

The maximum and minimum values of $D_{\mathbf{u}}U(x,y)$ are obtained when $\mathbf{u}$ is in the directions ∇U and $-\nabla U$, respectively. Thus, at a point (x,y), not $(0,0)$, the directions of maximum and minimum increase in U are $x\mathbf{i}+y\mathbf{j}$ and $-x\mathbf{i}-y\mathbf{j}$, respectively. A vector at (x,y) in the direction $\pm(x\mathbf{i}+y\mathbf{j})$ lies on a line through the origin.

37. $\nabla f = (3x^2 - 12)\mathbf{i} + (2y - 10)\mathbf{j}$. Setting $|\nabla f| = [(3x^2-12)^2 + (2y-10)^2]^{1/2} = 0$, we obtain $3x^2 - 12 = 0$ and $2y - 10 = 0$. The points where $|\nabla f| = 0$ are $(2,5)$ and $(-2,5)$.

38. Let $\nabla f(a,b) = \alpha\mathbf{i} + \beta\mathbf{j}$. Then

$$D_{\mathbf{u}}f(a,b) = \nabla f(a,b)\cdot\mathbf{u} = \frac{5}{13}\alpha - \frac{12}{13}\beta = 7 \quad \text{and} \quad D_{\mathbf{v}}f(a,b) = \nabla f(a,b)\cdot\mathbf{v} = \frac{5}{13}\alpha - \frac{12}{13}\beta = 3.$$

Solving for α and β, we obtain $\alpha = 13$ and $\beta = -13/6$. Thus, $\nabla f(a,b) = 13\mathbf{i} - (13/6)\mathbf{j}$.

39. $\nabla T = 4x\mathbf{i} + 2y\mathbf{j}$; $\nabla T(4,2) = 16\mathbf{i} + 4\mathbf{j}$. The minimum change in temperature (that is, the maximum decrease in temperature) is in the direction $-\nabla T(4,3) = -16\mathbf{i} - 4\mathbf{j}$.

40. Let $x(t)\mathbf{i} + y(t)\mathbf{j}$ be the vector equation of the path. At (x,y) on this curve, the direction of a tangent vector is $x'(t)\mathbf{i} + y'(t)\mathbf{j}$. Since we want the direction of motion to be $-\nabla T(x,y)$, we have $x'(t)\mathbf{i} + y'(t)\mathbf{j} = -\nabla T(x,y) = 4x\mathbf{i} + 2y\mathbf{j}$. Separating variables in $dx/dt = 4x$, we obtain $dx/x = 4\,dt$, $\ln x = 4t + c_1$, and $x = C_1e^{4t}$. Separating variables in $dy/dt = 2y$, we obtain $dy/y = 2\,dt$, $\ln y = 2t + c_2$, and $y = C_2e^{2t}$. Since $x(0) = 4$ and $y(0) = 2$, we have $x = 4e^{4t}$ and $y = 2e^{2t}$. The equation of the path is $4e^{4t}\mathbf{i} + 2e^{2t}\mathbf{j}$ for $t \geq 0$, or eliminating the parameter, $x = y^2$, $y \geq 0$.

41. Let $x(t)\mathbf{i} + y(t)\mathbf{j}$ be the vector equation of the path. At (x,y) on this curve, the direction of a tangent vector is $x'(t)\mathbf{i} + y'(t)\mathbf{j}$. Since we want the direction of motion to be $\nabla T(x,y)$, we have $x'(t)\mathbf{i} + y'(t)\mathbf{j} = \nabla T(x,y) = -4x\mathbf{i} - 2y\mathbf{j}$. Separating variables in $dx/dt = -4x$ we obtain $dx/x = -4\,dt$, $\ln x = -4t + c_1$ and $x = C_1e^{-4t}$. Separating variables in $dy/dt = -2y$ we obtain $dy/y = -2\,dt$, $\ln y = -2t + c_2$ and $y = C_2e^{-2t}$. Since $x(0) = 3$ and $y(0) = 4$, we have $x = 3e^{-4t}$ and $y = 4e^{-2t}$. The equation of the path is $3e^{-4t}\mathbf{i} + 4e^{-2t}\mathbf{j}$, or eliminating the parameter, $16x = 3y^2$, $y \geq 0$.

42. Substituting $x = 0$, $y = 0$, $z = 1$, and $T = 500$ into $T = k/(x^2+y^2+z^2)$ we see that $k = 500$ and $T(x,y,z) = 500/(x^2+y^2+z^2)$.

(a) $\mathbf{u} = \dfrac{1}{3}\langle 1,-2,-2\rangle = \dfrac{1}{3}\,\mathbf{i} - \dfrac{2}{3}\,\mathbf{j} - \dfrac{2}{3}\,\mathbf{k}$

$$\nabla T = -\frac{1000x}{(x^2+y^2+z^2)^2}\,\mathbf{i} - \frac{1000y}{(x^2+y^2+z^2)^2}\,\mathbf{j} - \frac{1000z}{(x^2+y^2+z^2)^2}\,\mathbf{k}$$

$$\nabla T(2,3,3) = -\frac{500}{121}\,\mathbf{i} - \frac{750}{121}\,\mathbf{j} - \frac{750}{121}\,\mathbf{k}$$

$$D_{\mathbf{u}}T(2,3,3) = \frac{1}{3}\left(-\frac{500}{121}\right) - \frac{2}{3}\left(-\frac{750}{121}\right) - \frac{2}{3}\left(-\frac{750}{121}\right) = \frac{2500}{363}$$

(b) The direction of maximum increase is $\nabla T(2,3,3) = -\dfrac{500}{121}\,\mathbf{i} - \dfrac{750}{121}\,\mathbf{j} - \dfrac{750}{121}\,\mathbf{k} = \dfrac{250}{121}(-2\mathbf{i} - 3\mathbf{j} - 3\mathbf{k})$.

(c) The maximum rate of change of T is $|\nabla T(2,3,3)| = \dfrac{250}{121}\sqrt{4+9+9} = \dfrac{250\sqrt{22}}{121}$.

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43. Since $\nabla f = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$, we have $\partial f/\partial x = 3x^2 + y^3 + ye^{xy}$. Integrating, we obtain $f(x,y) = x^3 + xy^3 + e^{xy} + g(y)$. Then $f_y = 3xy^2 + xe^{xy} + g'(y) = -2y^2 + 3xy^2 + xe^{xy}$. Thus, $g'(y) = -2y^2$, $g(y) = -\frac{2}{3}y^3 + c$, and $f(x,y) = x^3 + xy^3 + e^{xy} - \frac{2}{3}y^3 + C$.

44. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$. $D_{\mathbf{v}}f = (f_x\mathbf{i} + f_y\mathbf{j}) \cdot \mathbf{v} = v_1f_x + v_2f_y$

$$D_{\mathbf{u}}D_{\mathbf{v}}f = \left[\frac{\partial}{\partial x}(v_1f_x + v_2f_y)\mathbf{i} + \frac{\partial}{\partial y}(v_1f_x + v_2f_y)\mathbf{j}\right] \cdot \mathbf{u} = [(v_1f_{xx} + v_2f_{yx})\mathbf{i} + (v_1f_{xy} + v_2f_{yy})\mathbf{j}] \cdot \mathbf{u}$$
$$= u_1v_1f_{xx} + u_1v_2f_{yx} + u_2v_1f_{xy} + u_2v_2f_{yy}$$

$$D_{\mathbf{u}}f = (f_x\mathbf{i} + f_y\mathbf{j}) \cdot \mathbf{u} = u_1f_x + u_2f_y$$

$$D_{\mathbf{v}}D_{\mathbf{u}}f = \left[\frac{\partial}{\partial x}(u_1f_x + u_2f_y)\mathbf{i} + \frac{\partial}{\partial y}(u_1f_x + u_2f_y)\mathbf{j}\right] \cdot \mathbf{v} = [(u_1f_{xx} + u_2f_{yx})\mathbf{i} + (u_1f_{xy} + u_2f_{yy})\mathbf{j}] \cdot \mathbf{v}$$
$$= u_1v_1f_{xx} + u_2v_1f_{yx} + u_1v_2f_{xy} + u_2v_2f_{yy}$$

Since the second partial derivatives are continuous, $f_{xy} = f_{yx}$ and $D_{\mathbf{u}}D_{\mathbf{v}}f = D_{\mathbf{v}}D_{\mathbf{u}}f$. [Note that this result is a generalization of $f_{xy} = f_{yx}$ since $D_{\mathbf{i}}D_{\mathbf{j}}f = f_{yx}$ and $D_{\mathbf{j}}D_{\mathbf{i}}f = f_{xy}$.]

45. $\nabla(cf) = \dfrac{\partial}{\partial x}(cf)\mathbf{i} + \dfrac{\partial}{\partial y}(cf)\mathbf{j} = cf_x\mathbf{i} + cf_y\mathbf{j} = c(f_x\mathbf{i} + f_y\mathbf{j}) = c\nabla f$

46. $\nabla(f+g) = (f_x + g_x)\mathbf{i} + (f_y + g_y)\mathbf{j} = (f_x\mathbf{i} + f_y\mathbf{j}) + (g_x\mathbf{i} + g_y\mathbf{j}) = \nabla f + \nabla g$

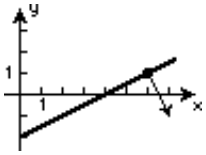
47. $\nabla(fg) = (fg_x + f_xg)\mathbf{i} + (fg_y + f_yg)\mathbf{j} = f(g_x\mathbf{i} + g_y\mathbf{j}) + g(f_x\mathbf{i} + f_y\mathbf{j}) = f\nabla g + g\nabla f$

48. $\nabla(f/g) = [(gf_x - fg_x)/g^2]\mathbf{i} + [(gf_y - fg_y)/g^2]\mathbf{j} = g(f_x\mathbf{i} + f_y\mathbf{j})/g^2 - f(g_x\mathbf{i} + g_y\mathbf{j})/g^2$
$= g\nabla f/g^2 - f\nabla g/g^2 = (g\nabla f - f\nabla g)/g^2$

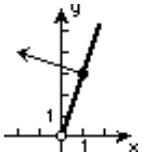
49. $\nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix} = \left(\dfrac{\partial f_3}{\partial y} - \dfrac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial f_1}{\partial z} - \dfrac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y}\right)\mathbf{k}$

Exercises 9.6

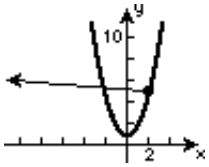
1. Since $f(6,1) = 4$, the level curve is $x - 2y = 4$.
$\nabla f = \mathbf{i} - 2\mathbf{j}$; $\nabla f(6,1) = \mathbf{i} - 2\mathbf{j}$



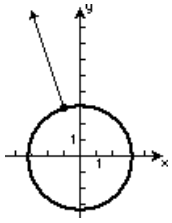
2. Since $f(1,3) = 5$, the level curve is $y + 2x = 5x$ or $y = 3x$, $x \neq 0$.
$\nabla f = -\dfrac{y}{x^2}\,\mathbf{i} + \dfrac{1}{x}\,\mathbf{j}$; $\nabla f(1,3) = -3\mathbf{i} + \mathbf{j}$



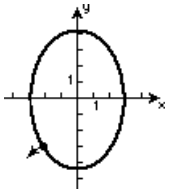
3. Since $f(2,5) = 1$, the level curve is $y = x^2 + 1$.
$\nabla f = -2x\mathbf{i} + \mathbf{j}$; $\nabla f(2,5) = -10\mathbf{i} + \mathbf{j}$



4. Since $f(-1,3) = 10$, the level curve is $x^2 + y^2 = 10$.
$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$; $\nabla f(-1,3) = -2\mathbf{i} + 6\mathbf{j}$



5. Since $f(-2,-3) = 2$, the level curve is $x^2/4 + y^2/9 = 2$
or $x^2/8 + y^2/18 = 1$. $\nabla f = \dfrac{x}{2}\,\mathbf{i} + \dfrac{2y}{9}\,\mathbf{j}$; $\nabla f(-2,-3) = -\mathbf{i} - \dfrac{2}{3}\mathbf{j}$



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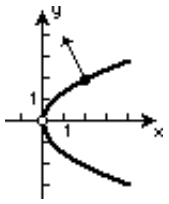
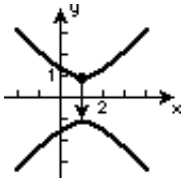
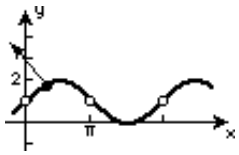
6. Since $f(2,2) = 2$, the level curve is $y^2 = 2x$, $x \neq 0$. $\nabla f = -\dfrac{y^2}{x^2}\,\mathbf{i} + \dfrac{2y}{x}\mathbf{j}$; $\nabla f(2,2) = -\mathbf{i} + 2\mathbf{j}$



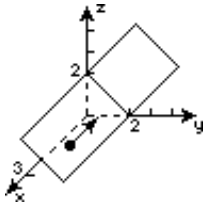
7. Since $f(1,1) = -1$, the level curve is $(x-1)^2 - y^2 = -1$ or $y^2 - (x-1)^2 = 1$. $\nabla f = 2(x-1)\mathbf{i} - 2y\mathbf{j}$; $\nabla f(1,1) = -2\mathbf{j}$



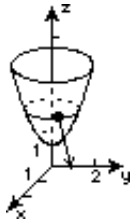
8. Since $f(\pi/6, 3/2) = 1$, the level curve is $y - 1 = \sin x$ or $y = 1 + \sin x$, $\sin x \neq 0$. $\nabla f = \dfrac{-(y-1)\cos x}{\sin^2 x}\,\mathbf{i} + \dfrac{1}{\sin x}\,\mathbf{j}$; $\nabla f(\pi/6, 3/2) = -\sqrt{3}\,\mathbf{i} + 2\mathbf{j}$



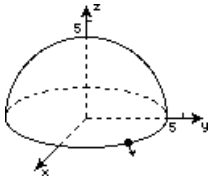
9. Since $F(3,1,1) = 2$, the level surface is $y + z = 2$. $\nabla F = \mathbf{j} + \mathbf{k}$; $\nabla F(3,1,1) = \mathbf{j} + \mathbf{k}$



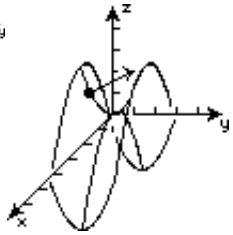
10. Since $F(1,1,3) = -1$, the level surface is $x^2 + y^2 - z = -1$ or $z = 1 + x^2 + y^2$. $\nabla F = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$; $\nabla F(1,1,3) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$



11. Since $F(3,4,0) = 5$, the level surface is $x^2 + y^2 + z^2 = 25$. $\nabla F = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}$; $\nabla F(3,4,0) = \dfrac{3}{5}\,\mathbf{i} + \dfrac{4}{5}\,\mathbf{j}$



12. Since $F(0,-1,1) = 0$, the level surface is $x^2 - y^2 + z = 0$ or $z = y^2 - x^2$. $\nabla F = 2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$; $\nabla F(0,-1,1) = 2\mathbf{j} + \mathbf{k}$



13. $F(x,y,z) = x^2 + y^2 - z$; $\nabla F = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$. We want $\nabla F = c(4\mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k})$ or $2x = 4c$, $2y = c$, $-1 = c/2$. From the third equation $c = -2$. Thus, $x = -4$ and $y = -1$. Since $z = x^2 + y^2 = 16 + 1 = 17$, the point on the surface is $(-4, -1, -17)$.

14. $F(x,y,z) = x^3 + y^2 + z$; $\nabla F = 3x^2\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$. We want $\nabla F = c(27\mathbf{i} + 8\mathbf{j} + \mathbf{k})$ or $3x^2 = 27c$, $2y = 8c$, $1 = c$. From $c = 1$ we obtain $x = \pm 3$ and $y = 4$. Since $z = 15 - x^3 - y^2 = 15 - (\pm 3)^3 - 16 = -1 \mp 27$, the points on the surface are $(3, 4, -28)$ and $(-3, 4, 26)$.

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15. $F(x,y,z) = x^2 + y^2 + z^2$; $\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. $\nabla F(-2,2,1) = -4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. The equation of the tangent plane is $-4(x+2) + 4(y-2) + 2(z-1) = 0$ or $-2x + 2y + z = 9$.

16. $F(x,y,z) = 5x^2 - y^2 + 4z^2$; $\nabla F = 10x\mathbf{i} - 2y\mathbf{j} + 8z\mathbf{k}$; $\nabla F(2,4,1) = 20\mathbf{i} - 8\mathbf{j} + 8\mathbf{k}$. The equation of the tangent plane is $20(x-2) - 8(y-4) + 8(z-1) = 0$ or $5x - 2y + 2z = 4$.

17. $F(x,y,z) = x^2 - y^2 - 3z^2$; $\nabla F = 2x\mathbf{i} - 2y\mathbf{j} - 6z\mathbf{k}$; $\nabla F(6,2,3) = 12\mathbf{i} - 4\mathbf{j} - 18\mathbf{k}$. The equation of the tangent plane is $12(x-6) - 4(y-2) - 18(z-3) = 0$ or $6x - 2y - 9z = 5$.

18. $F(x,y,z) = xy + yz + zx$; $\nabla F = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (y+x)\mathbf{k}$; $\nabla F(1,-3,-5) = -8\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$. The equation of the tangent plane is $-8(x-1) - 4(y+3) - 2(z+5) = 0$ or $4x + 2y + z = -7$.

19. $F(x,y,z) = x^2 + y^2 + z$; $\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$; $\nabla F(3,-4,0) = 6\mathbf{i} - 8\mathbf{j} + \mathbf{k}$. The equation of the tangent plane is $6(x-3) - 8(y+4) + z = 0$ or $6x - 8y + z = 50$.

20. $F(x,y,z) = xz$; $\nabla F = z\mathbf{i} + x\mathbf{k}$; $\nabla F(2,0,3) = 3\mathbf{i} + 2\mathbf{k}$. The equation of the tangent plane is $3(x-2) + 2(z-3) = 0$ or $3x + 2z = 12$.

21. $F(x,y,z) = \cos(2x+y) - z$; $\nabla F = -2\sin(2x+y)\mathbf{i} - \sin(2x+y)\mathbf{j} - \mathbf{k}$; $\nabla F(\pi/2, \pi/4, -1/\sqrt{2}) = \sqrt{2}\,\mathbf{i} + \dfrac{\sqrt{2}}{2}\,\mathbf{j} - \mathbf{k}$.

The equation of the tangent plane is $\sqrt{2}\left(x - \dfrac{\pi}{2}\right) + \dfrac{\sqrt{2}}{2}\left(y - \dfrac{\pi}{4}\right) - \left(z + \dfrac{1}{\sqrt{2}}\right) = 0$,

$2\left(x - \dfrac{\pi}{2}\right) + \left(y - \dfrac{\pi}{4}\right) - \sqrt{2}\left(z + \dfrac{1}{\sqrt{2}}\right) = 0$, or $2x + y - \sqrt{2}\,z = \dfrac{5\pi}{4} + 1$.

22. $F(x,y,z) = x^2y^3 + 6z$; $\nabla F = 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j} + 6\mathbf{k}$; $\nabla F(2,1,1) = 4\mathbf{i} + 12\mathbf{j} + 6\mathbf{k}$. The equation of the tangent plane is $4(x-2) + 12(y-1) + 6(z-1) = 0$ or $2x + 6y + 3z = 13$.

23. $F(x,y,z) = \ln(x^2+y^2) - z$; $\nabla F = \dfrac{2x}{x^2+y^2}\,\mathbf{i} + \dfrac{2y}{x^2+y^2}\,\mathbf{j} - \mathbf{k}$; $\nabla F(1/\sqrt{2}, 1/\sqrt{2}, 0) = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j} - \mathbf{k}$.

The equation of the tangent plane is $\sqrt{2}\left(x - \dfrac{1}{\sqrt{2}}\right) + \sqrt{2}\left(y - \dfrac{1}{\sqrt{2}}\right) - (z - 0) = 0$,

$2\left(x - \dfrac{1}{\sqrt{2}}\right) + 2\left(y - \dfrac{1}{\sqrt{2}}\right) - \sqrt{2}\,z = 0$, or $2x + 2y - \sqrt{2}\,z = 2\sqrt{2}$.

24. $F(x,y,z) = 8e^{-2y}\sin 4x - z$; $\nabla F = 32e^{-2y}\cos 4x\mathbf{i} - 16e^{-2y}\sin 4x\mathbf{j} - \mathbf{k}$; $\nabla F(\pi/24, 0, 4) = 16\sqrt{3}\,\mathbf{i} - 8\mathbf{j} - \mathbf{k}$.
The equation of the tangent plane is

$$16\sqrt{3}(x - \pi/24) - 8(y - 0) - (z - 4) = 0 \quad \text{or} \quad 16\sqrt{3}\,x - 8y - z = \frac{2\sqrt{3}\,\pi}{3} - 4.$$

25. The gradient of $F(x,y,z) = x^2 + y^2 + z^2$ is $\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, so the normal vector to the surface at (x_0, y_0, z_0) is $2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k}$. A normal vector to the plane $2x + 4y + 6z = 1$ is $2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. Since we want the tangent plane to be parallel to the given plane, we find c so that $2x_0 = 2c$, $2y_0 = 4c$, $2z_0 = 6c$ or $x_0 = c$, $y_0 = 2c$, $z_0 = 3c$. Now, (x_0, y_0, z_0) is on the surface, so $c^2 + (2c)^2 + (3c)^2 = 14c^2 = 7$ and $c = \pm 1/\sqrt{2}$. Thus, the points on the surface are $(\sqrt{2}/2, \sqrt{2}, 3\sqrt{2}/2)$ and $-\sqrt{2}/2, -\sqrt{2}, -3\sqrt{2}/2)$.

26. The gradient of $F(x,y,z) = x^2 - 2y^2 - 3z^2$ is $\nabla F(x,y,z) = 2x\mathbf{i} - 4y\mathbf{j} - 6z\mathbf{k}$, so a normal vector to the surface at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = 2x_0\mathbf{i} - 4y_0\mathbf{j} - 6z_0\mathbf{k}$. A normal vector to the plane $8x + 4y + 6z = 5$ is $8\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. Since we want the tangent plane to be parallel to the given plane, we find c so that $2x_0 = 8c$, $-4y_0 = 4c$, $-6z_0 = 6c$ or $x_0 = 4c$, $y_0 = -c$, $z_0 = -c$. Now, (x_0, y_0, z_0) is on the surface, so $(4c)^2 - 2(-c)^2 - 3(-c)^2 = 11c^2 = 33$ and $c = \pm\sqrt{3}$. Thus, the points on the surface are $(4\sqrt{3}, -\sqrt{3}, -\sqrt{3})$ and $(-4\sqrt{3}, \sqrt{3}, \sqrt{3})$.

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27. The gradient of $F(x, y, z) = x^2 + 4x + y^2 + z^2 - 2z$ is $\nabla F = (2x + 4)\mathbf{i} + 2y\mathbf{j} + (2z - 2)\mathbf{k}$, so a normal to the surface at (x_0, y_0, z_0) is $(2x_0 + 4)\mathbf{i} + 2y_0\mathbf{j} + (2z_0 - 2)\mathbf{k}$. A horizontal plane has normal $c\mathbf{k}$ for $c \neq 0$. Thus, we want $2x_0 + 4 = 0$, $2y_0 = 0$, $2z_0 - 2 = c$ or $x_0 = -2$, $y_0 = 0$, $z_0 = c + 1$. since (x_0, y_0, z_0) is on the surface, $(-2)^2 + 4(-2) + (c + 1)^2 - 2(c + 1) = c^2 - 5 = 11$ and $c = \pm 4$. The points on the surface are $(-2, 0, 5)$ and $(-2, 0, -3)$.

28. The gradient of $F(x, y, z) = x^2 + 3y^2 + 4z^2 - 2xy$ is $\nabla F = (2x - 2y)\mathbf{i} + (6y - 2x)\mathbf{j} + 8z\mathbf{k}$, so a normal to the surface at (x_0, y_0, z_0) is $2(x_0 - y_0)\mathbf{i} + 2(3y_0 - x_0)\mathbf{j} + 8z_0\mathbf{k}$.

(a) A normal to the xz plane is $c\mathbf{j}$ for $c \neq 0$. Thus, we want $2(x_0 - y_0) = 0$, $2(3y_0 - x_0) = c$, $8z_0 = 0$ or $x_0 = y_0$, $3y_0 - x_0 = c/2$, $z_0 = 0$. Solving the first two equations, we obtain $x_0 = y_0 = c/4$. Since (x_0, y_0, z_0) is on the surface, $(c/4)^2 + 3(c/4)^2 + 4(0)^2 - 2(c/4)(c/4) = 2c^2/16 = 16$ and $c = \pm 16/\sqrt{2}$. Thus, the points on the surface are $(4/\sqrt{2}, 4/\sqrt{2}, 0)$ and $(-4/\sqrt{2}, -4/\sqrt{2}, 0)$.

(b) A normal to the yz-plane is $c\mathbf{i}$ for $c \neq 0$. Thus, we want $2(x_0 - y_0) = c$, $2(3y_0 - x_0) = 0$, $8z_0 = 0$ or $x_0 - y_0 = c/2$, $x_0 = 3y_0$, $z_0 = 0$. Solving the first two equations, we obtain $x_0 = 3c/4$ and $y_0 = c/4$. Since (x_0, y_0, z_0) is on the surface, $(3c/4)^2 + 3(c/4)^2 + 4(0)^2 - 2(3c/4)(c/4) = 6c^2/16 = 16$ and $c = \pm 16\sqrt{6}$. Thus, the points on the surface are $(12/\sqrt{6}, 4/\sqrt{6}, 0)$ on the surface are $(12/\sqrt{6}, 4/\sqrt{6}, 0)$ and $(-12/\sqrt{6}, -4/\sqrt{6}, 0)$.

(c) A normal to the xy-plane is $c\mathbf{k}$ for $c \neq 0$. Thus, we want $2(x_0 - y_0) = 0$, $2(3y_0 - x_0) = 0$, $8z_0 = c$ or $x_0 = y_0$, $3y_0 - x_0 = 0$, $z_0 = c/8$. Solving the first two equations, we obtain $x_0 = y_0 = 0$. Since (x_0, y_0, z_0) is on the surface, $0^2 + 3(0)^2 + 4(c/8)^2 - 2(0)(0) = c^2/16 = 16$ and $c = \pm 16$. Thus, the points on the surface are $(0, 0, 2)$ and $(0, 0, -2)$.

29. If (x_0, y_0, z_0) is on $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, then $x_0^2/a^2 + y_0^2/b^2 + z_0^2/c^2 = 1$ and (x_0, y_0, z_0) is on the plane $xx_0/a^2 + yy_0/b^2 + zz_0/c^2 = 1$. A normal to the surface at (x_0, y_0, z_0) is

$$\nabla F(x_0, y_0, z_0) = (2x_0/a^2)\mathbf{i} + (2y_0/b^2)\mathbf{j} + (2z_0/c^2)\mathbf{k}.$$

A normal to the plane is $(x_0/a^2)\mathbf{i} + (y_0/b^2)\mathbf{j} + (z_0/c^2)\mathbf{k}$. Since the normal to the surface is a multiple of the normal to the plane, the normal vectors are parallel and the plane is tangent to the surface.

30. If (x_0, y_0, z_0) is on $x^2/a^2 - y^2/b^2 + z^2/c^2 = 1$, then $x_0^2/a^2 - y_0^2/b^2 + z_0^2/c^2 = 1$ and (x_0, y_0, z_0) is on the plane $xx_0/a^2 - yy_0/b^2 + zz_0/c^2 = 1$. A normal to the surface at (x_0, y_0, z_0) is

$$\nabla F(x_0, y_0, z_0) = (2x_0/a^2)\mathbf{i} - (2y_0/b^2)\mathbf{j} + (2z_0/c^2)\mathbf{k}.$$

A normal to the plane is $(x_0/a^2)\mathbf{i} - (y_0/b^2)\mathbf{j} + (z_0/c^2)\mathbf{k}$. Since the normal to the surface is a multiple of the normal to the plane, the normal vectors are parallel, and the plane is tangent to the surface.

31. Let $F(x, y, z) = x^2 + y^2 - z^2$. Then $\nabla F = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$ and a normal to the surface at (x_0, y_0, z_0) is $x_0\mathbf{i} + y_0\mathbf{j} - z_0\mathbf{k}$. An equation of the tangent plane at (x_0, y_0, z_0) is $x_0(x - x_0) + y_0(y - y_0) - z_0(z - z_0) = 0$ or $x_0x + y_0y - z_0z = x_0^2 + y_0^2 - z_0^2$. Since (x_0, y_0, z_0) is on the surface, $z_0^2 = x_0^2 + y_0^2$ and $x_0^2 + y_0^2 - z_0^2 = 0$. Thus, the equation of the tangent plane is $x_0x + y_0y - z_0z = 0$, which passes through the origin.

32. Let $F(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$. Then $\nabla F = \dfrac{1}{2\sqrt{x}}\,\mathbf{i} + \dfrac{1}{2\sqrt{y}}\,\mathbf{j} + \dfrac{1}{2\sqrt{z}}\,\mathbf{k}$ and a normal to the surface at (x_0, y_0, z_0) is $\dfrac{1}{2\sqrt{x_0}}\,\mathbf{i} + \dfrac{1}{2\sqrt{y_0}}\,\mathbf{j} + \dfrac{1}{2\sqrt{z_0}}\,\mathbf{k}$. An equation of the tangent plane at (x_0, y_0, z_0) is

$$\frac{1}{2\sqrt{x_0}}(x - x_0) + \frac{1}{2\sqrt{y_0}}(y - y_0) + \frac{1}{2\sqrt{z_0}}(z - z_0) = 0$$

or

$$\frac{1}{\sqrt{x_0}}\,x + \frac{1}{\sqrt{y_0}}\,y + \frac{1}{\sqrt{z_0}}\,z = \sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{a}\,.$$

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The sum of the intercepts is $\sqrt{x_0}\sqrt{a}+\sqrt{y_0}\sqrt{a}+\sqrt{z_0}\sqrt{a}=(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})\sqrt{a}=\sqrt{a}\cdot\sqrt{a}=a$.

33. $F(x,y,z)=x^2+2y^2+z^2$; $\nabla F=2x\mathbf{i}+4y\mathbf{j}+2z\mathbf{k}$; $\nabla F(1,-1,1)=2\mathbf{i}-4\mathbf{j}+2\mathbf{k}$. Parametric equations of the line are $x=1+2t$, $y=-1-4t$, $z=1+2t$.

34. $F(x,y,z)=2x^2-4y^2-z$; $\nabla F=4x\mathbf{i}-8y\mathbf{j}-\mathbf{k}$; $\nabla F(3,-2,2)=12\mathbf{i}+16\mathbf{j}-\mathbf{k}$. Parametric equations of the line are $x=3+12t$, $y=-2+16t$, $z=2-t$.

35. $F(x,y,z)=4x^2+9y^2-z$; $\nabla F=8x\mathbf{i}+18y\mathbf{j}-\mathbf{k}$; $\nabla F(1/2,1/3,3)=4\mathbf{i}+6\mathbf{j}-\mathbf{k}$. Symmetric equations of the line are $\dfrac{x-1/2}{4}=\dfrac{y-1/3}{6}=\dfrac{z-3}{-1}$.

36. $F(x,y,z)=x^2+y^2-z^2$; $\nabla F=2x\mathbf{i}+2y\mathbf{j}-2z\mathbf{k}$; $\nabla F(3,4,5)=6\mathbf{i}+8\mathbf{j}-10\mathbf{k}$. Symmetric equations of the line are $\dfrac{x-3}{6}=\dfrac{y-4}{8}=\dfrac{z-5}{-10}$.

37. A normal to the surface at (x_0,y_0,z_0) is $\nabla F(x_0,y_0,z_0)=2x_0\mathbf{i}+2y_0\mathbf{j}+2z_0\mathbf{k}$. Parametric equations of the normal line are $x=x_0+2x_0t$, $y=y_0+2y_0t$, $z=z_0+2z_0t$. Letting $t=-1/2$, we see that the normal line passes through the origin.

38. The normal lines to $F(x,y,z)=0$ and $G(x,y,z)=0$ are $F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k}$ and $G_x\mathbf{i}+G_y\mathbf{j}+G_z\mathbf{k}$, respectively. These vectors are orthogonal if and only if their dot product is 0. Thus, the surfaces are orthogonal at P if and only if $F_xG_x+F_yG_y+F_zG_z=0$.

39. Let $F(x,y,z)=x^2+y^2+z^2-25$ and $G(x,y,z)=-x^2+y^2+z^2$. Then

$$F_xG_x+F_yG_y+F_zg_z=(2x)(-2x)+(2y)(2y)+(2z)(2z)=4(-x^2+y^2+z^2).$$

For (x,y,z) on both surfaces, $F(x,y,z)=G(x,y,z)=0$. Thus, $F_xG_x+F_yG_y+F_zG_z=4(0)=0$ and the surfaces are orthogonal at points of intersection.

40. Let $F(x,y,z)=x^2-y^2+z^2-4$ and $G(x,y,z)=1/xy^2-z$. Then

$$\begin{aligned}F_xG_x+F_yG_y+F_zG_z&=(2x)(-1/x^2y^2)+(-2y)(-2/xy^3)+(2z)(-1)\\&=-2/xy^2+4/xy^2-2z=2(1/xy^2-z).\end{aligned}$$

For (x,y,z) on both surfaces, $F(x,y,z)=G(x,y,z)=0$. Thus, $F_xG_x+F_yG_y+F_zG_z=2(0)$ and the surfaces are orthogonal at points of intersection.

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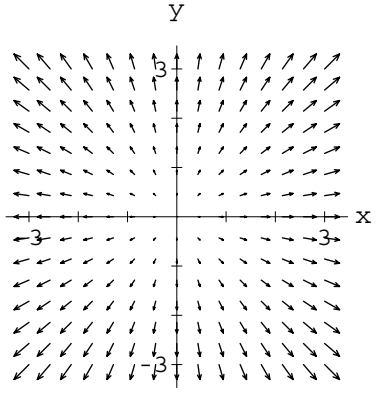
Exercises 9.7

1.



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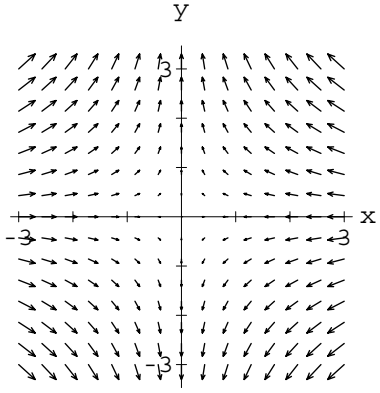

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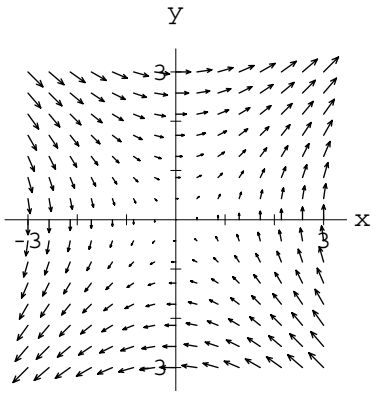

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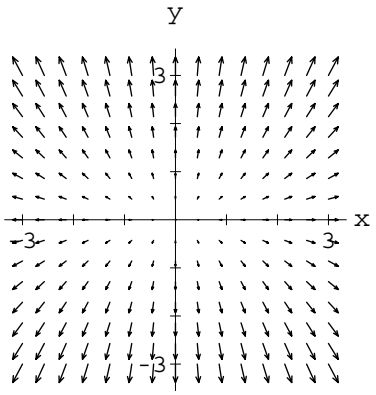

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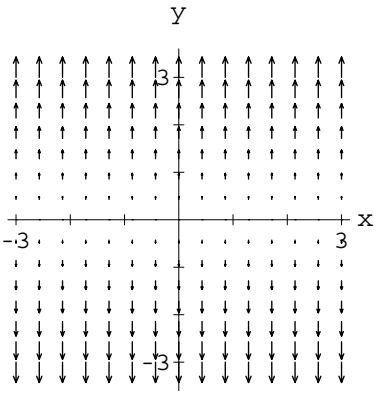
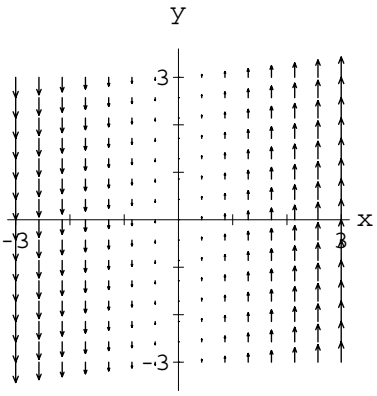


6.



7. curl $\mathbf{F} = (x-y)\mathbf{i} + (x-y)\mathbf{j}$; div $\mathbf{F} = 2z$

8. curl $\mathbf{F} = -2x^2\mathbf{i} + (10y - 18x^2)\mathbf{j} + (4xz - 10z)\mathbf{k}$; div $\mathbf{F} = 0$

9. curl $\mathbf{F} = \mathbf{0}$; div $\mathbf{F} = 4y + 8z$

10. curl $\mathbf{F} = (xe^{2y} + ye^{-yz} + 2xye^{2y})\mathbf{i} - ye^{2y}\mathbf{j} + 3(x-y)^2\mathbf{k}$; div $\mathbf{F} = 3(x-y)^2 - ze^{-yz}$

11. curl $\mathbf{F} = (4y^3 - 6xz^2)\mathbf{i} + (2z^3 - 3x^2)\mathbf{k}$; div $\mathbf{F} = 6xy$

12. curl $\mathbf{F} = -x^3z\mathbf{i} + (3x^2yz - z)\mathbf{j} + \left(\frac{3}{2}x^2y^2 - y - 15y^2\right)\mathbf{k}$; div $\mathbf{F} = (x^3y - x) - (x^3y - x) = 0$

13. curl $\mathbf{F} = (3e^{-z} - 8yz)\mathbf{i} - xe^{-z}\mathbf{j}$; div $\mathbf{F} = e^{-z} + 4z^2 - 3ye^{-z}$

14. curl $\mathbf{F} = (2xyz^3 + 3y)\mathbf{i} + (y\ln x - y^2z^3)\mathbf{j} + (2 - z\ln x)\mathbf{k}$; div $\mathbf{F} = \dfrac{yz}{x} - 3z + 3xy^2z^2$

15. curl $\mathbf{F} = (xy^2e^y + 2xye^y + x^3ye^z + x^3yze^z)\mathbf{i} - y^2e^y\mathbf{j} + (-3x^2yze^z - xe^x)\mathbf{k}$; div $\mathbf{F} = xye^x + ye^x - x^3ze^z$

16. curl $\mathbf{F} = (5xye^{5xy} + e^{5xy} + 3xz^3\sin xz^3 - \cos xz^3)\mathbf{i} + (x^2y\cos yz - 5y^2e^{5xy})\mathbf{j}$
$+ (-z^4\sin xz^3 - x^2z\cos yz)\mathbf{k}$; div $\mathbf{F} = 2x\sin yz$

17. div $\mathbf{r} = 1 + 1 + 1 = 3$

18. curl $\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

19. $\mathbf{a} \times \nabla = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \end{vmatrix} = \left(a_2\dfrac{\partial}{\partial z} - a_3\dfrac{\partial}{\partial y}\right)\mathbf{i} + \left(a_3\dfrac{\partial}{\partial x} - a_1\dfrac{\partial}{\partial z}\right)\mathbf{j} + \left(a_1\dfrac{\partial}{\partial y} - a_2\dfrac{\partial}{\partial x}\right)\mathbf{k}$

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$$(\mathbf{a}\times\nabla)\times\mathbf{r}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_2\dfrac{\partial}{\partial z}-a_3 & a_3\dfrac{\partial}{\partial x}-a_1\dfrac{\partial}{\partial z} & a_1\dfrac{\partial}{\partial y}-a_2\dfrac{\partial}{\partial x}\\ x & y & z\end{vmatrix}$$
$$=(-a_1-a_1)\mathbf{i}-(a_2+a_2)\mathbf{j}+(-a_3-a_3)\mathbf{k}=-2\mathbf{a}$$

20. $\nabla\times(\mathbf{a}\times\mathbf{r})=(\nabla\cdot\mathbf{r})\mathbf{a}-(\nabla\cdot\mathbf{a})\mathbf{r}=(1+1+1)\mathbf{a}-\left(a_1\dfrac{\partial}{\partial x}+a_2\dfrac{\partial}{\partial y}+a_3\dfrac{\partial}{\partial z}\right)\mathbf{r}=3\mathbf{a}-(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})=2\mathbf{a}$

21. $\nabla\cdot(\mathbf{a}\times\mathbf{r})=\begin{vmatrix}\partial/\partial x & \partial/\partial y & \partial/\partial z\\ a_1 & a_2 & a_3\\ x & y & z\end{vmatrix}=\dfrac{\partial}{\partial x}(a_2z-a_3y)-\dfrac{\partial}{\partial y}(a_1z-a_3x)+\dfrac{\partial}{\partial z}(a_1y-a_2x)=0$

22. $\nabla\times\mathbf{r}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x & y & z\end{vmatrix}=\mathbf{0};\ \ \mathbf{a}\times(\nabla\times\mathbf{r})=\mathbf{a}\times\mathbf{0}=\mathbf{0}$

23. $\mathbf{r}\times\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x & y & z\\ a_1 & a_2 & a_3\end{vmatrix}=(a_3y-a_2z)\mathbf{i}-(a_3x-a_1z)\mathbf{j}+(a_2x-a_1y)\mathbf{k};\ \ \mathbf{r}\cdot\mathbf{r}=x^2+y^2+z^2$

$$\nabla\times[(\mathbf{r}\cdot\mathbf{r})\mathbf{a}]=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ (\mathbf{r}\cdot\mathbf{r})a_1 & (\mathbf{r}\cdot\mathbf{r})a_2 & (\mathbf{r}\cdot\mathbf{r})a_3\end{vmatrix}$$
$$=(2ya_3-2za_2)\mathbf{i}-(2xa_3-2za_1)\mathbf{j}+(2xa_2-2ya_1)\mathbf{k}=2(\mathbf{r}\times\mathbf{a})$$

24. $\mathbf{r}\cdot\mathbf{a}=a_1x+a_2y+a_3z$; $\mathbf{r}\cdot\mathbf{r}=x^2+y^2+z^2$; $\nabla\cdot[(\mathbf{r}\cdot\mathbf{r})\mathbf{a}]=2xa_1+2ya_2+2za_3=2(\mathbf{r}\cdot\mathbf{a})$

25. Let $\mathbf{F}=P(x,y,z)\mathbf{i}+Q(x,y,z)\mathbf{j}+R(x,y,z)\mathbf{k}$ and $\mathbf{G}=S(x,y,z)\mathbf{i}+T(x,y,z)\mathbf{j}+U(x,y,z)\mathbf{k}$.

$$\nabla\cdot(\mathbf{F}+\mathbf{G})=\nabla\cdot[(P+S)\mathbf{i}+(Q+T)\mathbf{j}+(R+U)\mathbf{k}]=P_x+S_x+Q_y+T_y+R_z+U_z$$
$$=(P_x+Q_y+R_z)+(S_x+T_y+U_z)=\nabla\cdot\mathbf{F}+\nabla\cdot\mathbf{G}$$

26. Let $\mathbf{F}=P(x,y,z)\mathbf{i}+Q(x,y,z)\mathbf{j}+R(x,y,z)\mathbf{k}$ and $\mathbf{G}=S(x,y,z)\mathbf{i}+T(x,y,z)\mathbf{j}+U(x,y,z)\mathbf{k}$.

$$\nabla\times(\mathbf{F}+\mathbf{G})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ P+S & Q+T & R+U\end{vmatrix}$$
$$=(R_y+U_y-Q_z-T_z)\mathbf{i}-(R_x+U_x-P_z-S_z)\mathbf{j}+(Q_x+T_x-P_y-S_y)\mathbf{k}$$
$$=(R_y-Q_z)\mathbf{i}-(R_x-P_z)\mathbf{j}+(Q_x-P_y)\mathbf{k}+(U_y-T_z)\mathbf{i}-(U_x-S_z)\mathbf{j}+(T_x-S_y)\mathbf{k}$$
$$=\nabla\times\mathbf{F}+\nabla\times\mathbf{G}$$

27. $\nabla\cdot(f\mathbf{F})=\nabla\cdot(fP\mathbf{i}+fQ\mathbf{j}+fR\mathbf{k})=fP_x+Pf_x+fQ_y+Qf_y+fR_z+Rf_z$
$$=f(P_x+Q_y+R_z)+(Pf_x+Qf_y+Rf_z)=f(\nabla\cdot\mathbf{F})+\mathbf{F}\cdot(\nabla f)$$

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28. $\nabla \times (f\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ fP & fQ & fR \end{vmatrix}$

$= (fR_y + Rf_y - fQ_z - Qf_z)\mathbf{i} - (fR_x + Rf_x - fP_z - Pf_z)\mathbf{j} + (fQ_x + Qf_x - fP_y - Pf_y)\mathbf{k}$

$= (fR_y - fQ_z)\mathbf{i} - (fR_x - fP_z)\mathbf{j} + (fQ_x - fP_y)\mathbf{k} + (Rf_y - Qf_z)\mathbf{i}$

$\quad - (Rf_x - Pf_z)\mathbf{j} + (Qf_x - Pf_y)\mathbf{k}$

$= f[(R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ P & Q & R \end{vmatrix} = f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F}$

29. Assuming continuous second partial derivatives,

$$\text{curl (grad } f) = \nabla \times (f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix}$$
$$= (f_{zy} - f_{yz})\mathbf{i} - (f_{zx} - f_{xz})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k} = \mathbf{0}.$$

30. Assuming continuous second partial derivatives,

$$\text{div (curl } \mathbf{F}) = \nabla \cdot [(R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k}]$$
$$= (R_{yx} - Q_{zx} - (R_{xy} - P_{zy}) + (Q_{xz} - P_{yz}) = 0.$$

31. Let $\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ and $\mathbf{G} = S(x,y,z)\mathbf{i} + T(x,y,z)\mathbf{j} + U(x,y,z)\mathbf{k}$.

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P & Q & R \\ S & T & U \end{vmatrix} = (QU - RT)\mathbf{i} - (PU - RS)\mathbf{j} + (PT - QS)\mathbf{k}$$

$$\text{div } (\mathbf{F} \times \mathbf{G}) = (QU_x + Q_xU - RT_x - R_xT) - (PU_y + P_yU - RS_y - R_yS) + (PT_z + P_zT - QS_z - Q_zS)$$
$$= S(R_y - Q_z) + T(P_z - R_x) + U(Q_x - P_y) - P(U_y - T_z) - Q(S_z - U_x) - R(T_x - S_y)$$
$$= \mathbf{G} \cdot (\text{curl } \mathbf{F}) - \mathbf{F} \cdot (\text{curl } \mathbf{G})$$

32. Using Problems 26 and 29,

$$\text{curl (curl } \mathbf{F} + \text{grad } f) = \nabla \times (\text{curl } \mathbf{F} + \text{grad } f) = \nabla \times (\text{curl } \mathbf{F}) + \nabla \times (\text{grad } f)$$
$$= \text{curl (curl } \mathbf{F}) + \text{curl (grad } f) = \text{curl (curl } \mathbf{F}) + \mathbf{0} = \text{curl (curl } \mathbf{F}).$$

33. $\nabla \cdot \nabla f = \nabla \cdot (f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}) = f_{xx} + f_{yy} + f_{zz}$

34. Using Problem 27, $\nabla \cdot (f\nabla f) = f(\nabla \cdot \nabla f) + \nabla f \cdot \nabla f = f(\nabla^2 f) + |\nabla f|^2$.

35. curl $\mathbf{F} = -8yz\mathbf{i} - 2z\mathbf{j} - x\mathbf{k}$; curl (curl $\mathbf{F}$) $= 2\mathbf{i} - (8y - 1)\mathbf{j} + 8z\mathbf{k}$

36. **(a)** For $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$,

$$\text{curl (curl } \mathbf{F}) = (Q_{xy} - P_{yy} - P_{zz} + R_{xz})\mathbf{i} + (R_{yz} - Q_{zz} - Q_{xx} + P_{yx})\mathbf{j}$$
$$+ (P_{zx} - R_{xx} - R_{yy} + Q_{zy})\mathbf{k}$$

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$$\begin{aligned}-\nabla^2\mathbf{F} + \text{grad (div } \mathbf{F}) &= -(P_{xx}+P_{yy}+P_{zz})\mathbf{i} - (Q_{xx}+Q_{yy}+Q_{zz})\mathbf{j} - (R_{xx}+R_{yy}+R_{zz})\mathbf{k} \\ &\quad + \text{grad } (P_x + Q_y + R_z) \\ &= -P_{xx}\mathbf{i} - Q_{yy}\mathbf{j} - R_{zz}\mathbf{k} + (-P_{yy}-P_{zz})\mathbf{i} + (-Q_{xx}-Q_{zz})\mathbf{j} \\ &\quad + (-R_{xx}-R_{yy})\mathbf{k} + (P_{xx}+Q_{yx}+R_{zx})\mathbf{i} + (P_{xy}+Q_{yy}+R_{zy})\mathbf{j} \\ &\quad + (P_{xz}+Q_{yz}+R_{zz})\mathbf{k} \\ &= (-P_{yy}-P_{zz}+Q_{yx}+R_{zx})\mathbf{i} + (-Q_{xx}-Q_{zz}+P_{xy}+R_{zy})\mathbf{j} \\ &\quad + (-R_{xx}-R_{yy}+P_{xz}+Q_{yz})\mathbf{k}.\end{aligned}$$

Thus, curl (curl $\mathbf{F}$) $= -\nabla^2\mathbf{F} + \text{grad (div } \mathbf{F})$.

(b) For $\mathbf{F} = xy\mathbf{i} + 4yz^2\mathbf{j} + 2xz\mathbf{k}$, $\nabla^2\mathbf{F} = 0\mathbf{i} + 8y\mathbf{j} + 0\mathbf{k}$, div $\mathbf{F} = y + 4z^2 + 2x$, and grad (div $\mathbf{F}$) $= 2\mathbf{i} + \mathbf{j} + 8z\mathbf{k}$. Then curl (curl $\mathbf{F}$) $= -8y\mathbf{j} + 2\mathbf{i} + \mathbf{j} + 8z\mathbf{k} = 2\mathbf{i} + (1-8y)\mathbf{j} + 8z\mathbf{k}$.

37. $\dfrac{\partial f}{\partial x} = -x(x^2+y^2+z^2)^{-3/2}$

$\dfrac{\partial f}{\partial y} = -y(x^2+y^2+z^2)^{-3/2}$

$\dfrac{\partial f}{\partial z} = -z(x^2+y^2+z^2)^{-3/2}$

$\dfrac{\partial^2 f}{\partial x^2} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$

$\dfrac{\partial^2 f}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$

$\dfrac{\partial^2 f}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$

Adding the second partial derivatives gives

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} &= \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} - 3(x^2+y^2+z^2)^{-3/2} \\ &= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2} = 0\end{aligned}$$

except when $x = y = z = 0$.

38. $f_x = \dfrac{1}{1+\dfrac{4y^2}{(x^2+y^2-1)^2}}\left(-\dfrac{4xy}{(x^2+y^2-1)^2}\right) = -\dfrac{4xy}{(x^2+y^2-1)^2+4y^2}$

$$f_{xx} = -\frac{[(x^2+y^2-1)^2+4y^2]4y - 4xy[4x(x^2+y^2-1)]}{[(x^2+y^2-1)^2+4y^2]^2} = \frac{12x^4y - 4y^5 + 8x^2y^3 - 8x^2y - 8y^3 - 4y}{[(x^2+y^2-1)^2+4y^2]^2}$$

$$f_y = \frac{1}{1+\dfrac{4y^2}{(x^2+y^2-1)^2}}\left[\frac{2(x^2+y^2-1)-4y^2}{(x^2+y^2-1)^2}\right] = \frac{2(x^2-y^2-1)}{(x^2+y^2-1)^2+4y^2}$$

$$f_{yy} = \frac{[(x^2+y^2-1)^2+4y^2](-4y) - 2(x^2-y^2-1)[4y(x^2+y^2-1)+8y]}{[(x^2+y^2-1)^2+4y^2]^2}$$

$$= \frac{-12x^4y + 4y^5 - 8x^2y^3 + 8x^2y + 8y^3 + 4y}{[(x^2+y^2-1)^2+4y^2]^2}$$

$\nabla^2 f = f_{xx} + f_{yy} = 0$

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39. $\text{curl } \mathbf{F} = -Gm_1m_2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/|\mathbf{r}|^3 & y/|\mathbf{r}|^3 & z/|\mathbf{r}|^3 \end{vmatrix}$

$= -Gm_1m_2[(-3yz/|\mathbf{r}|^5 + 3yz/|\mathbf{r}|^5)\mathbf{i} - (-3xz/|\mathbf{r}|^5 + 3xz/|\mathbf{r}|^5)\mathbf{j} + (-3xy/|\mathbf{r}|^5 + 3xy/|\mathbf{r}|^5)\mathbf{k}]$

$= \mathbf{0}$

$$\text{div } \mathbf{F} = -Gm_1m_2\left[\frac{-2x^2+y^2+z^2}{|\mathbf{r}|^{5/2}} + \frac{x^2-2y^2+z^2}{|\mathbf{r}|^{5/2}} + \frac{x^2+y^2-2z^2}{|\mathbf{r}|^{5/2}}\right] = 0$$

40. $\dfrac{1}{2}\text{curl } \mathbf{v} = \dfrac{1}{2}\text{curl } (\boldsymbol{\omega} \times \mathbf{r}) = \dfrac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$

$= \dfrac{1}{2}[(\omega_1+\omega_1)\mathbf{i} - (-\omega_2-\omega_2)\mathbf{j} + (\omega_3+\omega_3)\mathbf{k}] = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k} = \boldsymbol{\omega}$

41. Using Problems 31 and 29,

$$\nabla \cdot \mathbf{F} = \text{div } (\nabla f \times \nabla g) = \nabla g \cdot (\text{curl } \nabla f) - \nabla f \cdot (\text{curl } \nabla g) = \nabla g \cdot \mathbf{0} - \nabla f \cdot \mathbf{0} = 0.$$

42. Recall that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$. Then, using Problems 31, 29, and 28,

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \text{div } (\nabla f \times f\nabla g) = f\nabla g \cdot (\text{curl } \nabla f) - \nabla f \cdot (\text{curl } f\nabla g) = f\nabla g \cdot \mathbf{0} - \nabla f \cdot (\nabla \times f\nabla g) \\ &= -\nabla f \cdot [f(\nabla \times \nabla g) + (\nabla f \times \nabla g)] = -\nabla f \cdot [f\text{curl } \nabla g + (\nabla f \times \nabla g] \\ &= -\nabla f \cdot [f\mathbf{0} + (\nabla f \times \nabla g)] = -\nabla f \cdot (\nabla f \times \nabla g) = 0. \end{aligned}$$

43. **(a)** Expressing the vertical component of $\mathbf{V}$ in polar coordinates, we have

$$\frac{2xy}{(x^2+y^2)^2} = \frac{2r^2\sin\theta\cos\theta}{r^4} = \frac{\sin 2\theta}{r^2}.$$

Similarly,

$$\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^4} = \frac{\cos 2\theta}{r^2}.$$

Since $\lim_{r\to\infty}(\sin 2\theta)/r^2 = \lim_{r\to\infty}(\cos 2\theta)/r^2 = 0$, $\mathbf{V} \approx A\mathbf{i}$ for r large or (x, y) far from the origin.

(b) Identifying $P(x,y) = A\left[1 - \dfrac{x^2-y^2}{(x^2-y^2)^2}\right]$, $Q(x,y) = -\dfrac{2Axy}{(x^2+y^2)^2}$, and $R(x,y) = 0$, we have

$$P_y = \frac{2Ay(3x^2-y^2)}{(x^2+y^2)^3}, \quad Q_x = \frac{2Ay(3x^2-y^2)}{(x^2+y^2)^3}, \quad \text{and} \quad P_z = Q_z = R_x = R_y = 0.$$

Thus, curl $\mathbf{V} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} = 0$ and $\mathbf{V}$ is irrotational.

(c) Since $P_x = \dfrac{2Ax(x^2-3y^2)}{(x^2+y^2)^3}$, $Q_y = \dfrac{2Ax(3y^2-x^2)}{(x^2+y^2)^3}$, and $R_z = 0$, $\nabla \cdot \mathbf{F} = P_x + Q_y + R_z = 0$ and $\mathbf{V}$ is incompressible.

44. We first note that curl $(\partial\mathbf{H}/\partial t) = \partial(\text{curl } \mathbf{H})/\partial t$ and curl $(\partial\mathbf{E}/\partial t) = \partial(\text{curl } \mathbf{E})/\partial t$. Then, from Problem 30,

$$\begin{aligned} -\nabla^2\mathbf{E} &= -\nabla^2\mathbf{E} + \mathbf{0} = -\nabla^2\mathbf{E} + \text{grad } 0 = -\nabla^2\mathbf{E} + \text{grad } (\text{div } \mathbf{E}) = \text{curl } (\text{curl } \mathbf{E}) \\ &= \text{curl } \left(-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right) = -\frac{1}{c}\frac{\partial}{\partial t}\text{curl } \mathbf{H} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t} \end{aligned}$$

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and $\nabla^2\mathbf{E} = \frac{1}{c^2}\,\partial^2\mathbf{E}/\partial t^2$. Similarly,

$$-\nabla^2\mathbf{H} = -\nabla^2\mathbf{H} + \text{grad (div } \mathbf{H}) = \text{curl (curl } \mathbf{H}) = \text{curl } \left(\frac{1}{c}\,\frac{\partial \mathbf{E}}{\partial t}\right) = \frac{1}{c}\,\frac{\partial}{\partial t}\,\text{curl } \mathbf{E}$$
$$= \frac{1}{c}\,\frac{\partial}{\partial t}\left(-\frac{1}{c}\,\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{1}{c^2}\,\frac{\partial^2 \mathbf{H}}{\partial t^2}$$

and $\nabla^2\mathbf{H} = \dfrac{1}{c^2}\,\partial^2\mathbf{H}/\partial t^2$.

45. We note that div $\mathbf{F} = 2xyz - 2xyz + 1 = 1 \neq 0$. If $\mathbf{F} = \text{curl } \mathbf{G}$, then div (curl $\mathbf{G}$) = div $\mathbf{F} = 1$. But, by Problem 30, for any vector field $\mathbf{G}$, div (curl $\mathbf{G}$) $= 0$. Thus, $\mathbf{F}$ cannot be the curl of $\mathbf{G}$.

Exercises 9.8

1. $\displaystyle\int_C 2xy\,dx = \int_0^{\pi/4} 2(5\cos t)(5\sin t)(-5\sin t)\,dt = -250\int_0^{\pi/4} \sin^2 t\cos t\,dt = -250\left(\frac{1}{3}\sin^3 t\right)\Big|_0^{\pi/4} = -\frac{125\sqrt{2}}{6}$

$\displaystyle\int_C 2xy\,dy = \int_0^{\pi/4} 2(5\cos t)(5\sin t)(5\cos t)\,dt = 250\int_0^{\pi/4} \cos^2 t\sin t\,dt = 250\left(-\frac{1}{3}\cos^3 t\right)\Big|_0^{\pi/4}$

$\displaystyle = \frac{250}{3}\left(1 - \frac{\sqrt{2}}{4}\right) = \frac{125}{6}(4 - \sqrt{2})$

$\displaystyle\int_C 2xy\,ds = \int_0^{\pi/4} 2(5\cos t)(5\sin t)\sqrt{25\sin^2 t + 25\cos^2 t}\,dt = 250\int_0^{\pi/4} \sin t\cos t\,dt$

$\displaystyle = 250\left(\frac{1}{2}\sin^2 t\right)\Big|_0^{\pi/4} = \frac{125}{2}$

2. $\displaystyle\int_C (x^3 + 2xy^2 + 2x)\,dx = \int_0^1 [8t^3 + 2(2t)(t^4) + 2(2t)]2\,dt = 2\int_0^1 (8t^3 + 4t^5 + 4t)\,dt$

$\displaystyle = 2\left(2t^4 + \frac{2}{3}t^6 + 2t^2\right)\Big|_0^1 = \frac{28}{3}$

$\displaystyle\int_C (x^3 + 2xy^2 + 2x)\,dy = \int_0^1 [8t^3 + 2(2t)(t^4) + 2(2t)]2t\,dt = 2\int_0^1 (8t^4 + 4t^6 + 4t^2)\,dt$

$\displaystyle = 2\left(\frac{8}{5}t^5 + \frac{4}{7}t^7 + \frac{4}{3}t^3\right)\Big|_0^1 = \frac{736}{105}$

$\displaystyle\int_C (x^3 + 2xy^2 + 2x)\,ds = \int_0^1 [8t^3 + 2(2t)(t^4) + 2(2t)]\sqrt{4 + 4t^2}\,dt = 8\int_0^1 t(1 + t^2)^{5/2}\,dt$

$\displaystyle = 8\left(\frac{1}{7}(1 + t^2)^{7/2}\right)\Big|_0^1 = \frac{8}{7}(2^{7/2} - 1)$

3. $\displaystyle\int_C (3x^2 + 6y^2)\,dx = \int_{-1}^0 [3x^2 + 6(2x + 1)^2]\,dx = \int_{-1}^0 (27x^2 + 24x + 6)\,dx = (9x^3 + 12x^2 + 6x)\Big|_{-1}^0$

$\displaystyle = -(-9 + 12 - 6) = 3$

$\displaystyle\int_C (3x^2 + 6y^2)\,dy = \int_{-1}^0 [3x^2 + 6(2x + 1)^2]2\,dx = 6$

$\displaystyle\int_C (3x^2 + 6y^2)\,ds = \int_{-1}^0 [3x^2 + 6(2x + 1)^2]\sqrt{1 + 4}\,dx = 3\sqrt{5}$

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4. $\displaystyle\int_C \frac{x^2}{y^3}\,dx = \int_1^8 \frac{x^2}{27x^2/8}\,dx = \frac{8}{27}\int_1^8 dx = \frac{56}{27}$

$\displaystyle\int_C \frac{x^2}{y^3}\,dy = \int_1^8 \frac{x^2}{27x^2/8}\,x^{-1/3}\,dx = \frac{8}{27}\int_1^8 x^{-1/3}\,dx = \frac{4}{9}x^{2/3}\,\Big|_1^8 = \frac{4}{3}$

$\displaystyle\int_C \frac{x^2}{y^3}\,ds = \int_1^8 \frac{x^2}{27x^2/8}\sqrt{1+x^{-2/3}}\,dx = \frac{8}{27}\int_1^8 x^{-1/3}\sqrt{1+x^{2/3}}\,dx = \frac{8}{27}(1+x^{2/3})^{3/2}\,\Big|_1^8 = \frac{8}{27}(5^{3/2}-2^{3/2})$

5. $\displaystyle\int_C z\,dx = \int_0^{\pi/2} t(-\sin t)\,dt$ Integration by parts

$= (t\cos t - \sin t)\,\Big|_0^{\pi/2} = -1$

$\displaystyle\int_C z\,dy = \int_0^{\pi/2} t\cos t\,dt$ Integration by parts

$= (t\sin t + \cos t)\,\Big|_0^{\pi/2} = \frac{\pi}{2} - 1$

$\displaystyle\int_C z\,dz = \int_0^{\pi/2} t\,dt = \frac{1}{2}t^2\,\Big|_0^{\pi/2} = \frac{\pi^2}{8}$

$\displaystyle\int_C z\,ds = \int_0^{\pi/2} t\sqrt{\sin^2 t + \cos^2 t + 1}\,dt = \sqrt{2}\int_0^{\pi/2} t\,dt = \frac{\pi^2\sqrt{2}}{8}$

6. $\displaystyle\int_C 4xyz\,dx = \int_0^1 4\left(\frac{1}{3}t^3\right)(t^2)(2t)t^2\,dt = \frac{8}{3}\int_0^1 t^8\,dt = \frac{8}{27}t^9\,\Big|_0^1 = \frac{8}{27}$

$\displaystyle\int_C 4xyz\,dy = \int_0^1 4\left(\frac{1}{3}\,t^3\right)(t^2)(2t)2t\,dt = \frac{16}{3}\int_0^1 t^7\,dt = \frac{2}{3}t^8\,\Big|_0^1 = \frac{2}{3}$

$\displaystyle\int_C 4xyz\,dz = \int_0^1 4\left(\frac{1}{3}t^3\right)(t^2)(2t)2\,dt = \frac{16}{3}\int_0^1 t^6\,dt = \frac{16}{21}t^7\,\Big|_0^1 = \frac{16}{21}$

$\displaystyle\int_C 4xyz\,ds = \int_0^1 4\left(\frac{1}{3}t^3\right)(t^2)(2t)\sqrt{t^4+4t^2+4}\,dt = \frac{8}{3}\int_0^1 t^6(t^2+2)\,dt = \frac{8}{3}\left(\frac{1}{9}t^9 + \frac{2}{7}t^7\right)\,\Big|_0^1 = \frac{200}{189}$

7. Using x as the parameter, $dy = dx$ and

$$\int_C (2x+y)\,dx + xy\,dy = \int_{-1}^2 (2x + x + 3 + x^2 + 3x)\,dx = \int_{-1}^2 (x^2 + 6x + 3)\,dx$$
$$= \left(\frac{1}{3}x^3 + 3x^2 + 3x\right)\,\Big|_{-1}^2 = 21.$$

8. Using x as the parameter, $dy = 2x\,dx$ and

$$\int_C (2x+y)\,dx + xy\,dy = \int_{-1}^2 (2x + x^2 + 1)\,dx + \int_{-1}^2 x(x^2+1)\,2x\,dx = \int_{-1}^2 (2x^4 + 3x^2 + 2x + 1)\,dx$$
$$= \left(\frac{2}{5}x^5 + x^3 + x^2 + x\right)\,\Big|_{-1}^2 = \frac{141}{5}\,.$$

9. From $(-1,2)$ to $(2,2)$ we use x as a parameter with $y = 2$ and $dy = 0$. From $(2,2)$ to $(2,5)$ we use y as a parameter with $x = 2$ and $dx = 0$.

$$\int_C (2x+y)\,dx + xy\,dy = \int_{-1}^2 (2x+2)\,dx + \int_2^5 2y\,dy = (x^2+2x)\,\Big|_{-1}^2 + y^2\,\Big|_2^5 = 9 + 21 = 30$$

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10. From $(-1,2)$ to $(-1,0)$ we use y as a parameter with $x=-1$ and $dx=0$. From $(-1,0)$ to $(2,0)$ we use x as a parameter with $y=dy=0$. From $(2,0)$ to $(2,5)$ we use y as a parameter with $x=2$ and $dx=0$.

$$\int_C (2x+y)\,dx + xy\,dy = \int_2^0 (-1)\,y\,dy + \int_{-1}^2 2x\,dx + \int_0^5 2y\,dy = -\frac{1}{2}\,y^2\,\Big|_2^0 + x^2\,\Big|_{-1}^2 + y^2\,\Big|_0^5$$
$$= 2+3+25 = 30$$

11. Using x as a the parameter, $dy = 2x\,dx$.

$$\int_C y\,dx + x\,dy = \int_0^1 x^2\,dx + \int_0^1 x(2x)\,dx = \int_0^1 3x^2\,dx = x^3\,\Big|_0^1 = 1$$

12. Using x as the parameter, $dy = dx$.

$$\int_C y\,dx + x\,dy = \int_0^1 x\,dx + \int_0^1 x\,dx = \int_0^1 2x\,dx = x^2\,\Big|_0^1 = 1$$

13. From $(0,0)$ to $(0,1)$ we use y as a parameter with $x=dx=0$. From $(0,1)$ to $(1,1)$ we use x as a parameter with $y=1$ and $dy=0$.

$$\int_C y\,dx + x\,dy = 0 + \int_0^1 1\,dx = 1$$

14. From $(0,0)$ to $(1,0)$ we use x as a parameter with $y=dy=0$. From $(1,0)$ to $(1,1)$ we use y as a parameter with $x=1$ and $dx=0$.

$$\int_C y\,dx + x\,dy = 0 + \int_0^1 1\,dy = 1$$

15. $$\int_C (6x^2+2y^2)\,dx + 4xy\,dy = \int_4^9 (6t+2t^2)\frac{1}{2}\,t^{-1/2}\,dt + \int_4^9 4\sqrt{t}\,t\,dt = \int_4^9 (3t^{1/2}+5t^{3/2})\,dt$$
$$= (2t^{3/2}+2t^{5/2})\,\Big|_4^9 = 460$$

16. $$\int_C (-y^2)\,dx + xy\,dy = \int_0^2 (-t^6)\,2\,dt + \int_0^2 (2t)(t^3)3t^2\,dt = \int_0^2 4t^6\,dt = \frac{4}{7}t^7\,\Big|_0^2 = \frac{512}{7}$$

17. $$\int_C 2x^3y\,dx + (3x+y)\,dy = \int_{-1}^1 2(y^6)y\,2y\,dy + \int_{-1}^1 (3y^2+y)\,dy = \int_{-1}^1 (4y^8+3y^2+y)\,dy$$
$$= \left(\frac{4}{9}y^9 + y^3 + \frac{1}{2}\,y^2\right)\,\Big|_{-1}^1 = \frac{26}{9}$$

18. $$\int_C 4x\,dx + 2y\,dy = \int_{-1}^2 4(y^3+1)3y^2\,dy + \int_{-1}^2 2y\,dy = \int_{-1}^2 (12y^5+12y^2+2y)\,dy$$
$$= (2y^6+4y^3+y^2)\,\Big|_{-1}^2 = 165$$

19. From $(-2,0)$ to $(2,0)$ we use x as a parameter with $y=dy=0$. From $(2,0)$ to $(-2,0)$ we parameterize the semicircle as $x=2\cos\theta$ and $y=2\sin\theta$ for $0\le\theta\le\pi$.

$$\oint_C (x^2+y^2)\,dx - 2xy\,dy = \int_{-2}^2 x^2\,dx + \int_0^\pi 4(-2\sin\theta\,d\theta) - \int_0^\pi 8\cos\theta\sin\theta(2\cos\theta\,d\theta)$$
$$= \frac{1}{3}x^3\,\Big|_{-2}^2 - 8\int_0^\pi (\sin\theta + 2\cos^2\theta\sin\theta)\,d\theta$$
$$= \frac{16}{3} - 8\left(-\cos\theta - \frac{2}{3}\cos^3\theta\right)\,\Big|_0^\pi = \frac{16}{3} - \frac{80}{3} = -\frac{64}{3}$$

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20. We start at $(0,0)$ and use x as a parameter.

$$\oint_C (x^2+y^2)\,dx - 2xy\,dy = \int_0^1 (x^2+x^4)\,dx - 2\int_0^1 xx^2(2x\,dx) + \int_1^0 (x^2+x)\,dx$$
$$- 2\int_1^0 x\sqrt{x}\left(\frac{1}{2}x^{-1/2}\right)dx$$
$$= \int_0^1 (x^2-3x^4)\,dx + \int_1^0 x^2\,dx = \int_0^1 (-3x^4)\,dx = -\frac{3}{5}x^5\Big|_0^1 = -\frac{3}{5}$$

21. From $(1,1)$ to $(-1,1)$ and $(-1,-1)$ to $(1,-1)$ we use x as a parameter with $y=1$ and $y=-1$, respectively, and $dy=0$. From $(-1,1)$ to $(-1,-1)$ and $(1,-1)$ to $(1,1)$ we use y as a parameter with $x=-1$ and $x=1$, respectively, and $dx=0$.

$$\oint_C x^2y^3\,dx - xy^2\,dy = \int_1^{-1} x^2(1)\,dx + \int_1^{-1} -(-1)y^2\,dy + \int_{-1}^1 x^2(-1)^3\,dx + \int_{-1}^1 -(1)y^2\,dy$$
$$= \frac{1}{3}x^3\Big|_1^{-1} + \frac{1}{3}y^3\Big|_1^{-1} - \frac{1}{3}x^3\Big|_{-1}^1 - \frac{1}{3}y^3\Big|_{-1}^1 = -\frac{8}{3}$$

22. From $(2,4)$ to $(0,4)$ we use x as a parameter with $y=4$ and $dy=0$. From $(0,4)$ to $(0,0)$ we use y as a parameter with $x=dx=0$. From $(0,0)$ to $(2,4)$ we use $y=2x$ and $dy=2\,dx$.

$$\oint_C x^2y^3\,dx - xy^2\,dy = \int_2^0 x^2(64)\,dx - \int_4^0 0\,dy + \int_0^2 x^2(8x^3)\,dx - \int_0^2 x(4x^2)2\,dx$$
$$= \frac{64}{3}x^3\Big|_2^0 + \frac{4}{3}x^6\Big|_0^2 - 2x^4\Big|_0^2 = -\frac{512}{3} + \frac{256}{3} - 32 = -\frac{352}{3}$$

23. $\displaystyle\oint_C (x^2-y^2)\,ds = \int_0^{2\pi} (25\cos^2\theta - 25\sin^2\theta)\sqrt{25\sin^2\theta + 25\cos^2\theta}\,d\theta = 125\int_0^{2\pi} (\cos^2\theta - \sin^2\theta)\,d\theta$

$$= 125\int_0^{2\pi} \cos 2\theta\,d\theta = \frac{125}{2}\sin 2\theta\Big|_0^{2\pi} = 0$$

24. $\displaystyle\oint_C y\,dx - x\,dy = \int_0^{\pi} 3\sin t(-2\sin t)\,dt - \int_0^{\pi} 2\cos t(3\cos t)\,dt = -6\int_0^{\pi} (\sin^2 t + \cos^2 t)\,dt = -6\int_0^{\pi} dt = -6\pi$

Thus, $\displaystyle\int_{-C} y\,dx - x\,dy = 6\pi$.

25. We parameterize the line segment from $(0,0,0)$ to $(2,3,4)$ by $x=2t$, $y=3t$, $z=4t$ for $0\le t\le 1$. We parameterize the line segment from $(2,3,4)$ to $(6,8,5)$ by $x=2+4t$, $y=3+5t$, $z=4+t$, $0\le t\le 1$.

$$\oint_C y\,dx + z\,dy + x\,dz = \int_0^1 3t(2\,dt) + \int_0^1 4t(3\,dt) + \int_0^1 2t(4\,dt) + \int_0^1 (3+5t)(4\,dt)$$
$$+ \int_0^1 (4+t)(5\,dt) + \int_0^1 (2+4t)\,dt$$
$$= \int_0^1 (55t+34)\,dt = \left(\frac{55}{2}t^2 + 34t\right)\Big|_0^1 = \frac{123}{2}$$

26. $\displaystyle\int_C y\,dx + z\,dy + x\,dz = \int_0^2 t^3(3\,dt) + \int_0^2 \left(\frac{5}{4}t^2\right)(3t^2\,dt) + \int_0^2 (3t)\left(\frac{5}{2}\,t\,dt\right)$

$$= \int_0^2 \left(3t^3 + \frac{15}{4}t^4 + \frac{15}{2}\,t^2\right)dt = \left(\frac{3}{4}t^4 + \frac{3}{4}t^5 + \frac{5}{2}t^3\right)\Big|_0^2 = 56$$

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27. From $(0,0,0)$ to $(6,0,0)$ we use x as a parameter with $y = dy = 0$ and $z = dz = 0$. From $(6,0,0)$ to $(6,0,5)$ we use z as a parameter with $x = 6$ and $dx = 0$ and $y = dy = 0$. From $(6,0,5)$ to $(6,8,5)$ we use y as a parameter with $x = 6$ and $dx = 0$ and $z = 5$ and $dz = 0$.

$$\int_C y\,dx + z\,dy + x\,dz = \int_0^6 0 + \int_0^5 6\,dz + \int_0^8 5\,dy = 70$$

28. We parametrize the line segment from $(0,0,0)$ to $(6,8,0)$ by $x = 6t$, $y = 8t$, $z = 0$ for $0 \le t \le 1$. From $(6,8,0)$ to $(6,8,5)$ we use z as a parameter with $x = 6$, $dx = 0$, and $y = 8$, $dy = 0$.

$$\int_C y\,dx + z\,dy + x\,dz = \int_0^1 8t(6\,dt) + \int_0^5 6\,dz = 24t^2\,\Big|_0^1 + 30 = 54$$

29. $\mathbf{F} = e^{3t}\mathbf{i} - (e^{-4t})e^t\mathbf{j} = e^{3t}\mathbf{i} - e^{-3t}\mathbf{j}$; $d\mathbf{r} = (-2e^{-2t}\mathbf{i} + e^t\mathbf{j})\,dt$; $\mathbf{F}\cdot d\mathbf{r} = (-2e^t - e^{-2t})\,dt$;

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\ln 2}(-2e^t - e^{-2t})\,dt = \left(-2e^t + \frac{1}{2}e^{-2t}\right)\Big|_0^{\ln 2} = -\frac{31}{8} - \left(-\frac{3}{2}\right) = -\frac{19}{8}$$

30. $\mathbf{F} = e^t\mathbf{i} + te^{t^3}\mathbf{j} + t^3e^{t^6}\mathbf{k}$; $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\,dt$;

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (e^t + 2t^2e^{t^3} + 3t^5e^{t^6})\,dt = \left(e^t + \frac{2}{3}e^{t^3} + \frac{1}{2}e^{t^6}\right)\Big|_0^1 = \frac{13}{6}(e-1)$$

31. Using x as a parameter, $\mathbf{r}(x) = x\mathbf{i} + \ln x\mathbf{j}$. Then $\mathbf{F} = \ln x\mathbf{i} + x\mathbf{j}$, $d\mathbf{r} = (\mathbf{i} + \frac{1}{x}\mathbf{j})\,dx$, and

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^e (\ln x + 1)\,dx = (x\ln x)\,\Big|_1^e = e.$$

32. Let $\mathbf{r}_1 = (-2+2t)\mathbf{i} + (2-2t)\mathbf{j}$ and $\mathbf{r}_2 = 2t\mathbf{i} + 3t\mathbf{j}$ for $0 \le t \le 1$. Then

$$d\mathbf{r}_1 = 2\mathbf{i} - 2\mathbf{j}, \qquad d\mathbf{r}_2 = 2\mathbf{i} + 3\mathbf{j},$$
$$\mathbf{F}_1 = 2(-2+2t)(2-2t)\mathbf{i} + 4(2-2t)^2\mathbf{j} = (-8t^2 + 16t - 8)\mathbf{i} + (16t^2 - 32t + 16)\mathbf{j},$$
$$\mathbf{F}_2 = 2(2t)(3t)\mathbf{i} + 4(3t)^2\mathbf{j} = 12t^2\mathbf{i} + 36t^2\mathbf{j},$$

and

$$W = \int_{C_1}\mathbf{F}_1\cdot d\mathbf{r}_1 + \int_{cC_2}\mathbf{F}_2\cdot d\mathbf{r}_2 = \int_0^1 (-16t^2 + 32t - 16 - 32t^2 + 64t - 32)\,dt + \int_0^1 (24t^2 + 108t^2)\,dt$$
$$= \int_0^1 (84t^2 + 96t - 48)\,dt = (28t^3 + 48t^2 - 48t)\,\Big|_0^1 = 28.$$

33. Let $\mathbf{r}_1 = (1+2t)\mathbf{i} + \mathbf{j}$, $\mathbf{r}_2 = 3\mathbf{i} + (1+t)\mathbf{j}$, and $\mathbf{r}_3 = (3-2t)\mathbf{i} + (2-t)\mathbf{j}$ for $0 \le t \le 1$.
Then

$$d\mathbf{r}_1 = 2\mathbf{i}, \qquad d\mathbf{r}_2 = \mathbf{j}, \qquad d\mathbf{r}_3 = -2\mathbf{i} - \mathbf{j},$$
$$\mathbf{F}_1 = (1+2t+2)\mathbf{i} + (6-2-4t)\mathbf{j} = (3+2t)\mathbf{i} + (4-4t)\mathbf{j},$$
$$\mathbf{F}_2 = (3+2+2t)\mathbf{i} + (6+6t-6)\mathbf{j} = (5+2t)\mathbf{i} + 6t\mathbf{j},$$
$$\mathbf{F}_3 = (3-2t+4-2t)\mathbf{i} + (12-6t-6+4t)\mathbf{j} = (7-4t)\mathbf{i} + (6-2t)\mathbf{j},$$

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and

$$\begin{aligned} W &= \int_{C_1} \mathbf{F}_1 \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F}_2 \cdot d\mathbf{r}_2 + \int_{C_3} \mathbf{F}_3 \cdot d\mathbf{r}_3 \\ &= \int_0^1 (6+4t)dt + \int_0^1 6t dt + \int_0^1 (-14+8t-6+2t)dt \\ &= \int_0^1 (-14+20t)\,dt = (-14t+10t^2)\,\Big|_0^1 = -4. \end{aligned}$$

34. $\mathbf{F} = t^3\mathbf{i} + t^4\mathbf{j} + t^5\mathbf{k}$; $d\mathbf{r} = 3t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{k}$; $W = \displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^3 (3t^5 + 2t^5 + t^5)\,dt = \int_1^3 6t^5\,dt = t^6\,\Big|_1^3 = 728$

35. $\mathbf{r} = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}$, $0 \le t \le 2\pi$; $d\mathbf{r} = -3\sin t\mathbf{i} + 3\cos t\mathbf{j}$; $\mathbf{F} = a\mathbf{i} + b\mathbf{j}$;

$W = \displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} (-3a\sin t + 3b\cos t)\,dt = (3a\cos t + 3b\sin t)\,\Big|_0^{2\pi} = 0$

36. Let $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$ for $1 \le t \le 3$. Then $d\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and

$$\mathbf{F} = \frac{c}{|\mathbf{r}|^3}(t\mathbf{i} + t\mathbf{j} + t\mathbf{k}) = \frac{ct}{(\sqrt{3t^2}\,)^3}(\mathbf{i}+\mathbf{j}+\mathbf{k}) = \frac{c}{3\sqrt{3}\,t^2}(\mathbf{i}+\mathbf{j}+\mathbf{k}),$$

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^3 \frac{c}{3\sqrt{3}\,t^2}(1+1+1)\,dt = \frac{c}{\sqrt{3}}\int_1^3 \frac{1}{t^2}\,dt = \frac{c}{\sqrt{3}}\left(-\frac{1}{t}\right)\Big|_1^3 = \frac{c}{\sqrt{3}}\left(-\frac{1}{3}+1\right) = \frac{2c}{3\sqrt{3}}.$$

37. $\displaystyle\int_{C_1} y^2\,dx + xy\,dy = \int_0^1 (4t+2)^2 2\,dt + \int_0^1 (2t+1)(4t+2)4\,dt = \int_0^1 (64t^2+64t+16)\,dt$

$$= \left(\frac{64}{3}t^3 + 32t^2 + 16t\right)\Big|_0^1 = \frac{64}{3} + 32 + 16 = \frac{208}{3}$$

$\displaystyle\int_{C_2} y^2\,dx + xy\,dy = \int_1^{\sqrt{3}} 4t^4(2t)\,dt + \int_1^{\sqrt{3}} 2t^4(4t)\,dt = \int_1^{\sqrt{3}} 16t^5\,dt = \frac{8}{3}t^6\,\Big|_1^{\sqrt{3}} = 72 - \frac{8}{3} = \frac{208}{3}$

$\displaystyle\int_{C_3} y^2\,dx + xy\,dy = \int_e^{e^3} 4(\ln t)^2\frac{1}{t}\,dt + \int_e^{e^3} 2(\ln t)^2\frac{2}{t}\,dt = \int_e^{e^3} \frac{8}{t}(\ln t)^2\,dt = \frac{8}{3}(\ln t)^3\,\Big|_e^{e^3} = \frac{8}{3}(27-1) = \frac{208}{3}$

38. $\displaystyle\int_{C_1} xy\,ds = \int_0^2 t(2t)\sqrt{1+4}\,dt = 2\sqrt{5}\int_0^2 t^2\,dt = 2\sqrt{5}\left(\frac{1}{3}t^3\right)\Big|_0^2 = \frac{16\sqrt{5}}{3}$

$\displaystyle\int_{C_2} xy\,ds = \int_0^2 t(t^2)\sqrt{1+4t^2}\,dt = \int_0^2 t^3\sqrt{1+4t^2}\,dt$ $\qquad \boxed{u = 1+4t^2,\ du = 8t\,dt,\ t^2 = \tfrac{1}{4}(u-1)}$

$$= \int_1^{17} \frac{1}{4}(u-1)u^{1/2}\frac{1}{8}\,du = \frac{1}{32}\int_1^{17}(u^{3/2} - u^{1/2})\,du = \frac{1}{32}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_1^{17} = \frac{391\sqrt{17}+1}{120}$$

$\displaystyle\int_{C_3} xy\,ds = \int_2^3 (2t-4)(4t-8)\sqrt{4+16}\,dt = 16\sqrt{5}\int_2^3 (t-2)^2\,dt = 16\sqrt{5}\left[\frac{1}{3}(t-2)^3\right]\Big|_2^3 = \frac{16\sqrt{5}}{3}$

C_1 and C_3 are different parameterizations of the same curve, while C_1 and C_2 are different curves.

39. Since $\mathbf{v}\cdot\mathbf{v} = v^2$, $\dfrac{d}{dt}v^2 = \dfrac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \mathbf{v}\cdot\dfrac{d\mathbf{v}}{dt} + \dfrac{d\mathbf{v}}{dt}\cdot\mathbf{v} = 2\dfrac{d\mathbf{v}}{dt}\cdot\mathbf{v}$. Then

$$\begin{aligned} W &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b m\mathbf{a}\cdot\left(\frac{d\mathbf{r}}{dt}\,dt\right) = m\int_a^b \frac{d\mathbf{v}}{dt}\cdot\mathbf{v}\,dt = m\int_a^b \frac{1}{2}\left(\frac{d}{dt}v^2\right)dt \\ &= \frac{1}{2}m(v^2)\,\Big|_a^b = \frac{1}{2}m[v(b)]^2 - \frac{1}{2}m[v(a)]^2. \end{aligned}$$

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40. We are given $\rho = kx$. Then

$$m = \int_C \rho\, ds = \int_0^\pi kx\, ds = k\int_0^\pi (1+\cos t)\sqrt{\sin^2 t + \cos^2 t}\, dt = k\int_0^\pi (1+\cos t)\, dt$$
$$= k(t+\sin t)\Big|_0^\pi = k\pi.$$

41. From Problem 40, $m = k\pi$ and $ds = dt$.

$$M_x = \int_C y\rho\, ds = \int_C kxy\, ds = k\int_0^\pi (1+\cos t)\sin t\, dt = k\left(-\cos t + \frac{1}{2}\sin^2 t\right)\Big|_0^\pi = 2k$$

$$M_y = \int_C x\rho\, ds = \int_C kx^2\, ds = k\int_0^\pi (1+\cos t)^2\, dt = k\int_0^\pi (1+2\cos t+\cos^2 t)\, dt$$
$$= k\left(t + 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t\right)\Big|_0^\pi = \frac{3}{2}k\pi$$

$\bar{x} = M_y/m = \dfrac{3k\pi/2}{k\pi} = \dfrac{3}{2}$; $\bar{y} = M_x/m = \dfrac{2k}{k\pi} = \dfrac{2}{\pi}$. The center of mass is $(3/2, 2/\pi)$.

42. On C_1, $\mathbf{T} = \mathbf{i}$ and $\mathbf{F}\cdot\mathbf{T} = \text{comp}_{\mathbf{T}}\mathbf{F} \approx 1$. On C_2, $\mathbf{T} = -\mathbf{j}$ and $\mathbf{F}\cdot\mathbf{T} = \text{comp}_{\mathbf{T}}\mathbf{F} \approx 2$. On C_3, $\mathbf{T} = -\mathbf{i}$ and $\mathbf{F}\cdot\mathbf{T} = \text{comp}_{\mathbf{T}}\mathbf{F} \approx 1.5$. Using the fact that the lengths of C_1, C_2, and C_3 are 4, 5, and 5, respectively, we have

$$W = \int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_{C_1} \mathbf{F}\cdot\mathbf{T}\, ds + \int_{C_2} \mathbf{F}\cdot\mathbf{T}\, ds + \int_{C_3} \mathbf{F}\cdot\mathbf{T}\, ds \approx 1(4) + 2(5) + 1.5(5) = 21.5 \text{ ft-lb}.$$

Exercises 9.9

1. **(a)** $P_y = 0 = Q_x$ and the integral is independent of path. $\phi_x = x^2$, $\phi = \dfrac{1}{3}x^3 + g(y)$,

$$\phi_y = g'(y) = y^2,\ g(y) = \frac{1}{3}y^3,\ \phi = \frac{1}{3}x^3 + \frac{1}{3}y^3,\ \int_{(0,0)}^{(2,2)} x^2\, dx + y^2\, dy = \frac{1}{3}(x^3+y^3)\Big|_{(0,0)}^{(2,2)} = \frac{16}{3}$$

(b) Use $y = x$ for $0 \le x \le 2$. $\displaystyle\int_{(0,0)}^{(2,2)} x^2\, dx + y^2\, dy = \int_0^2 (x^2+x^2)\, dx = \frac{2}{3}x^3\Big|_0^2 = \frac{16}{3}$

2. **(a)** $P_y = 2x = Q_x$ and the integral is independent of path. $\phi_x = 2xy$, $\phi = x^2y + g(y)$,

$$\phi_y = x^2 + g'(y) = x^2,\ g(y) = 0,\ \phi = x^2y,\ \int_{(1,1)}^{(2,4)} 2xy\, dx + x^2\, dy = x^2y\Big|_{(1,1)}^{(2,4)} = 16 - 1 = 15$$

(b) Use $y = 3x - 2$ for $1 \le x \le 2$.

$$\int_{(1,1)}^{(2,4)} 2xy\, dx + x^2\, dy = \int_1^2 [2x(3x-2) + x^2(3)]\, dx = \int_1^2 (9x^2 - 4x)\, dx = (3x^3 - 2x^2)\Big|_1^2 = 15$$

3. **(a)** $P_y = 2 = Q_x$ and the integral is independent of path. $\phi_x = x + 2y$, $\phi = \dfrac{1}{2}x^2 + 2xy + g(y)$,

$$\phi_y = 2x + g'(y) = 2x - y,\ g(y) = -\frac{1}{2}y^2,\ \phi = \frac{1}{2}x^2 + 2xy - \frac{1}{2}y^2,$$
$$\int_{(1,0)}^{(3,2)} (x+2y)\, dx + (2x-y)\, dy = \left(\frac{1}{2}x^2 + 2xy - \frac{1}{2}y^2\right)\Big|_{(1,0)}^{(3,2)} = 14$$

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(b) Use $y = x - 1$ for $1 \le x \le 3$.

$$\int_{(1,0)}^{(3,2)} (x + 2y)\,dx + (2x - y)\,dy = \int_1^3 [x + 2(x-1) + 2x - (x-1)\,dx$$
$$= \int_1^3 (4x - 1)\,dx = (2x^2 - x)\,\Big|_1^3 = 14$$

4. (a) $P_y = -\cos x \sin y = Q_x$ and the integral is independent of path. $\phi_x = \cos x \cos y$, $\phi = \sin x \cos y + g(y)$, $\phi_y = -\sin x \sin y + g'(y) = 1 - \sin x \sin y$, $g(y) = y$, $\phi = \sin x \cos y + y$,

$$\int_{(0,0)}^{(\pi/2,0)} \cos x \cos y\,dx + (1 - \sin x \sin y)\,dy = (\sin x \cos y + y)\,\Big|_{(0,0)}^{(\pi/2,0)} = 1$$

(b) Use $y = 0$ for $0 \le x \le \pi/2$.

$$\int_{(0,0)}^{(\pi/2,0)} \cos x \cos y\,dx + (1 - \sin x \sin y)\,dy = \int_0^{\pi/2} \cos x\,dx = \sin x\,\Big|_0^{\pi/2} = 1$$

5. (a) $P_y = 1/y^2 = Q_x$ and the integral is independent of path. $\phi_x = -\dfrac{1}{y}$, $\phi = -\dfrac{x}{y} + g(y)$, $\phi_y = \dfrac{x}{y^2} + g'(x) = \dfrac{x}{y^2}$, $g(y) = 0$, $\phi = -\dfrac{x}{y}$, $\displaystyle\int_{(4,1)}^{(4,4)} -\frac{1}{y}\,dx + \frac{x}{y^2}\,dy = \left(-\frac{x}{y}\right)\Big|_{(4,1)}^{(4,4)} = 3$

(b) Use $x = 4$ for $1 \le y \le 4$.

$$\int_{(4,1)}^{(4,4)} -\frac{1}{y}\,dx + \frac{x}{y^2}\,dy = \int_1^4 \frac{4}{y^2}\,dy = -\frac{4}{y}\,\Big|_1^4 = 3$$

6. (a) $P_y = -xy(x^2 + y^2)^{-3/2} = Q_x$ and the integral is independent of path. $\phi_x = \dfrac{x}{\sqrt{x^2 + y^2}}$, $\phi = \sqrt{x^2 + y^2} + g(y)$, $\phi_y = \dfrac{y}{\sqrt{x^2 + y^2}} + g'(y) = \dfrac{y}{\sqrt{x^2 + y^2}}$, $g(y) = 0$, $\phi = \sqrt{x^2 + y^2}$,

$$\int_{(1,0)}^{(3,4)} \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = \sqrt{x^2 + y^2}\,\Big|_{(1,0)}^{(3,4)} = 4$$

(b) Use $y = 2x - 2$ for $1 \le x \le 3$.

$$\int_{(1,0)}^{(3,4)} \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = \int_1^3 \frac{x + (2x-2)2}{\sqrt{x^2 + (2x-2)^2}}\,dx = \int_1^3 \frac{5x - 4}{\sqrt{5x^2 - 8x + 4}} = \sqrt{5x^2 - 8x + 4}\,\Big|_1^3 = 4$$

7. (a) $P_y = 4xy = Q_x$ and the integral is independent of path. $\phi_x = 2y^2x - 3$, $\phi = x^2y^2 - 3x + g(y)$, $\phi_y = 2x^2y + g'(y) = 2x^2y + 4$, $g(y) = 4y$, $\phi = x^2y^2 - 3x + 4y$,

$$\int_{(1,2)}^{(3,6)} (2y^2x - 3)\,dx + (2yx^2 + 4)\,dy = (x^2y^2 - 3x + 4y)\,\Big|_{(1,2)}^{(3,6)} = 330$$

(b) Use $y = 2x$ for $1 \le x \le 3$.

$$\int_{(1,2)}^{(3,6)} (2y^2x - 3)\,dx + (2yx^2 + 4)\,dy = \int_1^3 \big([2(2x)^2x - 3] + [2(2x)x^2 + 4]2\big)\,dx$$
$$= \int_1^3 (16x^3 + 5)\,dx = (4x^4 + 5x)\,\Big|_1^3 = 330$$

8. (a) $P_y = 4 = Q_x$ and the integral is independent of path. $\phi_x = 5x + 4y$, $\phi = \dfrac{5}{2}x^2 + 4xy + g(y)$, $\phi_y = 4x + g'(y) = 4x - 8y^3$, $g(y) = -2y^4$, $\phi = \dfrac{5}{2}x^2 + 4xy - 2y^4$,

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$$\int_{(-1,1)}^{(0,0)} (5x+4y)\,dx + (4x-8y^3)\,dy = \left(\frac{5}{2}x^2 + 4xy - 2y^4\right)\Big|_{(-1,1)}^{(0,0)} = \frac{7}{2}$$

(b) Use $y = -x$ for $-1 \le x \le 0$.

$$\int_{(-1,1)}^{(0,0)} (5x+4y)\,dx + (4x-8y^3)\,dy = \int_{-1}^{0} [(5x-4x) + (4x+8x^3)(-1)]\,dx$$
$$= \int_{-1}^{0} (-3x - 8x^3)\,dx = \left(-\frac{3}{2}x^2 - 2x^4\right)\Big|_{-1}^{0} = \frac{7}{2}$$

9. (a) $P_y = 3y^2 + 3x^2 = Q_x$ and the integral is independent of path. $\phi_x = y^3 + 3x^2y$,
$\phi = xy^3 + x^3y + g(y)$, $\phi_y = 3xy^2 + x^3 + g'(y) = x^3 + 3y^2x + 1$, $g(y) = y$, $\phi = xy^3 + x^3y + y$,
$\displaystyle\int_{(0,0)}^{(2,8)} (y^3 + 3x^2y)\,dx + (x^3 + 3y^2x + 1)\,dy = (xy^3 + x^3y + y)\Big|_{(0,0)}^{(2,8)} = 1096$

(b) Use $y = 4x$ for $0 \le x \le 2$.

$$\int_{(0,0)}^{(2,8)} (y^3 + 3x^2y)\,dx + (x^3 + 3y^2x + 1)\,dy = \int_0^2 [(64x^3 + 12x^3) + (x^3 + 48x^3 + 1)(4)]\,dx$$
$$= \int_0^2 (272x^3 + 4)\,dx = (68x^4 + 4x)\Big|_0^2 = 1096$$

10. (a) $P_y = -xy\cos xy - \sin xy - 20y^3 = Q_x$ and the integral is independent of path.
$\phi_x = 2x - y\sin xy - 5y^4$, $\phi = x^2 + \cos xy - 5xy^4 + g(y)$,
$\phi_y = -x\sin xy - 20xy^3 + g'(y) = -20xy^3 - x\sin xy$, $g(y) = 0$, $\phi = x^2 + \cos xy - 5xy^4$,
$\displaystyle\int_{(-2,0)}^{(1,0)} (2x - y\sin xy - 5y^4)\,dx - (20xy^3 + x\sin xy)\,dy = (x^2 + \cos xy - 5xy^4)\Big|_{(-2,0)}^{(1,0)} = -3$

(b) Use $y = 0$ for $-2 \le x \le 1$.

$$\int_{(-2,0)}^{(1,0)} (2x - y\sin xy - 5y^4)\,dx - (20xy^3 + x\sin xy)\,dy = \int_{-2}^{1} 2x\,dx = x^2\Big|_{-2}^{1} = -3$$

11. $P_y = 12x^3y^2 = Q_x$ and the vector field is a gradient field. $\phi_x = 4x^3y^3 + 3$, $\phi = x^4y^3 + 3x + g(y)$,
$\phi_y = 3x^4y^2 + g'(y) = 3x^4y^2 + 1$, $g(y) = y$, $\phi = x^4y^3 + 3x + y$

12. $P_y = 6xy^2 = Q_x$ and the vector field is a gradient field. $\phi_x = 2xy^3$, $\phi = x^2y^3 + g(y)$,
$\phi_y = 3x^2y^2 + g'(y) = 3x^2y^2 + 3y^2$, $g(y) = y^3$, $\phi = x^2y^3 + y^3$

13. $P_y = -2xy^3\sin xy^2 + 2y\cos xy^2$, $Q_x = -2xy^3\cos xy^2 - 2y\sin xy^2$ and the vector field is not a gradient field.

14. $P_y = -4xy(x^2+y^2+1)^{-3} = Q_x$ and the vector field is a gradient field.
$\phi_x = x(x^2+y^2+1)^{-2}$, $\phi = -\frac{1}{2}(x^2+y^2+1)^{-1} + g(y)$, $\phi_y = y(x^2+y^2+1)^{-2} + g'(y) = y(x^2+y^2+1)^{-2}$,
$g(y) = 0$, $\phi = -\frac{1}{2}(x^2+y^2+1)^{-1}$

15. $P_y = 1 = Q_x$ and the vector field is a gradient field. $\phi_x = x^3 + y$, $\phi = \frac{1}{4}x^4 + xy + g(y)$, $\phi_y = x + g'(y) = x + y^3$,
$g(y) - \frac{1}{4}y^4$, $\phi = \frac{1}{4}x^4 + xy + \frac{1}{4}y^4$

16. $P_y = 4e^{2y}$, $Q_x = e^{2y}$ and the vector field is not a gradient field.

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17. Since $P_y = -e^{-y} = Q_x$, $\mathbf{F}$ is conservative and $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path. Thus, instead of the given curve we may use the simpler curve C_1: $y = x$, $0 \le x \le 1$. Then

$$\begin{aligned} W &= \int_{C_1} (2x + e^{-y})\,dx + (4y - xe^{-y})\,dy \\ &= \int_0^1 (2x + e^{-x})\,dx + \int_0^1 (4x - xe^{-x})\,dx \qquad \boxed{\text{Integration by parts}} \\ &= (x^2 - e^{-x})\,\Big|_0^1 + (2x^2 + xe^{-x} + e^{-x})\,\Big|_0^1 \\ &= [(1 - e^{-1}) - (-1)] + [(2 + e^{-1} + e^{-1}) - (1)] = 3 + e^{-1}. \end{aligned}$$

18. Since $P_y = -e^{-y} = Q_x$, $\mathbf{F}$ is conservative and $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path. Thus, instead of the given curve we may use the simpler curve C_1: $y = 0$, $-2 \le -x \le 2$. Then $dy = 0$ and

$$W = \int_{C_1} (2x + e^{-y})\,dx + (4y - xe^{-y})\,dy = \int_2^{-2} (2x + 1)\,dx = (x^2 + x)\,\Big|_2^{-2} = (4 - 2) - (4 + 2) = -4.$$

19. $P_y = z = Q_x$, $Q_z = x = R_y$, $R_x = y = P_z$, and the integral is independent of path. Parameterize the line segment between the points by $x = 1 + t$, $y = 1 + 3t$, $z = 1 + 7t$, for $0 \le t \le 1$. Then $dx = dt$, $dy = 3\,dt$, $dz = 7\,dt$ and

$$\begin{aligned} \int_{(1,1,1)}^{(2,4,8)} yz\,dx + xz\,dy + xy\,dz &= \int_0^1 [(1 + 3t)(1 + 7t) + (1 + t)(1 + 7t)(3) + (1 + t)(1 + 3t)(7)]\,dt \\ &= \int_0^1 (11 + 62t + 63t^2)\,dt = (11t + 31t^2 + 21t^3)\,\Big|_0^1 = 63. \end{aligned}$$

20. $P_y = 0 = Q_x$, $Q_z = 0 = R_y$, $R_x = 0 = P_z$ and the integral is independent of path. Parameterize the line segment between the points by $x = t$, $y = t$, $z = t$, for $0 \le t \le 1$. Then $dx = dy = dz = dt$ and

$$\int_{(0,0,0)}^{(1,1,1)} 2x\,dx + 3y^2\,dy + 4z^3\,dz = \int_0^1 (2t + 3t^2 + 4t^3)\,dt = (t^2 + t^3 + t^4)\,\Big|_0^1 = 3.$$

21. $P_y = 2x\cos y = Q_x$, $Q_z = 0 = R_y$, $R_x = 3e^{3z} = P_z$, and the integral is independent of path. Integrating $\phi_x = 2x\sin y + e^{3z}$ we find $\phi = x^2\sin y + xe^{3z} + g(y, z)$. Then $\phi_y = x^2\cos y + g_y = Q = x^2\cos y$, so $g_y = 0$, $g(y, z) = h(z)$, and $\phi = x^2\sin y + xe^{3z} + h(z)$. Now $\phi_z = 3xe^{3z} + h'(z) = R = 3xe^{3z} + 5$, so $h'(z) = 5$ and $h(z) = 5z$. Thus $\phi = x^2\sin y + xe^{3z} + 5z$ and

$$\begin{aligned} &\int_{(1,0,0)}^{(2,\pi/2,1)} (2x\sin y + e^{3z})\,dx + x^2\cos y\,dy + (3xe^{3z} + 5)\,dz \\ &\qquad = (x^2\sin y + xe^{3z} + 5z)\,\Big|_{(1,0,0)}^{(2,\pi/2,1)} = [4(1) + 2e^3 + 5] - [0 + 1 + 0] = 8 + 2e^3. \end{aligned}$$

22. $P_y = 0 = Q_x$, $Q_z = 0 = R_y$, $R_x = 0 = P_z$, and the integral is independent of path. Parameterize the line segment between the points by $x = 1 + 2t$, $y = 2 + 2t$, $z = 1$, for $0 \le t \le 1$. Then $dx = 2\,dt$, $dz = 0$ and

$$\begin{aligned} \int_{(1,2,1)}^{(3,4,1)} (2x + 1)\,dx + 3y^2\,dy + \frac{1}{z}\,dz &= \int_0^1 [(2 + 4t + 1)2 + 3(2 + 2t)^2 2]\,dt \\ &= \int_0^1 (24t^2 + 56t + 30)\,dt = (8t^3 + 28t^2 + 30t)\,\Big|_0^1 = 66. \end{aligned}$$

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23. $P_y = 0 = Q_x$, $Q_z = 0 = R_y$, $R_x = 2e^{2z} = P_z$ and the integral is independent of path. Parameterize the line segment between the points by $x = 1 + t$, $y = 1 + t$, $z = \ln 3$, for $0 \le t \le 1$. Then $dx = dy = dt$, $dz = 0$ and

$$\int_{(1,1,\ln 3)}^{(2,2\ln 3)} e^{2z}\,dx + 3y^2\,dy + 2xe^{2z}\,dz = \int_0^1 [e^{2\ln 3} + 3(1+t)^2]\,dt = [9t + (1+t)^3]\,\Big|_0^1 = 16.$$

24. $P_y = 0 = Q_x$, $Q_z = 2y = R_y$, $R_x = 2x = P_z$ and the integral is independent of path. Parameterize the line segment between the points by $x = -2(1-t)$, $y = 3(1-t)$, $z = 1-t$, for $0 \le t \le 1$. Then $dx = 2\,dt$, $dy = -3\,dt$, $dz = -dt$, and

$$\begin{aligned}\int_{(-2,3,1)}^{(0,0,0)} &2xz\,dx + 2yz\,dy + (x^2+y^2)\,dz \\ &= \int_0^1 [-4(1-t)^2(2) + 6(1-t)^2(-3) + 4(1-t)^2(-1) + 9(1-t)^2(-1)]\,dt \\ &= \int_0^1 -39(1-t)^2\,dt = 13(1-t)^3\,\Big|_0^1 = -13.\end{aligned}$$

25. $P_y = 1 - z\sin x = Q_x$, $Q_z = \cos x = R_y$, $R_x = -y\sin x = P_z$ and the integral is independent of path. Integrating $\phi_x = y - yz\sin x$ we find $\phi = xy + yz\cos x + g(y,z)$. Then $\phi_y = x + z\cos x + g_y(y,z) = Q = x + z\cos x$, so $g_y = 0$, $g(y,z) = h(z)$, and $\phi = xy + yz\cos x + h(z)$. Now $\phi_z = y\cos x + h(z) = R = y\cos x$, so $h(z) = 0$ and $\phi = xy + yz\cos x$. Since $\mathbf{r}(0) = 4\mathbf{j}$ and $\mathbf{r}(\pi/2) = \pi\mathbf{i} + \mathbf{j} + 4\mathbf{k}$,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = (xy + yz\cos x)\,\Big|_{(0,4,0)}^{(\pi,1,4)} = (\pi - 4) - (0 + 0) = \pi - 4.$$

26. $P_y = 0 = Q_x$, $Q_z = 0 = R_y$ $R_x = -e^z = P_z$ and the integral is independent of path. Integrating $\phi_x = 2 - e^z$ we find $\phi = 2x - xe^z + g(y,z)$. Then $\phi_y = g_y = 2y - 1$, so $g(y,z) = y^2 - y + h(z)$ and $\phi = 2x - xe^z + y^2 - y + h(z)$. Now $\phi_z = -xe^z + h'(z) = R = 2 - xe^z$, so $h'(z) = 2$, $h(z) = 2z$, and $\phi = 2x - xe^z + y^2 - y + 2z$. Thus

$$\int_C \mathbf{F}\cdot d\mathbf{r} = (2x - xe^z + y^2 - y + 2z)\,\Big|_{(-1,1,-1)}^{(2,2,2)} = (4 - 2e^2 + 4 - 2 + 4) - (-2 + e^{-1} + 1 - 1 - 2) = 14 - 2e^2 - e^{-1}.$$

27. Since $P_y = Gm_1m_2(2xy/|\mathbf{r}|^5) = Q_x$, $Q_z = Gm_1m_2(2yz/|\mathbf{r}|^5) = R_y$, and $R_x = Gm_1m_2(2xz/|\mathbf{r}|^5) = P_z$, the force field is conservative.

$\phi_x = -Gm_1m_2\dfrac{x}{(x^2+y^2+z^2)^{3/2}}$, $\phi = Gm_1m_2(x^2+y^2+z^2)^{-1/2} + g(y,z)$,

$\phi_y = -Gm_1m_2\dfrac{y}{(x^2+y^2+z^2)^{3/2}} + g_y(y,z) = -Gm_1m_2\dfrac{y}{(x^2+y^2+z^2)^{3/2}}$, $g(y,z) = h(z)$,

$\phi = Gm_1m_2(x^2+y^2+z^2)^{-1/2} + h(z)$,

$\phi_z = -Gm_1m_2\dfrac{z}{(x^2+y^2+z^2)^{3/2}} + h'(z) = -Gm_1m_2\dfrac{z}{(x^2+y^2+z^2)^{3/2}}$,

$h(z) = 0$, $\phi = \dfrac{Gm_1m_2}{\sqrt{x^2+y^2+z^2}} = \dfrac{Gm_1m_2}{|\mathbf{r}|}$

28. Since $P_y = 24xy^2z = Q_x$, $Q_z = 12x^2y^2 = R_y$, and $R_x = 8xy^3 = P_z$, $\mathbf{F}$ is conservative. Thus, the work done between two points is independent of the path. From $\phi_x = 8xy^3z$ we obtain $\phi = 4x^2y^3z$ which is a potential function for $\mathbf{F}$. Then

$$W = \int_{(2,0,0)}^{(1,\sqrt{3},\pi/3)} \mathbf{F}\cdot d\mathbf{r} = 4x^2y^3z\,\Big|_{(2,0,0)}^{(1,\sqrt{3},\pi/3)} = 4\sqrt{3}\,\pi \quad \text{and} \quad W = \int_{(2,0,0)}^{(0,2,\pi/2)} \mathbf{F}\cdot d\mathbf{r} = 0.$$

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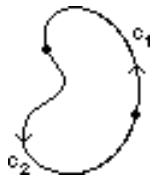
29. Since $\mathbf{F}$ is conservative, $\displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{-C_2} \mathbf{F}\cdot d\mathbf{r}$. Then, since the simply closed curve C is composed of C_1 and C_2,



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} - \int_{-C_2} \mathbf{F}\cdot d\mathbf{r} = 0.$$

30. From $\mathbf{F} = (x^2+y^2)^{n/2}(x\mathbf{i}+y\mathbf{j})$ we obtain $P_y = nxy(x^2+y^2)^{n/2-1} = Q_x$, so that $\mathbf{F}$ is conservative. From $\phi_x = x(x^2+y^2)^{n/2}$ we obtain the potential function $\phi = (x^2+y^2)^{(n+2)/2}/(n+2)$. Then

$$W = \int_{(x_1,y_1)}^{(x_2,y_2)} \mathbf{F}\cdot d\mathbf{r} = \left(\frac{(x^2+y^2)^{(n+2)/2}}{n+2}\right)\Bigg|_{(x_1,y_1)}^{(x_2,y_2)} = \frac{1}{n+2}\left[(x_2^2+y_2^2)^{(n+2)/2} - (x_1^2+y_1^2)^{(n+2)/2}\right].$$

31. From Problem 39 in Exercises 9.8, $\dfrac{d\mathbf{v}}{dt}\cdot\dfrac{d\mathbf{r}}{dt} = \dfrac{d\mathbf{v}}{dt}\cdot\mathbf{v} = \dfrac{1}{2}\dfrac{d}{dt}v^2$. Then, using $\dfrac{dp}{dt} = \dfrac{\partial p}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial p}{\partial y}\dfrac{dy}{dt} = \nabla p\cdot\dfrac{d\mathbf{r}}{dt}$, we have

$$\int m\frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{r}}{dt}\,dt + \int \nabla p\cdot\frac{d\mathbf{r}}{dt} = \int 0\,dt$$
$$\frac{1}{2}m\int\frac{d}{dt}v^2\,dt + \int\frac{dp}{dt}\,dt = \text{constant}$$
$$\frac{1}{2}mv^2 + p = \text{constant}.$$

32. By Problem 31, the sum of kinetic and potential energies in a conservative force field is constant. That is, it is independent of points A and B, so $p(B)+K(B) = p(A)+K(A)$.

Exercises 9.10

1. $\displaystyle\int_{-1}^{3}(6xy-5e^y)\,dx = (3x^2y-5xe^y)\Big|_{-1}^{3} = (27y-15e^y)-(3y+5e^y) = 24y-20e^y$

2. $\displaystyle\int_1^2 \tan xy\,dy = \frac{1}{x}\ln|\sec xy|\Big|_1^2 = \frac{1}{x}\ln|\sec 2x - \sec x|$

3. $\displaystyle\int_1^{3x} x^3e^{xy}\,dy = x^2e^{xy}\Big|_1^{3x} = x^2(e^{3x^2}-e^x)$

4. $\displaystyle\int_{\sqrt{y}}^{y^3}(8x^3y-4xy^2)\,dx = (2x^4y-2x^2y^2)\Big|_{\sqrt{y}}^{y^3} = (2y^{13}-2y^8)-(2y^3-2y^3) = 2y^{13}-2y^8$

5. $\displaystyle\int_0^{2x}\frac{xy}{x^2+y^2}\,dy = \frac{x}{2}\ln(x^2+y^2)\Big|_0^{2x} = \frac{x}{2}[\ln(x^2+4x^2)-\ln x^2] = \frac{x}{2}\ln 5$

6. $\displaystyle\int_{x^3}^{x} e^{2y/x}\,dy = \frac{x}{2}e^{2y/x}\Big|_{x^3}^{x} = \frac{x}{2}(e^{2x/x}-e^{2x^3/x}) = \frac{x}{2}(e^2-e^{2x^2})$

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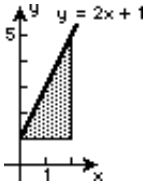
7. $\displaystyle\int_{\tan y}^{\sec y}(2x+\cos y)\,dx=(x^2+x\cos y)\,\Big|_{\tan y}^{\sec y}=\sec^2 y+\sec y\cos y-\tan^2 y-\tan y\cos y$

$$=\sec^2 y+1-\tan^2 y-\sin y=2-\sin y$$

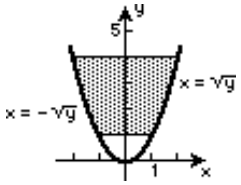
8. $\displaystyle\int_{\sqrt{y}}^{1} y\ln x\,dx$ Integration by parts

$$=y(x\ln x-x)\,\Big|_{\sqrt{y}}^{1}=y(0-1)-y(\sqrt{y}\,\ln\sqrt{y}-\sqrt{y}\,)=-y-y\sqrt{y}\left(\frac{1}{2}\ln y-1\right)$$

9.



10.



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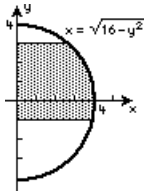
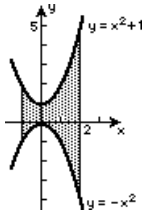
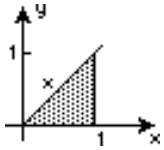


12.



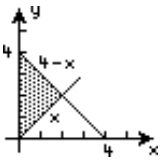
13. $\displaystyle\iint_R x^3y^2\,dA=\int_0^1\int_0^x x^3y^2\,dy\,dx=\int_0^1\frac{1}{3}x^3y^3\,\Big|_0^x\,dx=\frac{1}{3}\int_0^1 x^6\,dx=\frac{1}{21}x^7\,\Big|_0^1=\frac{1}{21}$



14. $\displaystyle\iint_R (x+1)\,dA=\int_0^2\int_x^{4-x}(x+1)\,dy\,dx=\int_0^2(xy+y)\,\Big|_x^{4-x}\,dx$

$$=\int_0^2[(4x-x^2+4-x)-(x^2+x)]\,dx=\int_0^2(2x-2x^2+4)\,dx$$

$$=\left(x^2-\frac{2}{3}x^3+4x\right)\,\Big|_0^2=\frac{20}{3}$$

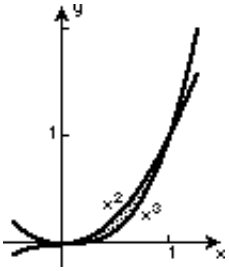


15. $\displaystyle\iint_R (2x+4y+1)\,dA=\int_0^1\int_{x^3}^{x^2}(2x+4y+1)\,dy\,dx=\int_0^1(2xy+2y^2+y)\,\Big|_{x^3}^{x^2}\,dx$

$$=\int_0^1[(2x^3+2x^4+x^2)-(2x^4+2x^6+x^3)]\,dx$$

$$=\int_0^1(x^3+x^2-2x^6)\,dx=\left(\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{2}{7}x^7\right)\,\Big|_0^1$$

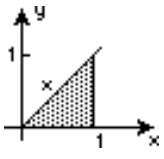
$$=\frac{1}{4}+\frac{1}{3}-\frac{2}{7}=\frac{25}{84}$$



16. $\displaystyle\iint_R xe^y\,dA=\int_0^1\int_0^x xe^y\,dy\,dx=\int_0^1 xe^y\,\Big|_0^x\,dx=\int_0^1(xe^x-x)\,dx$

Integration by parts

$$=\left(xe^x-e^x-\frac{1}{2}x^2\right)\,\Big|_0^1=\left(e-e-\frac{1}{2}\right)-(-1)=\frac{1}{2}$$



17. $\displaystyle\iint_R 2xy\,dA=\int_0^2\int_{x^3}^{8}2xy\,dy\,dx=\int_0^2 xy^2\,\Big|_{x^3}^{8}\,dx=\int_0^2(64x-x^7)dx=\left(32x^2-\frac{1}{8}x^8\right)\,\Big|_0^2=96$

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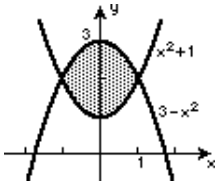
18. $\displaystyle\iint_R \frac{x}{\sqrt{y}}\,dA = \int_{-1}^{1}\int_{x^2+1}^{3-x^2} xy^{-1/2}\,dy\,dx = \int_{-1}^{1} 2x\sqrt{y}\,\Big|_{x^2+1}^{3-x^2} dx$

$= 2\displaystyle\int_{-1}^{1}(x\sqrt{3-x^2} - x\sqrt{x^2+1}\,)\,dx$

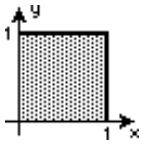
$= 2\left[-\frac{1}{3}(3-x^2)^{3/2} - \frac{1}{3}(x^2+1)^{3/2}\right]\Big|_{-1}^{1}$

$= -\frac{2}{3}[(2^{3/2}+2^{3/2}) - (2^{3/2}+2^{3/2})] = 0$



19. $\displaystyle\iint_R \frac{y}{1+xy}\,dA = \int_0^1\int_0^1 \frac{y}{1+xy}\,dx\,dy = \int_0^1 \ln(1+xy)\,\Big|_0^1 dy = \int_0^1 \ln(1+y)\,dy$

$= [(1+y)\ln(1+y) - (1+y)]\,\Big|_0^1 = (2\ln 2 - 2) - (-1) = 2\ln 2 - 1$

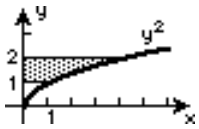


20. $\displaystyle\iint_R \sin\frac{\pi x}{y}\,dA = \int_1^2\int_0^{y^2} \sin\frac{\pi x}{y}\,dx\,dy = \int_1^2 \left(-\frac{y}{\pi}\cos\frac{\pi x}{y}\right)\Big|_0^{y^2} dy$

$= \displaystyle\int_1^2 \left(-\frac{y}{\pi}\cos\pi y + \frac{y}{\pi}\right) dy$ Integration by parts

$= \left(-\frac{y}{\pi^2}\sin\pi y - \frac{1}{\pi^3}\cos\pi y + \frac{y^2}{2\pi}\right)\Big|_1^2 = \left(-\frac{1}{\pi^3} + \frac{2}{\pi}\right) - \left(\frac{1}{\pi^3} + \frac{1}{2\pi}\right)$

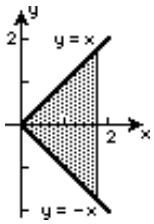
$= \frac{3\pi^2 - 4}{2\pi^3}$



21. $\displaystyle\iint_R \sqrt{x^2+1}\,dA = \int_0^{\sqrt{3}}\int_{-x}^{x} \sqrt{x^2+1}\,dy\,dx = \int_0^{\sqrt{3}} y\sqrt{x^2+1}\,\Big|_{-x}^{x} dx$

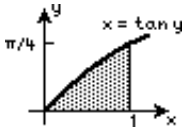
$= \displaystyle\int_0^{\sqrt{3}}(x\sqrt{x^2+1} + x\sqrt{x^2+1}\,)\,dx = \int_0^{\sqrt{3}} 2x\sqrt{x^2+1}\,dx$

$= \frac{2}{3}(x^2+1)^{3/2}\,\Big|_0^{\sqrt{3}} = \frac{2}{3}(4^{3/2} - 1^{3/2}) = \frac{14}{3}$



22. $\displaystyle\iint_R x\,dA = \int_0^{\pi/4}\int_{\tan y}^{1} x\,dx\,dy = \int_0^{\pi/4} \frac{1}{2}x^2\,\Big|_{\tan y}^{1} dy = \frac{1}{2}\int_0^{\pi/4}(1 - \tan^2 y)\,dy$

$= \displaystyle\frac{1}{2}\int_0^{\pi/4}(2 - \sec^2 y)\,dy = \frac{1}{2}(2y - \tan y)\,\Big|_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{2} - 1\right) = \frac{\pi}{4} - \frac{1}{2}$



23. The correct integral is **(c)**.

$V = 2\displaystyle\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} (4-y)\,dx\,dy = 2\int_{-2}^{2}(4-y)x\,\Big|_0^{\sqrt{4-y^2}}\,dy = 2\int_{-2}^{2}(4-y)\sqrt{4-y^2}\,dy$

$= 2\left[2y\sqrt{4-y^2} + 8\sin^{-1}\frac{y}{2} + \frac{1}{3}(4-y^2)^{3/2}\right]\Big|_{-2}^{2} = 2(4\pi - (-4\pi)] = 16\pi$

24. The correct integral is **(b)**.

$A = 8\displaystyle\int_0^2\int_0^{\sqrt{4-y^2}} (4-y^2)^{1/2}dx\,dy = 8\int_0^2 (4-y^2)^{1/2}x\,\Big|_0^{\sqrt{4-y^2}}\,dy = 8\int_0^2 (4-y^2)\,dy$

$= 8\left(4y - \frac{1}{3}y^3\right)\Big|_0^2 = \frac{128}{3}$

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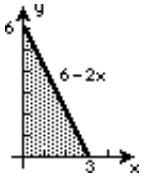
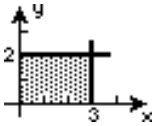
25. Setting $z = 0$ we have $y = 6 - 2x$.



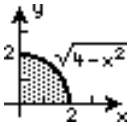
$$V = \int_0^3 \int_0^{6-2x} (6 - 2x - y)\, dy\, dx = \int_0^3 \left(6y - 2xy - \frac{1}{2}y^2\right)\Big|_0^{6-2x} dx$$
$$= \int_0^3 [6(6-2x) - 2x(6-2x) - \frac{1}{2}(6-2x)^2]\, dx = \int_0^3 (18 - 12x + 2x^2)\, dx$$
$$= \left(18x - 6x^2 + \frac{2}{3}x^3\right)\Big|_0^3 = 18$$

26. Setting $z = 0$ we have $y = \pm 2$.



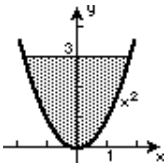
$$V = \int_0^3 \int_0^2 (4 - y^2)\, dy\, dx = \int_0^3 \left(4y - \frac{1}{3}y^3\right)\Big|_0^2 dx = \int_0^3 \frac{16}{3}\, dx = 16$$

27. Solving for z, we have $x = 2 - \frac{1}{2}x + \frac{1}{2}y$. Setting $z = 0$, we see that this surface (plane) intersects the xy-plane in the line $y = x - 4$. since $z(0,0) = 2 > 0$, the surface lies above the xy-plane over the quarter-circular region.



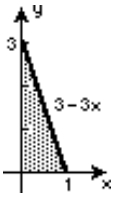
$$V = \int_0^2 \int_0^{\sqrt{4-x^2}} \left(2 - \frac{1}{2}x + \frac{1}{2}y\right) dy\, dx = \int_0^2 \left(2y - \frac{1}{2}xy + \frac{1}{4}y^2\right)\Big|_0^{\sqrt{4-x^2}} dx$$
$$= \int_0^2 \left(2\sqrt{4-x^2} - \frac{1}{2}x\sqrt{4-x^2} + 1 - \frac{1}{4}x^2\right) dx = \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2} + \frac{1}{6}(4-x^2)^{3/2} + x - \frac{1}{12}x^3\right]\Big|_0^2$$
$$= \left(2\pi + 2 - \frac{2}{3}\right) - \frac{4}{3} = 2\pi$$

28. Setting $z = 0$ we have $y = 3$. Using symmetry,



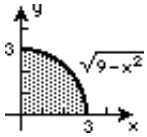
$$V = 2\int_0^{\sqrt{3}} \int_{x^2}^3 (3 - y) dy\, dx = 2\int_0^{\sqrt{3}} \left(3y - \frac{1}{2}y^2\right)\Big|_{x^2}^3 dx = 2\int_0^{\sqrt{3}} \left(\frac{9}{2} - 3x^2 + \frac{1}{2}x^4\right) dx$$
$$= 2\left(\frac{9}{2}x - x^3 + \frac{1}{10}x^5\right)\Big|_0^{\sqrt{3}} = 2\left(\frac{9}{2}\sqrt{3} - 3\sqrt{3} + \frac{9}{10}\sqrt{3}\right) = \frac{24\sqrt{3}}{5}.$$

29. Note that $z = 1 + x^2 + y^2$ is always positive. Then



$$V = \int_0^1 \int_0^{3-3x} (1 + x^2 + y^2) dy\, dx = \int_0^1 \left(y + x^2 y + \frac{1}{3}y^3\right)\Big|_0^{3-3x} dx$$
$$= \int_0^1 [(3 - 3x) + x^2(3 - 3x) + 9(1 - x)^3]\, dx = \int_0^1 (12 - 30x + 30x^2 - 12x^3)\, dx$$
$$= (12x - 15x^2 + 10x^3 - 3x^4)\Big|_0^1 = 4.$$

30. In the first octant, $z = x + y$ is nonnegative. Then



$$V = \int_0^3 \int_0^{\sqrt{9-x^2}} (x + y)\, dy\, dx = \int_0^3 \left(xy + \frac{1}{2}y^2\right)\Big|_0^{\sqrt{9-x^2}} dx$$
$$= \int_0^3 \left(x\sqrt{9-x^2} + \frac{9}{2} - \frac{1}{2}x^2\right) dx = \left[-\frac{1}{3}(9-x^2)^{3/2} + \frac{9}{2}x - \frac{1}{6}x^3\right]\Big|_0^3 = \left(\frac{27}{2} - \frac{9}{2}\right) - (-9) = 18.$$

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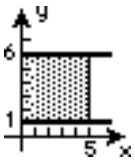
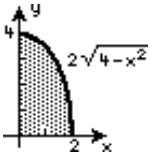
31. In the first octant $z = 6/y$ is positive. Then



$$V = \int_1^6 \int_0^5 \frac{6}{y}\,dx\,dy = \int_1^6 \frac{6x}{y}\,\Big|_0^5\,dy = 30\int_1^6 \frac{dy}{y} = 30\ln y\,\Big|_1^6 = 30\ln 6.$$

32. Setting $z = 0$, we have $x^2/4 + y^2/16 = 1$. Using symmetry,



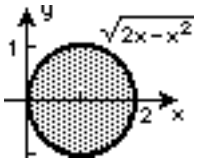
$$V = 4\int_0^2 \int_0^{2\sqrt{4-x^2}} \left(4 - x^2 - \frac{1}{4}y^2\right) dy\,dx = 4\int_0^2 \left(4y - x^2y - \frac{1}{12}y^3\right)\Big|_0^{2\sqrt{4-x^2}}\,dx$$

$$= 4\int_0^2 \left[8\sqrt{4-x^2} - 2x^2\sqrt{4-x^2} - \frac{2}{3}(4-x^2)^{3/2}\right] dx \qquad \boxed{\text{Trig substitution}}$$

$$= 4\left[4x\sqrt{4-x^2} + 16\sin^{-1}\frac{x}{2} - \frac{1}{4}x(2x^2-4)\sqrt{4-x^2} - 4\sin^{-1}\frac{x}{2} + \frac{1}{12}x(2x^2-20)\sqrt{4-x^2} - 4\sin\frac{x}{2}\right]\Big|_0^2$$

$$= 4\left(\frac{16\pi}{2} - \frac{4\pi}{2} - \frac{4\pi}{2}\right) - (0) = 16\pi.$$

33. Note that $z = 4 - y^2$ is positive for $|y| \le 1$. Using symmetry,



$$V = 2\int_0^2 \int_0^{\sqrt{2x-x^2}} (4 - y^2)\,dy\,dx = 2\int_0^2 \left(4y - \frac{1}{3}y^3\right)\Big|_0^{\sqrt{2x-x^2}}\,dx$$

$$= 2\int_0^2 \left[4\sqrt{2x-x^2} - \frac{1}{3}(2x-x^2)\sqrt{2x-x^2}\right] dx$$

$$= 2\int_0^2 \left(4\sqrt{1-(x-1)^2} - \frac{1}{3}[1-(x-1)^2]\sqrt{1-(x-1)^2}\right) dx \qquad \boxed{u = x-1,\ du = dx}$$

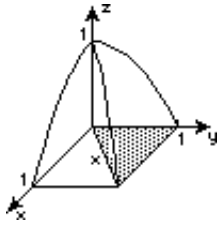
$$= 2\int_{-1}^1 \left[4\sqrt{1-u^2} - \frac{1}{3}(1-u^2)\sqrt{1-u^2}\right] du = 2\int_{-1}^1 \left(\frac{11}{3}\sqrt{1-u^2} + \frac{1}{3}u^2\sqrt{1-u^2}\right) du$$

$\boxed{\text{Trig substitution}}$

$$= 2\left[\frac{11}{6}u\sqrt{1-u^2} + \frac{11}{6}\sin u + \frac{1}{24}x(2x^2-1)\sqrt{1-u^2} + \frac{1}{24}\sin^{-1}u\right]\Big|_{-1}^1$$

$$= 2\left[\left(\frac{11}{6}\,\frac{\pi}{2} + \frac{1}{24}\,\frac{\pi}{2}\right) - \left(-\frac{11}{6}\,\frac{\pi}{2} - \frac{1}{24}\,\frac{\pi}{2}\right)\right] = \frac{15\pi}{4}.$$

34. From $z = 1 - x^2$ and $z = 1 - y^2$ we have $1 - x^2 = 1 - y^2$ or $y = x$ (in the first octant). Thus, the surfaces intersect in the plane $y = x$. Using symmetry,

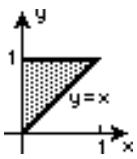


$$V = 2\int_0^1 \int_x^1 (1 - y^2)\,dy\,dx = 2\int_0^1 \left(y - \frac{1}{3}y^3\right)\Big|_x^1\,dx = 2\int_0^1 \left(\frac{2}{3} - x + \frac{1}{3}x^3\right) dx$$

$$= 2\left(\frac{2}{3}x - \frac{1}{2}x^2 + \frac{1}{12}x^4\right)\Big|_0^1 = \frac{1}{2}.$$

35. $\displaystyle\int_0^1 \int_x^1 x^2\sqrt{1+y^4}\,dy\,dx = \int_0^1 \int_0^y x^2\sqrt{1+y^4}\,dx\,dy = \int_0^1 \frac{1}{3}x^3\sqrt{1+y^4}\,\Big|_0^y\,dy$

$$= \frac{1}{3}\int_0^1 y^3\sqrt{1+y^4}\,dy = \frac{1}{3}\left[\frac{1}{6}(1+y^4)^{3/2}\right]\Big|_0^1 = \frac{1}{18}(2\sqrt{2} - 1)$$



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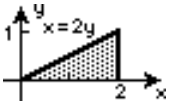
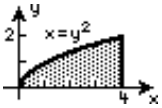
36. $\displaystyle\int_0^1\int_{2y}^2 e^{-y/x}dx\,dy=\int_0^2\int_0^{x/2} e^{-y/x}\,dy\,dx=\int_0^2 -xe^{-y/x}\,\Big|_0^{x/2}=\int_0^2(-xe^{-1/2}+x)\,dx$

$\displaystyle=\int_0^2(1-e^{-1/2})x\,dx=\frac{1}{2}(1-e^{-1/2})x^2\,\Big|_0^2=2(1-e^{-1/2})$



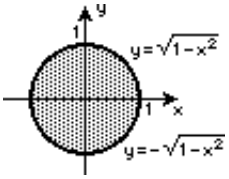
37. $\displaystyle\int_0^2\int_{y^2}^4\cos x^{3/2}\,dx\,dy=\int_0^4\int_0^{\sqrt{x}}\cos x^{3/2}\,dy\,dx=\int_0^4 y\cos x^{3/2}\,\Big|_0^{\sqrt{x}}\,dx$

$\displaystyle=\int_0^4\sqrt{x}\,\cos x^{3/2}\,dx=\frac{2}{3}\sin x^{3/2}\,\Big|_0^4=\frac{2}{3}\sin 8$



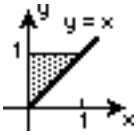
38. $\displaystyle\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}x\sqrt{1-x^2-y^2}\,dy\,dx=\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}x\sqrt{1-x^2-y^2}\,dx\,dy$

$\displaystyle=\int_{-1}^1\left[-\frac{1}{3}(1-x^2-y^2)^{3/2}\right]\Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}dy=-\frac{1}{3}\int_{-1}^1(0-0)dy=0$



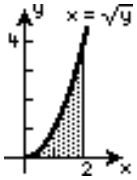
39. $\displaystyle\int_0^1\int_x^1\frac{1}{1+y^4}\,dy\,dx=\int_0^1\int_0^y\frac{1}{1+y^4}\,dx\,dy=\int_0^1\frac{x}{1+y^4}\,\Big|_0^y\,dy=\int_0^1\frac{y}{1+y^4}\,dy$

$\displaystyle=\frac{1}{2}\tan^{-1}y^2\,\Big|_0^1=\frac{\pi}{8}$



40. $\displaystyle\int_0^4\int_{\sqrt{y}}^2\sqrt{x^3+1}\,dx\,dy=\int_0^2\int_0^{x^2}\sqrt{x^3+1}\,dy\,dx=\int_0^2 y\sqrt{x^3+1}\,\Big|_0^{x^2}\,dx$

$\displaystyle=\int_0^2 x^2\sqrt{x^3+1}\,dx=\frac{2}{9}(x^3+1)^{3/2}\,\Big|_0^2=\frac{2}{9}(9^{3/2}-1^{3/2})=\frac{52}{9}$

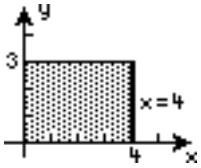


41. $\displaystyle m=\int_0^3\int_0^4 xy\,dx\,dy=\int_0^3\frac{1}{2}x^2y\,\Big|_0^4\,dy=\int_0^3 8y\,dy=4y^2\,\Big|_0^3=36$

$\displaystyle M_y=\int_0^3\int_0^4 x^2y\,dx\,dy=\int_0^3\frac{1}{3}x^3y\,\Big|_0^4\,dy=\int_0^3\frac{64}{3}\,y\,dy=\frac{32}{3}y^2\,\Big|_0^3=96$

$\displaystyle M_x=\int_0^3\int_0^4 xy^2\,dx\,dy=\int_0^3\frac{1}{2}x^2y^2\,\Big|_0^4=\int_0^3 8y^2\,dy=\frac{8}{3}y^3\,\Big|_0^3=72$

$\bar{x}=M_y/m=96/36=8/3;\ \ \bar{y}=M_x/m=72/36=2$. The center of mass is $(8/3,2)$.



42. $\displaystyle m=\int_0^2\int_0^{4-2x}x^2\,dy\,dx=\int_0^2 x^2y\,\Big|_0^{4-2x}\,dx=\int_0^2 x^2(4-2x)\,dx$

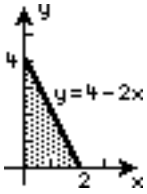
$\displaystyle=\int_0^2(4x^2-2x^3)\,dx=\left(\frac{4}{3}x^3-\frac{1}{2}x^4\right)\Big|_0^2=\frac{32}{3}-8=\frac{8}{3}$

$\displaystyle M_y=\int_0^2\int_0^{4-2x}x^3\,dy\,dx=\int_0^2 x^3y\,\Big|_0^{4-2x}\,dx=\int_0^2 x^3(4-2x)\,dx=\int_0^2(4x^3-2x^4)\,dx$

$\displaystyle=\left(x^4-\frac{2}{5}x^5\right)\Big|_0^2=16-\frac{64}{5}=\frac{16}{5}$

$\displaystyle M_x=\int_0^2\int_0^{4-2x}x^2y\,dy\,dx=\int_0^2\frac{1}{2}x^2y^2\,\Big|\,\int_0^{4-2x}dx=\frac{1}{2}\int_0^2 x^2(4-2x)^2\,dx=\frac{1}{2}\int_0^2(16x^2-16x^3+4x^4)\,dx$

$\displaystyle=2\int_0^2(4x^2-4x^3+x^4)\,dx=2\left(\frac{4}{3}x^3-x^4+\frac{1}{5}x^5\right)\Big|_0^2=2\left(\frac{32}{3}-16+\frac{32}{5}\right)=\frac{32}{15}$



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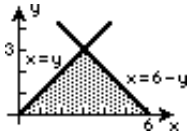
$\bar{x} = M_y/m = \dfrac{16/5}{8/3} = 6/5$; $\bar{y} = M_x/m = \dfrac{32/15}{8/3} = 4/5$. The center of mass is $(6/5, 4/5)$.

43. Since both the region and ρ are symmetric with respect to the line $x = 3$, $\bar{x} = 3$.



$$m = \int_0^3 \int_y^{6-y} 2y\,dx\,dy = \int_0^3 2xy\,\Big|_y^{6-y}\,dy = \int_0^3 2y(6-y-y)\,dy = \int_0^3 (12y - 4y^2)\,dy$$

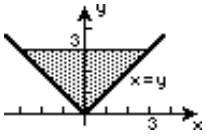
$$= \left(6y^2 - \frac{4}{3}y^3\right)\Big|_0^3 = 18$$

$$M_x = \int_0^3 \int_y^{6-y} 2y^2\,dx\,dy = \int_0^3 2xy^2\,\Big|_y^{6-y}\,dx\,dy = \int_0^3 2y^2(6-y-y)\,dy = \int_0^3 (12y^2 - 4y^3)\,dy$$

$$= (4y^3 - y^4)\Big|_0^3 = 27$$

$\bar{y} = M_x/m = 27/18 = 3/2$. The center of mass is $(3, 3/2)$.

44. Since both the region and ρ are symmetric with respect to the y-axis, $\bar{x} = 0$. Using symmetry,



$$m = \int_0^3 \int_0^y (x^2 + y^2)\,dx\,dy = \int_0^3 \left(\frac{1}{3}x^3 + xy^2\right)\Big|_0^y\,dy = \int_0^3 \left(\frac{1}{3}y^3 + y^3\right)dy$$

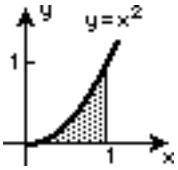
$$= \frac{4}{3}\int_0^3 y^3\,dy = \frac{1}{3}y^4\Big|_0^3 = 27.$$

$$M_x = \int_0^3 \int_0^y (x^2y + y^3)\,dx\,dy = \int_0^3 \left(\frac{1}{3}x^3y + xy^3\right)\Big|_0^y\,dy = \int_0^3 \left(\frac{1}{3}y^4 + y^4\right)dy = \frac{4}{3}\int_0^3 y^4\,dy$$

$$= \frac{4}{15}y^5\Big|_0^3 = \frac{324}{5}$$

$\bar{y} = M_x/m = \dfrac{324/5}{27} = 12/5$. The center of mass is $(0, 12/5)$.

45. $m = \displaystyle\int_0^1 \int_0^{x^2} (x+y)\,dy\,dx = \int_0^1 \left(xy + \frac{1}{2}y^2\right)\Big|_0^{x^2} dx = \int_0^1 \left(x^3 + \frac{1}{2}x^4\right)dx$



$$= \left(\frac{1}{4}x^4 + \frac{1}{10}x^5\right)\Big|_0^1 = \frac{7}{20}$$

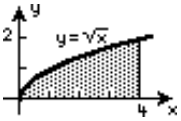
$$M_y = \int_0^1 \int_0^{x^2} (x^2 + xy)\,dy\,dx = \int_0^1 \left(x^2y + \frac{1}{2}xy^2\right)\Big|_0^{x^2} dx = \int_0^1 \left(x^4 + \frac{1}{2}x^5\right)dx$$

$$= \left(\frac{1}{5}x^5 + \frac{1}{12}x^6\right)\Big|_0^1 = \frac{17}{60}$$

$$M_x = \int_0^1 \int_0^{x^2} (xy + y^2)\,dy\,dx = \int_0^1 \left(\frac{1}{2}xy^2 + \frac{1}{3}y^3\right)\Big|_0^{x^2} = \int_0^1 \left(\frac{1}{2}x^5 + \frac{1}{3}x^6\right)dx = \left(\frac{1}{12}x^6 + \frac{1}{21}x^7\right)\Big|_0^1 = \frac{11}{84}$$

$\bar{x} = M_y/m = \dfrac{17/60}{7/20} = 17/21$; $\bar{y} = M_x/m = \dfrac{11/84}{7/20} = 55/147$. The center of mass is $(17/21, 55/147)$.

46. $m = \displaystyle\int_0^4 \int_0^{\sqrt{x}} (y+5)\,dy\,dx = \int_0^4 \left(\frac{1}{2}y^2 + 5y\right)\Big|_0^{\sqrt{x}} dx = \int_0^4 \left(\frac{1}{2}x + 5\sqrt{x}\right)dx$



$$= \left(\frac{1}{4}x^2 + \frac{10}{3}x^{3/2}\right)\Big|_0^4 = \frac{92}{3}$$

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$$M_y = \int_0^4 \int_0^{\sqrt{x}} (xy + 5x)\, dy\, dx = \int_0^4 \left(\frac{1}{2}xy^2 + 5xy\right)\Big|_0^{\sqrt{x}} dx = \int_0^4 \left(\frac{1}{2}x^2 + 5x^{3/2}\right) dx$$

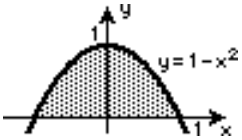
$$= \left(\frac{1}{6}x^3 + 2x^{5/2}\right)\Big|_0^4 = \frac{224}{3}$$

$$M_x = \int_0^4 \int_0^{\sqrt{x}} (y^2 + 5y)\, dy\, dx = \int_0^4 \left(\frac{1}{3}y^3 + \frac{5}{2}y^2\right)\Big|_0^{\sqrt{x}} dx = \int_0^4 \left(\frac{1}{3}x^{3/2} + \frac{5}{2}x\right) dx$$

$$= \left(\frac{2}{15}x^{5/2} + \frac{5}{4}x^2\right)\Big|_0^4 = \frac{364}{15}$$

$\bar{x} = M_y/m = \dfrac{224/3}{92/3} = 56/23$; $\bar{y} = M_x/m = \dfrac{364/15}{92/3} = 91/115$. The center of mass is $(56/23, 91/115)$.

47. The density is $\rho = ky$. Since both the region and ρ are symmetric with respect to the y-axis, $\bar{x} = 0$. Using symmetry,



$$m = 2\int_0^1 \int_0^{1-x^2} ky\, dy\, dx = 2k\int_0^1 \frac{1}{2}y^2\Big|_0^{1-x^2} dx = k\int_0^1 (1-x^2)^2\, dx$$

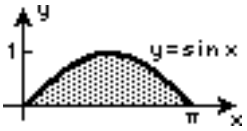
$$= k\int_0^1 (1 - 2x^2 + x^4)\, dx = k\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1 = k\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{8}{15}k$$

$$M_x = 2\int_0^1 \int_0^{1-x^2} ky^2\, dy\, dx = 2k\int_0^1 \frac{1}{3}y^3\Big|_0^{1-x^2} dx = \frac{2}{3}k\int_0^1 (1-x^2)^3\, dx = \frac{2}{3}k\int_0^1 (1 - 3x^2 + 3x^4 - x^6)\, dx$$

$$= \frac{2}{3}k\left(x - x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7\right)\Big|_0^1 = \frac{2}{3}k\left(1 - 1 + \frac{3}{5} - \frac{1}{7}\right) = \frac{32}{105}\,k$$

$\bar{y} = M_x/m = \dfrac{32k/105}{8k/15} = 4/7$. The center of mass is $(0, 4/7)$.

48. The density is $\rho = kx$.



$$m = \int_0^\pi \int_0^{\sin x} kx\, dy\, dx = \int_0^\pi kxy\Big|_0^{\sin x} dx = \int_0^\pi kx\sin x\, dx$$

Integration by parts

$$= k(\sin x - x\cos x)\Big|_0^\pi = k\pi$$

$$M_y = \int_0^\pi \int_0^{\sin x} kx^2\, dy\, dx = \int_0^\pi kx^2 y\Big|_0^{\sin x} dx = \int_0^\pi kx^2\sin x\, dx$$ Integration by parts

$$= k(-x^2\cos x + 2\cos x + 2x\sin x)\Big|_0^\pi = k[(\pi^2 - 2) - 2] = k(\pi^2 - 4)$$

$$M_x = \int_0^\pi \int_0^{\sin x} kxy\, dy\, dx = \int_0^\pi \frac{1}{2}kxy^2\Big|_0^{\sin x} dx = \int_0^\pi \frac{1}{2}kx\sin^2 x\, dx = \int_0^\pi \frac{1}{4}kx(1 - \cos 2x)\, dx$$

$$= \frac{1}{4}k\left[\int_0^\pi x\, dx - \int_0^\pi x\cos 2x\, dx\right]$$ Integration by parts

$$= \frac{1}{4}k\left[\frac{1}{2}x^2\Big|_0^\pi - \frac{1}{4}(\cos 2x + 2x\sin 2x)\Big|_0^\pi\right] = \frac{1}{4}k\left(\frac{1}{2}\pi^2\right) = \frac{1}{8}k\pi^2$$

$\bar{x} = M_y/m = \dfrac{k(\pi^2 - 4)}{k\pi} = \pi - 4/\pi$; $\bar{y} = M_x/m = \dfrac{k\pi^2/8}{k\pi} = \pi/8$. The center of mass is $(\pi - 4/\pi, \pi/8)$.

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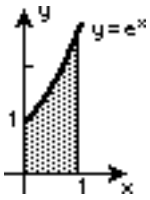
49. $m = \displaystyle\int_0^1 \int_0^{e^x} y^3\,dy\,dx = \int_0^1 \frac{1}{4}y^4 \Big|_0^{e^x} dx = \int_0^1 \frac{1}{4}e^{4x}\,dx = \frac{1}{16}e^{4x}\Big|_0^1 = \frac{1}{16}(e^4-1)$



$M_y = \displaystyle\int_0^1 \int_0^{e^x} xy^3\,dy\,dx = \int_0^1 \frac{1}{4}xy^4 \Big|_0^{e^x} dx = \int_0^1 \frac{1}{4}xe^{4x}\,dx$ Integration by parts

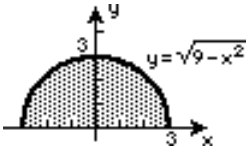
$= \dfrac{1}{4}\left(\dfrac{1}{4}xe^{4x} - \dfrac{1}{16}e^{4x}\right)\Big|_0^1 = \dfrac{1}{4}\left(\dfrac{3}{16}e^4 + \dfrac{1}{16}\right) = \dfrac{1}{64}(3e^4+1)$

$M_x = \displaystyle\int_0^1 \int_0^{e^x} y^4\,dy\,dx = \int_0^1 \frac{1}{5}y^5 \Big|_0^{e^x} dx = \int_0^1 \frac{1}{5}e^{5x}\,dx = \frac{1}{25}e^{5x}\Big|_0^1 = \frac{1}{25}(e^5-1)$

$\bar{x} = M_y/m = \dfrac{(3e^4+1)/64}{(e^4-1)/16} = \dfrac{3e^4+1}{4(e^4-1)}$; $\bar{y} = M_x/m = \dfrac{(e^5-1)/25}{(e^4-1)/16} = \dfrac{16(e^5-1)}{25(e^4-1)}$

The center of mass is $\left(\dfrac{3e^4+1}{4(e^4-1)}, \dfrac{16(e^5-1)}{25(e^4-1)}\right) \approx (0.77, 1.76)$.

50. Since both the region and ρ are symmetric with respect to the y-axis, $\bar{x} = 0$. Using symmetry,



$m = 2\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} x^2\,dy\,dx = 2\int_0^3 x^2 y\Big|_0^{\sqrt{9-x^2}} dx = 2\int_0^3 x^2\sqrt{9-x^2}\,dx$

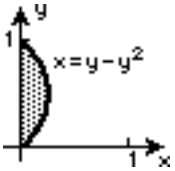
Trig substitution

$= 2\left[\dfrac{x}{8}(2x^2-9)\sqrt{9-x^2} + \dfrac{81}{8}\sin^{-1}\dfrac{x}{3}\right]\Big|_0^3 = \dfrac{81}{4}\,\dfrac{\pi}{2} = \dfrac{81\pi}{2}$.

$M_x = 2\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} x^2 y\,dy\,dx = 2\int_0^3 \frac{1}{2}x^2y^2\Big|_0^{\sqrt{9-x^2}} dy\,dx = \int_0^3 x^2(9-x^2)\,dx = \left(3x^2 - \frac{1}{5}x^5\right)\Big|_0^3 = \frac{162}{5}$

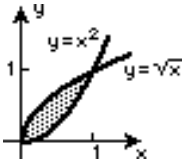
$\bar{y} = M_x/m = \dfrac{162/5}{81\pi/8} = 16/5\pi$. The center of mass is $(0, 16/5\pi)$.

51. $I_x = \displaystyle\int_0^1 \int_0^{y-y^2} 2xy^2\,dx\,dy = \int_0^1 x^2y^2\Big|_0^{y-y^2} dy = \int_0^1 (y-y^2)^2\,y^2\,dy$



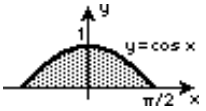
$= \displaystyle\int_0^1 (y^4 - 2y^5 + y^6)\,dy = \left(\frac{1}{5}y^5 - \frac{1}{3}y^6 + \frac{1}{7}y^7\right)\Big|_0^1 = \frac{1}{105}$

52. $I_x = \displaystyle\int_0^1 \int_{x^2}^{\sqrt{x}} x^2y^2\,dy\,dx = \int_0^1 \frac{1}{3}x^2y^3\Big|_{x^2}^{\sqrt{x}} dx = \frac{1}{3}\int_0^1 (x^{7/2} - x^8)\,dx$



$= \dfrac{1}{3}\left(\dfrac{2}{9}x^{9/2} - \dfrac{1}{9}x^9\right)\Big|_0^1 = \dfrac{1}{27}$

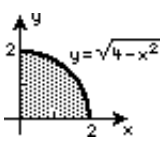
53. Using symmetry,



$I_x = 2\displaystyle\int_0^{\pi/2} \int_0^{\cos x} ky^2\,dy\,dx = 2k\int_0^{\pi/2} \frac{1}{3}y^3\Big|_0^{\cos x} dx = \frac{2}{3}k\int_0^{\pi/2} \cos^3 x\,dx$

$= \dfrac{2}{3}k\displaystyle\int_0^{\pi/2} \cos x(1-\sin^2 x)\,dx = \frac{2}{3}k\left(\sin x - \frac{1}{3}\sin^3 x\right)\Big|_0^{\pi/2} = \frac{4}{9}k.$

54. $I_x = \displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} y^3\,dy\,dx = \int_0^2 \frac{1}{4}y^4\Big|_0^{\sqrt{4-x^2}} dx = \frac{1}{4}\int_0^2 (4-x^2)^2\,dx$



$= \dfrac{1}{4}\displaystyle\int_0^2 (16 - 8x^2 + x^4)\,dx = \frac{1}{4}\left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^2 = \frac{1}{4}\left(32 - \frac{64}{3} + \frac{32}{5}\right)$

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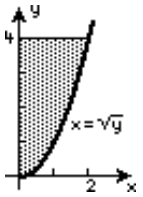
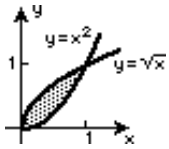
$$= 8\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{64}{15}$$

55. $I_y = \displaystyle\int_0^4 \int_0^{\sqrt{y}} x^2 y \, dx \, dy = \int_0^4 \frac{1}{3}x^3 y \Big|_0^{\sqrt{y}} dy = \frac{1}{3}\int_0^4 y^{3/2} y \, dy = \frac{1}{3}\int_0^4 y^{5/2} \, dy$

$= \dfrac{1}{3}\left(\dfrac{2}{7}y^{7/2}\right)\Big|_0^4 = \dfrac{2}{21}(4^{7/2}) = \dfrac{256}{21}$

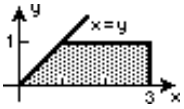


56. $I_y = \displaystyle\int_0^1 \int_{x^2}^{\sqrt{x}} x^4 \, dy \, dx = \int_0^1 x^4 y \Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 (x^{9/2} - x^6)\, dx = \left(\frac{2}{11}x^{11/2} - \frac{1}{7}x^7\right)\Big|_0^1 = \frac{3}{77}$



57. $I_y = \displaystyle\int_0^1 \int_y^3 (4x^3 + 3x^2 y)\, dx \, dy = \int_0^1 (x^4 + x^3 y)\Big|_y^3 dy = \int_0^1 (81 + 27y - 2y^4)\, dy$

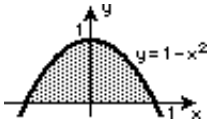
$= \left(81y + \dfrac{27}{2}y^2 - \dfrac{2}{5}y^5\right)\Big|_0^1 = \dfrac{941}{10}$



58. The density is $\rho = ky$. Using symmetry,

$I_y = 2\displaystyle\int_0^1 \int_0^{1-x^2} kx^2 y \, dy \, dx = 2\int_0^1 \frac{1}{2}kx^2 y^2 \Big|_0^{1-x^2} dx = k\int_0^1 x^2(1-x^2)^2\, dx$

$= k\displaystyle\int_0^1 (x^2 - 2x^4 + x^6)\, dx = k\left(\frac{1}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7\right)\Big|_0^1 = \frac{8k}{105}\,.$



59. Using symmetry,

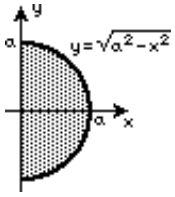
$m = 2\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} x \, dx \, dy = 2\int_0^a \frac{1}{2}x^2 \Big|_0^{\sqrt{a^2-y^2}} dy = \int_0^a (a^2 - y^2)\, dy$

$= \left(a^2 y - \dfrac{1}{3}y^3\right)\Big|_0^a = \dfrac{2}{3}a^3.$

$I_y = 2\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} x^3 \, dx \, dy = 2\int_0^a \frac{1}{4}x^4 \Big|_0^{\sqrt{a^2-y^2}} dy = \frac{1}{2}\int_0^a (a^2 - y^2)^2\, dy$

$= \dfrac{1}{2}\displaystyle\int_0^a (a^4 - 2a^2y^2 + y^4)\, dy = \frac{1}{2}\left(a^4 y - \frac{2}{3}a^2 y^3 + \frac{1}{5}y^5\right)\Big|_0^a = \frac{4}{15}a^5$

$R_g = \sqrt{\dfrac{I_y}{m}} = \sqrt{\dfrac{4a^5/15}{2a^3/3}} = \sqrt{\dfrac{2}{5}}\; a$

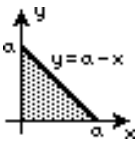


60. $m = \displaystyle\int_0^a \int_0^{a-x} k \, dy \, dx = \int_0^a ky\Big|_0^{a-x} dx = k\int_0^a (a-x)\, dx = k\left(ax - \frac{1}{2}x^2\right)\Big|_0^a = \frac{1}{2}ka^2$

$I_x = \displaystyle\int_0^a \int_0^{a-x} ky^2 \, dy \, dx = \int_0^a \frac{1}{3}ky^3 \Big|_0^{a-x} dx = \frac{1}{3}k\int_0^a (a-x)^3\, dx$

$= \dfrac{1}{3}k\displaystyle\int_0^a (a^3 - 3a^2x + 3ax^2 - x^3)\, dx = \frac{1}{3}k\left(a^3 x - \frac{3}{2}a^2x^2 + ax^3 - \frac{1}{4}x^4\right)\Big|_0^a = \frac{1}{12}ka^4$

$R_g = \sqrt{\dfrac{I_x}{m}} = \sqrt{\dfrac{ka^4/12}{ka^2/2}} = \sqrt{\dfrac{1}{6}}\; a$



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61. **(a)** Using symmetry,

$$I_x = 4\int_0^a \int_0^{b\sqrt{a^2-x^2}/a} y^2\,dy\,dx = \frac{4b^3}{3a^3}\int_0^a (a^2-x^2)^{3/2}dx \qquad \boxed{x = a\sin\theta,\ dx = a\cos\theta\,d\theta}$$

$$= \frac{4}{3}ab^3\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{4}{3}ab^3\int_0^{\pi/2}\frac{1}{4}(1+\cos 2\theta)^2 d\theta = \frac{1}{3}ab^3\int_0^{\pi/2}\left(1+\cos 2\theta+\frac{1}{2}+\frac{1}{2}\cos 4\theta\right)d\theta$$

$$= \frac{1}{3}ab^3\left(\frac{3}{2}\theta+\frac{1}{2}\sin 2\theta+\frac{1}{8}\sin 4\theta\right)\Big|_0^{\pi/2} = \frac{ab^3\pi}{4}.$$

(b) Using symmetry,

$$I_y = 4\int_0^a \int_0^{b\sqrt{a^2-x^2}/a} x^2\,dy\,dx = \frac{4b}{a}\int_0^a x^2\sqrt{a^2-x^2}\,dx \qquad \boxed{x = a\sin\theta,\ dx = a\cos\theta\,d\theta}$$

$$= 4a^3b\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta = 4a^3b\int_0^{\pi/2}\frac{1}{4}(1-\cos^2 2\theta)\,d\theta$$

$$= a^3b\int_0^{\pi/2}\left(1-\frac{1}{2}-\frac{1}{2}\cos 4\theta\right)d\theta = a^3b\left(\frac{1}{2}\theta-\frac{1}{8}\sin 4\theta\right)\Big|_0^{\pi/2} = \frac{a^3b\pi}{4}.$$

(c) Using $m = \pi ab$, $R_g = \sqrt{I_x/m} = \frac{1}{2}\sqrt{ab^3\pi/\pi ab} = \frac{1}{2}b$.

(d) $R_g = \sqrt{I_y/m} = \frac{1}{2}\sqrt{a^3b\pi/\pi ab} = \frac{1}{2}a$

62. The equation of the ellipse is $9x^2/a^2 + 4y^2/b^2 = 1$ and the equation of the parabola is $y = \pm(9bx^2/8a^2 - b/2)$. Letting I_e and I_p represent the moments of inertia of the ellipse and parabola, respectively, about the x-axis, we have

$$I_e = 2\int_{-a/3}^0 \int_0^{b\sqrt{a^2-9x^2}/2a} y^2\,dy\,dx = \frac{b^3}{12a^3}\int_{-a/3}^0 (a^2-9x^2)^{3/2}dx \qquad \boxed{x = \frac{a}{3}\sin\theta,\ dx = \frac{a}{3}\cos\theta\,d\theta}$$

$$= \frac{b^3}{12a^3}\,\frac{a^4}{3}\int_{-\pi/3}^0 \cos^4\theta\,d\theta = \frac{b^3a}{36}\,\frac{3\pi}{16} = \frac{ab^3\pi}{192}$$

and

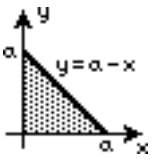
$$I_p = 2\int_0^{2a/3}\int_0^{b/2-9bx^2/8a^2} y^2\,dy\,dx = \frac{2}{3}\int_0^{2a/3}\left(\frac{b}{2}-\frac{9b}{8a^2}x^2\right)^3 dx = \frac{2}{3}\,\frac{b^3}{8}\int_0^{2a/3}\left(1-\frac{9}{4a^2}x^2\right)^3 dx$$

$$= \frac{b^3}{12}\int_0^{2a/3}\left(1-\frac{27}{4a^2}x^2+\frac{243}{16a^4}x^4-\frac{729}{64a^6}x^6\right)dx = \frac{b^3}{12}\left(x-\frac{9}{4a^2}x^3+\frac{243}{80a^4}x^5-\frac{729}{64a^6}x^7\right)\Big|_0^{2a/3}$$

$$= \frac{b^3}{12}\,\frac{32a}{105} = \frac{8ab^3}{315}.$$

Then $I_x = I_e + I_p = \dfrac{ab^3\pi}{192} + \dfrac{8ab^3}{315}$.

63. From Problem 60, $m = \frac{1}{2}ka^2$ and $I_x = \frac{1}{12}ka^4$.



$$I_y = \int_0^a \int_0^{a-x} kx^2\,dy\,dx = \int_0^a kx^2 y\Big|_0^{a-x}\,dx = k\int_0^a x^2(a-x)\,dx$$

$$= k\left(\frac{1}{3}ax^3-\frac{1}{4}x^4\right)\Big|_0^a = \frac{1}{12}ka^4$$

$$I_0 = I_x + I_y = \frac{1}{12}ka^4 + \frac{1}{12}ka^4 = \frac{1}{6}ka^4$$

64. From Problem 52, $I_x = \dfrac{1}{27}$, and from Problem 56, $I_y = \dfrac{3}{77}$. Thus, $I_0 = I_x + I_y = \dfrac{1}{27} + \dfrac{3}{77} = \dfrac{158}{2079}$.

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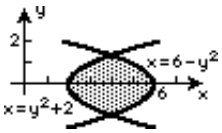
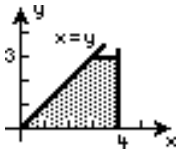
65. The density is $\rho = k/(x^2+y^2)$. Using symmetry,



$$I_0 = 2\int_0^{\sqrt{2}}\int_{y^2+2}^{6-y^2}(x^2+y^2)\,\frac{k}{x^2+y^2}\,dx\,dy = 2\int_0^{\sqrt{2}} kx\,\Big|_{y^2+2}^{6-y^2}\,dy$$

$$= 2k\int_0^{\sqrt{2}}(6-y^2-y^2-2)\,dy = 2k\left(4y-\frac{2}{3}y^3\right)\Big|_0^{\sqrt{2}} = 2k\left(\frac{8}{3}\sqrt{2}\right) = \frac{16\sqrt{2}}{3}\,k.$$

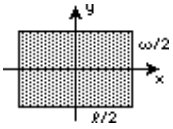
66. $I_0 = \displaystyle\int_0^3\int_y^4 k(x^2+y^2)\,dx\,dy = k\int_0^3\left(\frac{1}{3}x^3+xy^2\right)\Big|_y^4\,dy$



$$= k\int_0^3\left(\frac{64}{3}+4y^2-\frac{1}{3}y^3-y^3\right)dy = k\left(\frac{64}{3}y+\frac{4}{3}y^3-\frac{1}{3}y^4\right)\Big|_0^3 = 73k$$

67. From Problem 60, $m = \frac{1}{2}ka^2$, and from Problem 61, $I_0 = \frac{1}{6}ka^4$. Then $R_g = \sqrt{I_0/m} = \sqrt{\dfrac{ka^4/6}{ka^2/2}} = \sqrt{\dfrac{1}{3}}\,a$.

68. Since the plate is homogeneous, the density is $\rho = m/\ell\omega$. Using symmetry,

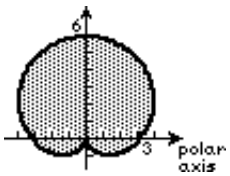


$$I_0 = 4\int_0^{\ell/2}\int_0^{\omega/2}\frac{m}{\ell\omega}(x^2+y^2)\,dy\,dx = \frac{4m}{\ell\omega}\int_0^{\ell/2}\left(x^2y+\frac{1}{3}y^3\right)\Big|_0^{\omega/2}\,dx$$

$$= \frac{4m}{\ell\omega}\int_0^{\ell/2}\left(\frac{\omega}{2}\,x^2+\frac{\omega^3}{24}\right)dx = \frac{4m}{\ell\omega}\left(\frac{\omega}{6}\,x^3+\frac{\omega^3}{24}\,x\right)\Big|_0^{\ell/2} = \frac{4m}{\ell\omega}\left(\frac{\omega\ell^3}{48}+\frac{\ell\omega^3}{48}\right) = m\,\frac{\ell^2+\omega^2}{12}\,.$$

Exercises 9.11

1. Using symmetry,

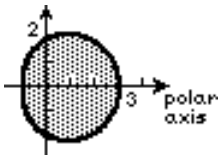


$$A = 2\int_{-\pi/2}^{\pi/2}\int_0^{3+3\sin\theta} r\,dr\,d\theta = 2\int_{-\pi/2}^{\pi/2}\frac{1}{2}r^2\,\Big|_0^{3+3\sin\theta}\,d\theta = \int_{-\pi/2}^{\pi/2}9(1+\sin\theta)^2\,d\theta$$

$$= 9\int_{-\pi/2}^{\pi/2}(1+2\sin\theta+\sin^2\theta)\,d\theta = 9\left(\theta-2\cos\theta+\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)\Big|_{-\pi/2}^{\pi/2}$$

$$= 9\left[\frac{3}{2}\,\frac{\pi}{2}-\frac{3}{2}\left(-\frac{\pi}{2}\right)\right] = \frac{27\pi}{2}$$

2. Using symmetry,



$$A = 2\int_0^{\pi}\int_0^{2+\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi}\frac{1}{2}r^2\,\Big|_0^{2+\cos\theta}\,d\theta = \int_0^{\pi}(2+\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi}(4+4\cos\theta+\cos^2\theta)\,d\theta = \left(4\theta+4\sin\theta+\frac{1}{2}\theta+\frac{1}{4}\cos 2\theta\right)\Big|_0^{\pi}$$

$$= \left(4\pi+\frac{\pi}{2}+\frac{1}{4}\right)-\left(\frac{1}{4}\right) = \frac{9\pi}{2}\,.$$

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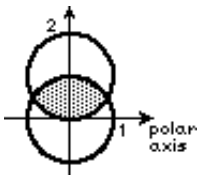
3. Solving $r = 2\sin\theta$ and $r = 1$, we obtain $\sin\theta = 1/2$ or $\theta = \pi/6$. Using symmetry,

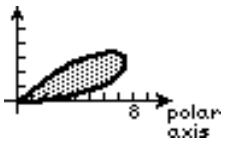


$$A = 2\int_0^{\pi/6}\int_0^{2\sin\theta} r\,dr\,d\theta + 2\int_{\pi/6}^{\pi/2}\int_0^{1} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/6}\frac{1}{2}r^2\Big|_0^{2\sin\theta}d\theta + 2\int_{\pi/6}^{\pi/2}\frac{1}{2}r^2\Big|_0^{1}d\theta = \int_0^{\pi/6}4\sin^2\theta\,d\theta + \int_{\pi/6}^{\pi/2}d\theta$$

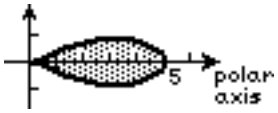
$$= (2\theta - \sin 2\theta)\Big|_0^{\pi/6} + \left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\pi}{3} = \frac{4\pi - 3\sqrt{3}}{6}$$

4. $A = \displaystyle\int_0^{\pi/4}\int_0^{8\sin 4\theta} r\,dr\,d\theta = \int_0^{\pi/4}\frac{1}{2}r^2\Big|_0^{8\sin 4\theta}d\theta = \frac{1}{2}\int_0^{\pi/4}64\sin^2 4\theta\,d\theta$



$$= 32\left(\frac{1}{2}\theta - \frac{1}{16}\sin 8\theta\right)\Big|_0^{\pi/4} = 4\pi$$

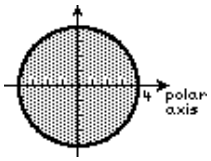
5. Using symmetry,



$$V = 2\int_0^{\pi/6}\int_0^{5\cos 3\theta} 4r\,dr\,d\theta = 4\int_0^{\pi/6} r^2\Big|_0^{5\cos 3\theta}d\theta = 4\int_0^{\pi/6}25\cos^2 3\theta\,d\theta$$

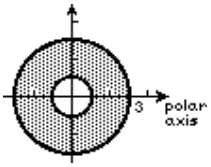
$$= 100\left(\frac{1}{2}\theta + \frac{1}{12}\sin 6\theta\right)\Big|_0^{\pi/6} = \frac{25\pi}{3}$$

6. $V = \displaystyle\int_0^{2\pi}\int_0^{2}\sqrt{9-r^2}\,r\,dr\,d\theta = \int_0^{2\pi} -\frac{1}{3}(9-r^2)^{3/2}\Big|_0^{2}d\theta$



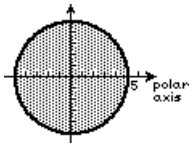
$$= -\frac{1}{3}\int_0^{2\pi}(5^{3/2} - 27)\,d\theta = \frac{1}{3}(27 - 5^{3/2})2\pi = \frac{2\pi(27 - 5\sqrt{5}\,)}{3}$$

7. $V = \displaystyle\int_0^{2\pi}\int_1^{3}\sqrt{16-r^2}\,r\,dr\,d\theta = \int_0^{2\pi} -\frac{1}{3}(16-r^2)^{3/2}\Big|_1^{3}d\theta$

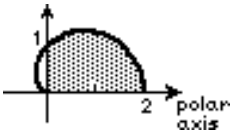


$$= -\frac{1}{3}\int_0^{2\pi}(7^{3/2} - 15^{3/2})\,d\theta = \frac{1}{3}(15^{3/2} - 7^{3/2})2\pi = \frac{2\pi(15\sqrt{15} - 7\sqrt{7}\,)}{3}$$

8. $V = \displaystyle\int_0^{2\pi}\int_0^{5}\sqrt{r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{3}r^3\Big|_0^{5}d\theta = \int_0^{2\pi}\frac{125}{3}\,d\theta = \frac{250\pi}{3}$



9. $V = \displaystyle\int_0^{\pi/2}\int_0^{1+\cos\theta}(r\sin\theta)r\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{3}r^3\sin\theta\Big|_0^{1+\cos\theta}d\theta$



$$= \frac{1}{3}\int_0^{\pi/2}(1+\cos\theta)^3\sin\theta\,d\theta = \frac{1}{3}\left[-\frac{1}{4}(1+\cos\theta)^4\right]\Big|_0^{\pi/2} = -\frac{1}{12}(1-2^4) = \frac{5}{4}$$

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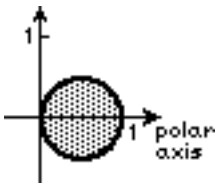
10. Using symmetry,



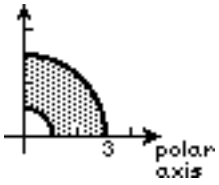
$$V = 2\int_0^{\pi/2}\int_0^{\cos\theta}(2+r^2)\,r\,dr\,d\theta = \int_0^{\pi/2}\left(r^2+\frac{1}{4}r^4\right)\Big|_0^{\cos\theta}d\theta$$

$$= 2\int_0^{\pi/2}\left(\cos^2\theta+\frac{1}{4}\cos^4\theta\right)d\theta = 2\int_0^{\pi/2}\left[\cos^2\theta+\frac{1}{4}\left(\frac{1+\cos 2\theta}{2}\right)^2\right]d\theta$$

$$= \int_0^{\pi/2}\left(2\cos^2\theta+\frac{1}{8}+\frac{1}{4}\cos 2\theta+\frac{1}{8}\cos^2 2\theta\right)d\theta$$

$$= \left(\theta+\frac{1}{2}\sin 2\theta+\frac{1}{8}\theta+\frac{1}{8}\sin 2\theta+\frac{1}{16}\theta+\frac{1}{64}\sin 4\theta\right)\Big|_0^{\pi/2} = \frac{19\pi}{32}.$$

11.



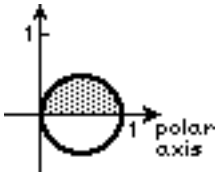
$$m = \int_0^{\pi/2}\int_1^3 kr\,dr\,d\theta = k\int_0^{\pi/2}\frac{1}{2}r^2\Big|_1^3 d\theta = \frac{1}{2}k\int_0^{\pi/2}8\,d\theta = 2k\pi$$

$$M_y = \int_0^{\pi/2}\int_1^3 kxr\,dr\,d\theta = k\int_0^{\pi/2}\int_1^3 r^2\cos\theta\,dr\,d\theta = k\int_0^{\pi/2}\frac{1}{3}r^3\cos\theta\Big|_1^3 d\theta$$

$$= \frac{1}{3}k\int_0^{\pi/2}26\cos\theta\,d\theta = \frac{26}{3}k\sin\theta\Big|_0^{\pi/2} = \frac{26}{3}k$$

$\bar{x} = M_y/m = \dfrac{26k/3}{2k\pi} = \dfrac{13}{3\pi}$. Since the region and density function are symmetric about the ray $\theta = \pi/4$, $\bar{y} = \bar{x} = 13/3\pi$ and the center of mass is $(13/3\pi, 13/3\pi)$.

12. The interior of the upper-half circle is traced from $\theta = 0$ to $\pi/2$. The density is kr. Since both the region and the density are symmetric about the polar axis, $\bar{y} = 0$.



$$m = \int_0^{\pi/2}\int_0^{\cos\theta} kr^2\,dr\,d\theta = k\int_0^{\pi/2}\frac{1}{3}r^3\Big|_0^{\cos\theta}d\theta = \frac{k}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta$$

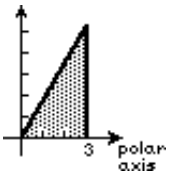
$$= \frac{k}{3}\left(\frac{2}{3}+\frac{1}{3}\cos^2\theta\right)\sin\theta\Big|_0^{\pi/2} = \frac{2k}{9}$$

$$M_y = k\int_0^{\pi/2}\int_0^{\cos\theta}(r\cos\theta)(r)(r\,dr\,d\theta) = k\int_0^{\pi/2}\int_0^{\cos\theta} r^3\cos\theta\,dr\,d\theta = k\int_0^{\pi/2}\frac{1}{4}r^4\cos\theta\Big|_0^{\cos\theta}d\theta$$

$$= \frac{k}{4}\int_0^{\pi/2}\cos^5\theta\,d\theta = \frac{k}{4}\left(\sin\theta-\frac{2}{3}\sin^3\theta+\frac{1}{5}\sin^5\theta\right)\Big|_0^{\pi/2} = \frac{2k}{15}$$

Thus, $\bar{x} = \dfrac{2k/15}{2k/9} = 3/5$ and the center of mass is $(3/5, 0)$.

13. In polar coordinates the line $x = 3$ becomes $r\cos\theta = 3$ or $r = 3\sec\theta$. The angle of inclination of the line $y = \sqrt{3}\,x$ is $\pi/3$.



$$m = \int_0^{\pi/3}\int_0^{3\sec\theta} r^2 r\,dr\,d\theta = \int_0^{\pi/3}\frac{1}{4}r^4\Big|_0^{3\sec\theta}d\theta = \frac{81}{4}\int_0^{\pi/3}\sec^4\theta\,d\theta$$

$$= \frac{81}{4}\int_0^{\pi/3}(1+\tan^2\theta)\sec^2\theta\,d\theta = \frac{81}{4}\left(\tan\theta+\frac{1}{3}\tan^3\theta\right)\Big|_0^{\pi/3} = \frac{81}{4}(\sqrt{3}+\sqrt{3}) = \frac{81}{2}\sqrt{3}$$

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$$M_y = \int_0^{\pi/3}\int_0^{3\sec\theta} xr^2 r\,dr\,d\theta = \int_0^{\pi/3}\int_0^{3\sec\theta} r^4\cos\theta\,dr\,d\theta = \int_0^{\pi/3}\frac{1}{5}r^5\cos\theta\,\Big|_0^{3\sec\theta}\,d\theta$$

$$= \frac{243}{5}\int_0^{\pi/3}\sec^5\theta\cos\theta\,d\theta = \frac{243}{5}\int_0^{\pi/3}\sec^4\theta\,d\theta = \frac{243}{5}(2\sqrt{3}\,) = \frac{486}{5}\sqrt{3}$$

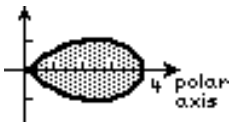
$$M_x = \int_0^{\pi/3}\int_0^{3\sec\theta} yr^2 r\,dr\,d\theta = \int_0^{\pi/3}\int_0^{3\sec\theta} r^4\sin\theta\,d\theta = \int_0^{\pi/3}\frac{1}{5}r^5\sin\theta\,\Big|_0^{3\sec\theta}$$

$$= \frac{243}{5}\int_0^{\pi/3}\sec^5\theta\sin\theta\,d\theta = \frac{243}{5}\int_0^{\pi/3}\tan\theta\sec^4\theta\,d\theta = \frac{243}{5}\int_0^{\pi/3}\tan\theta(1+\tan^2\theta)\sec^2\theta\,d\theta$$

$$= \frac{243}{5}\int_0^{\pi/3}(\tan\theta+\tan^3\theta)\sec^2\theta\,d\theta = \frac{243}{5}\left(\frac{1}{2}\tan^2\theta+\frac{1}{4}\tan^4\theta\right)\Big|_0^{\pi/3} = \frac{243}{5}\left(\frac{3}{2}+\frac{9}{4}\right) = \frac{729}{4}$$

$\bar{x} = M_y/m = \dfrac{486\sqrt{3}/5}{81\sqrt{3}/2} = 12/5;\ \ \bar{y} = M_x/m = \dfrac{729/4}{81\sqrt{3}/2} = 3\sqrt{3}/2$. The center of mass is $(12/5, 3\sqrt{3}/2)$.

14. Since both the region and the density are symmetric about the x-axis, $\bar{y} = 0$. Using symmetry,



$$m = 2\int_0^{\pi/4}\int_0^{4\cos2\theta} kr\,dr\,d\theta = 2k\int_0^{\pi/4}\frac{1}{2}r^2\,\Big|_0^{4\cos2\theta}\,d\theta = 16k\int_0^{\pi/4}\cos^2 2\theta\,d\theta$$

$$= 16k\left(\frac{1}{2}\theta+\frac{1}{8}\sin4\theta\right)\Big|_0^{\pi/4} = 2k\pi$$

$$M_y = 2\int_0^{\pi/4}\int_0^{4\cos2\theta} kxr\,dr\,d\theta = 2k\int_0^{\pi/4}\int_0^{4\cos2\theta} r^2\cos\theta\,dr\,d\theta = 2k\int_0^{\pi/4}\frac{1}{3}r^3\cos\theta\,\Big|_0^{4\cos2\theta}\,d\theta$$

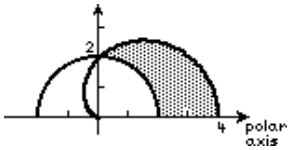
$$= \frac{128}{3}k\int_0^{\pi/4}\cos^3 2\theta\cos\theta\,d\theta = \frac{128}{3}k\int_0^{\pi/4}(1-2\sin^2\theta)^3\cos\theta\,d\theta$$

$$= \frac{128}{3}k\int_0^{\pi/4}(1-6\sin^2\theta+12\sin^4\theta-8\sin^6\theta)\cos\theta\,d\theta = \frac{128}{3}k\left(\sin\theta-2\sin^3\theta+\frac{12}{5}\sin^5\theta-\frac{8}{7}\sin^7\theta\right)\Big|_0^{\pi/4}$$

$$= \frac{128}{3}k\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}+\frac{3\sqrt{2}}{10}-\frac{\sqrt{2}}{14}\right) = \frac{1024}{105}\sqrt{2}\,k$$

$\bar{x} = M_y/m = \dfrac{1024\sqrt{2}\,k/105}{2k\pi} = \dfrac{512\sqrt{2}}{105\pi}$. The center of mass is $(512\sqrt{2}/105\pi, 0)$ or approximately $(2.20, 0)$.

15. The density is $\rho = k/r$.



$$m = \int_0^{\pi/2}\int_2^{2+2\cos\theta}\frac{k}{r}\,r\,dr\,d\theta = k\int_0^{\pi/2}\int_2^{2+2\cos\theta} dr\,d\theta$$

$$= k\int_0^{\pi/2} 2\cos\theta\,d\theta = 2k(\sin\theta)\,\Big|_0^{\pi/2} = 2k$$

$$M_y = \int_0^{\pi/2}\int_2^{2+2\cos\theta} x\,\frac{k}{r}\,r\,dr\,d\theta = k\int_0^{\pi/2}\int_2^{2+2\cos\theta} r\,\cos\theta\,dr\,d\theta = k\int_0^{\pi/2}\frac{1}{2}r^2\,\Big|_2^{2+2\cos\theta}\cos\theta\,d\theta$$

$$= \frac{1}{2}k\int_0^{\pi/2}(8\cos\theta+4\cos^2\theta)\cos\theta\,d\theta = 2k\int_0^{\pi/2}(2\cos^2\theta+\cos\theta-\sin^2\theta\cos\theta)\,d\theta$$

$$= 2k\left(\theta+\frac{1}{2}\sin2\theta+\sin\theta-\frac{1}{3}\sin^3\theta\right)\Big|_0^{\pi/2} = 2k\left(\frac{\pi}{2}+\frac{2}{3}\right) = \frac{3\pi+4}{3}\,k$$

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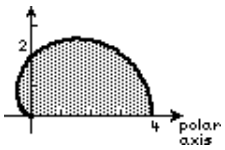
$$M_x = \int_0^{\pi/2}\int_2^{2+2\cos\theta} y\,\frac{k}{r}\,r\,dr\,d\theta = k\int_0^{\pi/2}\int_2^{2+2\cos\theta} r\,\sin\theta\,dr\,d\theta = k\int_0^{\pi/2}\frac{1}{2}r^2\Big|_2^{2+2\cos\theta}\sin\theta\,d\theta$$

$$= \frac{1}{2}k\int_0^{\pi/2}(8\cos\theta + 4\cos^2\theta)\sin\theta\,d\theta = \frac{1}{2}k\left(-4\cos^2\theta - \frac{4}{3}\cos^3\theta\right)\Big|_0^{\pi/2} = \frac{1}{2}k\left[-\left(-4-\frac{4}{3}\right)\right] = \frac{8}{3}k$$

$\bar{x} = M_y/m = \dfrac{(3\pi+4)k/3}{2k} = \dfrac{3\pi+4}{6}$; $\bar{y} = M_x/m = \dfrac{8k/3}{2k} = \dfrac{4}{3}$. The center of mass is $((3\pi+4)/6, 4/3)$.

16. $$m = \int_0^{\pi}\int_0^{2+2\cos\theta} kr\,dr\,d\theta = k\int_0^{\pi}\frac{1}{2}r^2\Big|_0^{2+2\cos\theta}d\theta = 2k\int_0^{\pi}(1+\cos\theta)^2 d\theta$$



$$= 2k\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta = 2k\left(\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi} = 3\pi k$$

$$M_y = \int_0^{\pi}\int_0^{2+2\cos\theta} kxr\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+2\cos\theta} r^2\cos\theta\,dr\,d\theta = k\int_0^{\pi}\frac{1}{3}r^3\Big|_0^{2+2\cos\theta}\cos\theta\,d\theta$$

$$= \frac{8}{3}k\int_0^{\pi}(1+\cos\theta)^3\cos\theta\,d\theta = \frac{8}{3}k\int_0^{\pi}(\cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)\,d\theta$$

$$= \frac{8}{3}k\left[\sin\theta + \left(\frac{3}{2}\theta + \frac{3}{4}\sin 2\theta\right) + (3\sin\theta - \sin^3\theta) + \left(\frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right)\right]\Big|_0^{\pi} = \frac{8}{3}k\left(\frac{15}{8}\pi\right) = 5\pi k$$

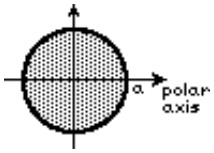
$$M_x = \int_0^{\pi}\int_0^{2+2\cos\theta} kyr\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+2\cos\theta} r^2\sin\theta\,dr\,d\theta = k\int_0^{\pi}\frac{1}{3}r^3\Big|_0^{2+2\cos\theta}\sin\theta\,d\theta$$

$$= \frac{8}{3}k\int_0^{\pi}(1+\cos\theta)^3\sin\theta\,d\theta = \frac{8}{3}k\int_0^{\pi}(1+3\cos\theta+3\cos^2\theta+\cos^3\theta)\sin\theta\,d\theta$$

$$= \frac{8}{3}k\left(-\cos\theta - \frac{3}{2}\cos^2\theta - \cos^3\theta - \frac{1}{4}\cos^4\theta\right)\Big|_0^{\pi} = \frac{8}{3}k\left[\frac{1}{4} - \left(-\frac{15}{4}\right)\right] = \frac{32}{3}k$$

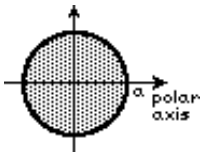
$\bar{x} = M_y/m = \dfrac{5\pi k}{3\pi k} = 5/3$; $\bar{y} = M_x/m = \dfrac{32k/3}{3\pi k} = 32/9\pi$. The center of mass is $(5/3, 32/9\pi)$.

17. $$I_x = \int_0^{2\pi}\int_0^{a} y^2 kr\,dr\,d\theta = k\int_0^{2\pi}\int_0^{a} r^3\sin^2\theta\,dr\,d\theta = k\int_0^{2\pi}\frac{1}{4}r^4\sin^2\theta\Big|_0^{a}d\theta$$



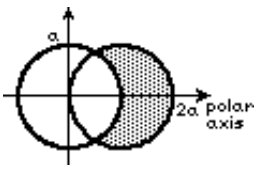
$$= \frac{ka^4}{4}\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{ka^4}{4}\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{k\pi a^4}{4}$$

18. $$I_x = \int_0^{2\pi}\int_0^{a} y^2\,\frac{1}{1+r^4}\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{a}\frac{r^3}{1+r^4}\sin^2\theta\,dr\,d\theta$$



$$= \int_0^{2\pi}\frac{1}{4}\ln(1+r^4)\Big|_0^{a}\sin^2\theta\,d\theta = \frac{1}{4}\ln(1+a^4)\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{\pi}{4}\ln(1+a^4)$$

19. Solving $a = 2a\cos\theta$, $\cos\theta = 1/2$ or $\theta = \pi/3$. The density is k/r^3. Using symmetry,



$$I_y = 2\int_0^{\pi/3}\int_a^{2a\cos\theta} x^2\,\frac{k}{r^3}\,r\,dr\,d\theta = 2k\int_0^{\pi/3}\int_a^{2a\cos\theta}\cos^2\theta\,dr\,d\theta$$

$$= 2k\int_0^{\pi/3}(2a\cos^3\theta - a\cos^2\theta)\,d\theta = 2ak\left(2\sin\theta - \frac{2}{3}\sin^3\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/3}$$

$$= 2ak\left(\sqrt{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) = \frac{5ak\sqrt{3}}{4} - \frac{ak\pi}{3}$$

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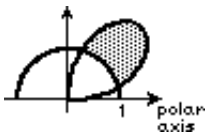
20. Solving $1 = 2\sin 2\theta$, we obtain $\sin 2\theta = 1/2$ or $\theta = \pi/12$ and $\theta = 5\pi/12$.

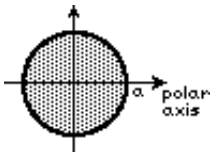


$$I_y = \int_{\pi/12}^{5\pi/12}\int_1^{2\sin 2\theta} x^2\sec^2\theta\, r\, dr\, d\theta = \int_{\pi/12}^{5\pi/12}\int_1^{2\sin 2\theta} r^3\, dr\, d\theta$$

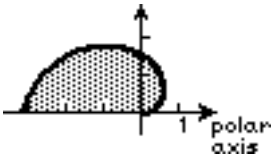
$$= \int_{\pi/12}^{5\pi/12} \frac{1}{4}r^4\Big|_1^{2\sin 2\theta} d\theta = 4\int_{\pi/12}^{5\pi/12}\sin^4 2\theta\, d\theta = 2\left(\frac{3}{4}\theta - \frac{1}{4}\sin 4\theta + \frac{1}{32}\sin 8\theta\right)\Big|_{\pi/12}^{5\pi/12}$$

$$= 2\left[\left(\frac{5\pi}{16} + \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{64}\right) - \left(\frac{\pi}{16} - \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{64}\right)\right] = \frac{8\pi + 7\sqrt{3}}{16}$$

21. From Problem 17, $I_x = k\pi a^4/4$. By symmetry, $I_y = I_x$. Thus $I_0 = k\pi a^4/2$.



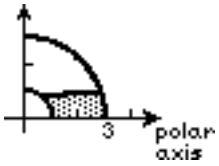
22. The density is $\rho = kr$.



$$I_0 = \int_0^{\pi}\int_0^{\theta} r^2(kr)r\, dr\, d\theta = k\int_0^{\pi}\int_0^{\theta} r^4\, dr\, d\theta = k\int_0^{\pi}\frac{1}{5}r^5\Big|_0^{\theta} d\theta$$

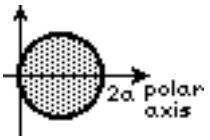
$$= \frac{1}{5}k\int_0^{\pi}\theta^5\, d\theta = \frac{1}{5}k\left(\frac{1}{6}\theta^6\right)\Big|_0^{\pi} = \frac{k\pi^6}{30}$$

23. The density is $\rho = k/r$.



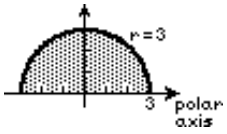
$$I_0 = \int_1^3\int_0^{1/r} r^2\frac{k}{r}\, r\, d\theta\, dr = k\int_1^3\int_0^{1/r} r^2\, d\theta\, dr = k\int_1^3 r^2\left(\frac{1}{r}\right) dr = k\left(\frac{1}{2}r^2\right)\Big|_1^3 = 4k$$

24. $I_0 = \displaystyle\int_0^{\pi}\int_0^{2a\cos\theta} r^2 kr\, dr\, d\theta = k\int_0^{\pi}\frac{1}{4}r^4\Big|_0^{2a\cos\theta} d\theta = 4ka^4\int_0^{\pi}\cos^4\theta\, d\theta$

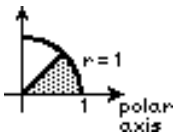


$$= 4ka^4\left(\frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right)\Big|_0^{\pi} = 4ka^4\left(\frac{3\pi}{8}\right) = \frac{3k\pi a^4}{2}$$

25. $\displaystyle\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\sqrt{x^2+y^2}\, dy\, dx = \int_0^{\pi}\int_0^3 |r|r\, dr\, d\theta = \int_0^{\pi}\frac{1}{3}r^3\Big|_0^3 d\theta = 9\int_0^{\pi} d\theta = 9\pi$



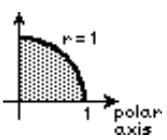
26. $\displaystyle\int_0^{\sqrt{2}/2}\int_y^{\sqrt{1-y^2}}\frac{y^2}{\sqrt{x^2+y^2}}\, dx\, dy = \int_0^{\pi/4}\int_0^1\frac{r^2\sin^2\theta}{|r|}\, r\, dr\, d\theta$



$$= \int_0^{\pi/4}\int_0^1 r^2\sin^2\theta\, dr\, d\theta = \int_0^{\pi/4}\frac{1}{3}r^3\sin^2\theta\Big|_0^1 d\theta = \frac{1}{3}\int_0^{\pi/4}\sin^2\theta\, d\theta$$

$$= \frac{1}{3}\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/4} = \frac{\pi - 2}{24}$$

27. $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\, dx\, dy = \int_0^{\pi/2}\int_0^1 e^{r^2} r\, dr\, d\theta = \int_0^{\pi/2}\frac{1}{2}e^{r^2}\Big|_0^1 d\theta$



$$= \frac{1}{2}\int_0^{\pi/2}(e-1)\, d\theta = \frac{\pi(e-1)}{4}$$

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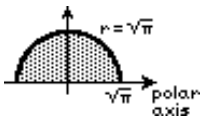
28. $\displaystyle\int_{-\sqrt{\pi}}^{\sqrt{\pi}}\int_0^{\sqrt{\pi-x^2}} \sin(x^2+y^2)\,dy\,dx = \int_0^{\pi}\int_0^{\sqrt{\pi}} (\sin r^2) r\,dr\,d\theta = \int_0^{\pi} -\frac{1}{2}\cos r^2 \Big|_0^{\sqrt{\pi}}\,d\theta$

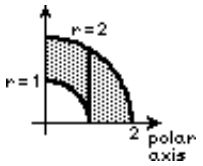
$$= -\frac{1}{2}\int_0^{\pi}(-1-1)\,d\theta = \pi$$



29. $\displaystyle\int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \frac{x^2}{x^2+y^2}\,dy\,dx + \int_1^2\int_0^{\sqrt{4-x^2}} \frac{x^2}{x^2+y^2}\,dy\,dx$

$$= \int_0^{\pi/2}\int_1^2 \frac{r^2\cos^2\theta}{r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\int_1^2 r\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\frac{1}{2}r^2\Big|_1^2\cos^2\theta\,d\theta = \frac{3}{2}\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{3}{2}\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2} = \frac{3\pi}{8}$$



30. $\displaystyle\int_0^1\int_0^{\sqrt{2y-y^2}} (1-x^2-y^2)\,dx\,dy$

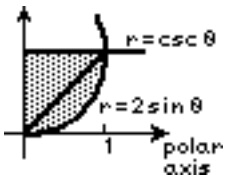
$$= \int_0^{\pi/4}\int_0^{2\sin\theta}(1-r^2)r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{\csc\theta}(1-r^2)r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\left(\frac{1}{2}r^2-\frac{1}{4}r^4\right)\Big|_0^{2\sin\theta}d\theta + \int_{\pi/4}^{\pi/2}\left(\frac{1}{2}r^2-\frac{1}{4}r^4\right)\Big|_0^{\csc\theta}d\theta$$

$$= \int_0^{\pi/4}(2\sin^2\theta-4\sin^4\theta)\,d\theta + \int_{\pi/4}^{\pi/2}\left(\frac{1}{2}\csc^2\theta-\frac{1}{4}\csc^4\theta\right)d\theta$$

$$= \left[\theta-\frac{1}{2}\sin 2\theta-\left(\frac{3}{2}\theta-\sin 2\theta+\frac{1}{8}\sin 4\theta\right)\right]+\left[-\frac{1}{2}\cot\theta-\frac{1}{4}\left(-\cot\theta-\frac{1}{3}\cot^3\theta\right)\right]\Big|_{\pi/4}^{\pi/2}$$

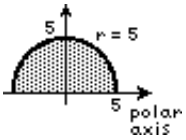
$$= \left(-\frac{\pi}{8}+\frac{1}{2}\right)+\left[0-\left(-\frac{1}{4}+\frac{1}{12}\right)\right] = \frac{16-3\pi}{24}$$



31. $\displaystyle\int_{-5}^{5}\int_0^{\sqrt{25-x^2}} (4x+3y)\,dy\,dx = \int_0^{\pi}\int_0^5 (4r\cos\theta+3r\sin\theta)r\,dr\,d\theta$

$$= \int_0^{\pi}\int_0^5 (4r^2\cos\theta+3r^2\sin\theta)\,dr\,d\theta = \int_0^{\pi}\left(\frac{4}{3}r^3\cos\theta+r^3\sin\theta\right)\Big|_0^5 d\theta$$

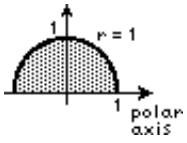
$$= \int_0^{\pi}\left(\frac{500}{3}\cos\theta+125\sin\theta\right)d\theta = \left(\frac{500}{3}\sin\theta-125\cos\theta\right)\Big|_0^{\pi} = 250$$



32. $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}} \frac{1}{1+\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{\pi/2}\int_0^1 \frac{1}{1+r}\,r\,dr\,d\theta$

$$= \int_0^{\pi/2}\int_0^1\left(1-\frac{1}{1+r}\right)dr\,d\theta = \int_0^{\pi/2}[r-\ln(1+r)]\Big|_0^1 d\theta$$

$$= \int_0^{\pi/2}(1-\ln 2)d\theta = \frac{\pi}{2}(1-\ln 2)$$



33. The volume of the cylindrical portion of the tank is $V_c = \pi(4.2)^2 19.3 \approx 1069.56$ m^3. We take the equation of the ellipsoid to be

$$\frac{x^2}{(4.2)^2}+\frac{z^2}{(5.15)^2} = 1 \quad \text{or} \quad z = \pm\frac{5.15}{4.2}\sqrt{(4.2)^2-x^2-y^2}\,.$$

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The volume of the ellipsoid is

$$V_e = 2\left(\frac{5.15}{4.2}\right)\iint_R \sqrt{(4.2)^2 - x^2 - y^2}\,dx\,dy = \frac{10.3}{4.2}\int_0^{2\pi}\int_0^{4.2}[(4.2)^2 - r^2]^{1/2} r\,dr\,d\theta$$
$$= \frac{10.3}{4.2}\int_0^{2\pi}\left[\left(-\frac{1}{2}\right)\frac{2}{3}[(4.2)^2 - r^2]^{3/2}\,\Big|_0^{4.2}\right]d\theta = \frac{10.3}{4.2}\,\frac{1}{3}\int_0^{2\pi}(4.2)^3\,d\theta$$
$$= \frac{2\pi}{3}\,\frac{10.3}{4.2}(4.2)^3 \approx 380.53.$$

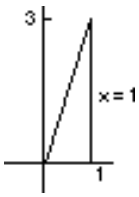
The volume of the tank is approximately $1069.56 + 380.53 = 1450.09$ m^3.

34. $$\iint_R (x+y)\,dA = \int_0^{\pi/2}\int_{2\sin\theta}^{2}(r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = \int_0^{\pi/2}\int_{2\sin\theta}^{2} r^2(\cos\theta + \sin\theta)\,dr\,d\theta$$
$$= \int_0^{\pi/2}\frac{1}{3}r^3(\cos\theta + \sin\theta)\,\Big|_{2\sin\theta}^{2}\,d\theta = \frac{8}{3}\int_0^{\pi/2}(\cos\theta + \sin\theta - \sin^3\theta\cos\theta - \sin^4\theta)\,d\theta$$
$$= \frac{8}{3}\left(\sin\theta - \cos\theta - \frac{1}{4}\sin^4\theta + \frac{1}{4}\sin^3\theta\cos\theta - \frac{3}{8}\theta + \frac{3}{16}\sin 2\theta\right)\Big|_0^{\pi/2}$$
$$= \frac{8}{3}\left[\left(1 - \frac{1}{4} - \frac{3\pi}{16}\right) - (-1)\right] = \frac{28 - 3\pi}{6}$$

35. $$I^2 = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dx\,dy = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta = \int_0^{\pi/2}\lim_{t\to\infty} -\frac{1}{2}e^{-r^2}\,\Big|_0^{t}\,d\theta$$
$$= \int_0^{\pi/2}\lim_{t\to\infty}\left(-\frac{1}{2}e^{-t^2} + \frac{1}{2}\right)d\theta = \int_0^{\pi/2}\frac{1}{2}\,d\theta = \frac{\pi}{4}; \quad I = \frac{\sqrt{\pi}}{2}$$

Exercises 9.12

1. The sides of the triangle are C_1: $y = 0$, $0 \le x \le 1$; C_2: $x = 1$, $0 \le y \le 3$; C_3: $y = 3x$, $0 \le -x \le 1$.



$$\oint_C (x-y)\,dx + xy\,dy = \int_0^1 x\,dx + \int_0^3 y\,dy + \int_1^0 (x-3x)\,dx + \int_1^0 x(3x)\,3\,dx$$
$$= \left(\frac{1}{2}x^2\right)\Big|_0^1 + \left(\frac{1}{2}y^2\right)\Big|_0^3 + (-x^2)\,\Big|_0^1 + (3x^2)\,\Big|_1^0 = \frac{1}{2} + \frac{9}{2} + 1 - 3 = 3$$

$$\iint_R (y+1)\,dA = \int_0^1\int_0^{3x}(y+1)\,dy\,dx = \int_0^1\left(\frac{1}{2}y^2 + y\right)\Big|_0^{3x}\,dx = \int_0^1\left(\frac{9}{2}x^2 + 3x\right)dx$$
$$= \left(\frac{3}{2}x^3 + \frac{3}{2}x^2\right)\Big|_0^1 = 3$$

2. The sides of the rectangle are C_1: $y = 0$, $-1 \le x \le 1$; C_2: $x = 1$, $0 \le y \le 1$; C_3: $y = 1$, $1 \ge x \ge -1$; C_4: $x = -1$, $1 \ge y \ge 0$.

$$\oint_C 3x^2y\,dx + (x^2 - 5y)\,dy = \int_{-1}^1 0\,dx + \int_0^1 (1-5y)\,dy = \int_{-1}^1 0\,dx + \int_0^1 (1-5y)\,dy$$
$$= \int_1^{-1} 3x^2\,dx + \int_1^0 (1-5y)\,dy = \left(y - \frac{5}{2}y^2\right)\Big|_0^1 + x^3\,\Big|_1^{-1} + \left(y - \frac{5}{2}y^2\right)\Big|_1^0 = -2$$

$$\iint_R (2x - 3x^2)\,dA = \int_0^1\int_{-1}^1 (2x - 3x^2)\,dx\,dy = \int_0^1 (x^2 - x^3)\,\Big|_{-1}^1\,dy = \int_0^1 (-2)\,dy = -2$$

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3. $\displaystyle\oint_C -y^2\,dx + x^2\,dy = \int_0^{2\pi} (-9\sin^2 t)(-3\sin t)\,dt + \int_0^{2\pi} 9\cos^2 t(3\cos t)\,dt$

$$= 27\int_0^{2\pi} [(1-\cos^2 t)\sin t + (1-\sin^2 t)\cos t]\,dt$$

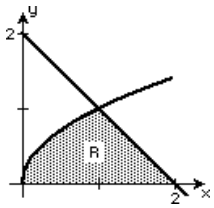
$$= 27\left(-\cos t + \frac{1}{3}\cos^3 t + \sin t - \frac{1}{3}\sin^3 t\right)\Big|_0^{2\pi} = 27(0) = 0$$

$$\iint_R (2x+2y)\,dA = 2\int_0^{2\pi}\int_0^3 (r\cos\theta + r\sin\theta) r\,dr\,d\theta = 2\int_0^{2\pi}\int_0^3 r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\left[\frac{1}{3}r^3(\cos\theta+\sin\theta)\right]\Big|_0^3 d\theta = 18\int_0^{2\pi}(\cos\theta+\sin\theta)\,d\theta$$

$$= 18(\sin\theta - \cos\theta)\Big|_0^{2\pi} = 18(0) = 0$$

4. The sides of the region are C_1: $y = 0$, $0 \le x \le 2$; C_2: $y = -x+2$, $2 \ge x \ge 1$; C_3: $y = \sqrt{x}$, $1 \ge x \ge 0$.



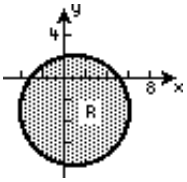
$$\oint_C -2y^2\,dx + 4xy\,dy = \int_0^2 0\,dx + \int_2^1 -2(-x+2)^2\,dx + \int_2^1 4x(-x+2)(-dx)$$

$$+ \int_1^0 -2x\,dx + \int_1^0 4x\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx$$

$$= 0 + \frac{2}{3} + \frac{8}{3} + 1 - 1 = \frac{10}{3}$$

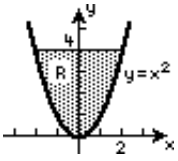
$$\iint_R 8y\,dA = \int_0^1\int_{y^2}^{2-y} 8y\,dx\,dy = \int_0^1 8y(2-y-y^2)\,dy = \left(8y^2 - \frac{8}{3}y^3 - 2y^4\right)\Big|_0^1 = \frac{10}{3}$$

5. $P = 2y$, $P_y = 2$, $Q = 5x$, $Q_x = 5$



$$\oint_C 2y\,dx + 5x\,dy = \iint_R (5-2)\,dA = 3\iint_R dA = 3(25\pi) = 75\pi$$

6. $P = x + y^2$, $P_y = 2y$, $Q = 2x^2 - y$, $Q_x = 4x$

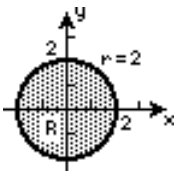


$$\oint_C (x+y^2)\,dx + (2x^2-y)\,dy = \iint_R (4x-2y)\,dA = \int_{-2}^2\int_{x^2}^4 (4x-2y)\,dy\,dx$$

$$= \int_{-2}^2 (4xy - y^2)\Big|_{x^2}^4\,dx = \int_{-2}^2 (16x - 16 - 4x^3 + x^4)\,dx$$

$$= \left(8x^2 - 16x - x^4 + \frac{1}{5}x^5\right)\Big|_{-2}^2 = -\frac{96}{5}$$

7. $P = x^4 - 2y^3$, $P_y = -6y^2$, $Q = 2x^3 - y^4$, $Q_x = 6x^2$. Using polar coordinates,



$$\oint_C (x^4 - 2y^3)\,dx + (2x^3 - y^4)\,dy = \iint_R (6x^2+6y^2)\,dA = \int_0^{2\pi}\int_0^2 6r^2 r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{3}{2}r^4\right)\Big|_0^2 d\theta = \int_0^{2\pi} 24\,d\theta = 48\pi.$$

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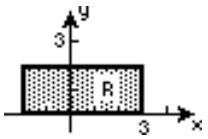
8. $P = x - 3y$, $P_y = -3$, $Q = 4x + y$, $Q_x = 4$

$$\oint_C (x - 3y)\,dx + 4(x + y)\,dy = \iint_R (4 + 3)\,dA = 7(10) = 70$$

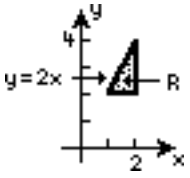


9. $P = 2xy$, $P_y = 2x$, $Q = 3xy^2$, $Q_x = 3y^2$

$$\oint_C 2xy\,dx + 3xy^2\,dy = \iint_R (3y^2 - 2x)\,dA = \int_1^2 \int_2^{2x} (3y^2 - 2x)\,dy\,dx$$

$$= \int_1^2 (y^3 - 2xy)\,\Big|_2^{2x}\,dx = \int_1^2 (8x^3 - 4x^2 - 8 + 4x)\,dx$$

$$= \left(2x^4 - \frac{4}{3}x^3 - 8x + 2x^2\right)\Big|_1^2 = \frac{40}{3} - \left(-\frac{16}{3}\right) = \frac{56}{3}$$



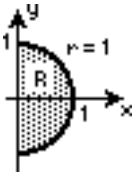
10. $P = e^{2x}\sin 2y$, $P_y = 2e^{2x}\cos 2y$, $Q = e^{2x}\cos 2y$, $Q_x = 2e^{2x}\cos 2y$

$$\oint_C = e^{2x}\sin 2y\,dx + e^{2x}\cos 2y\,dy = \iint_R 0\,dA = 0$$

11. $P = xy$, $P_y = x$, $Q = x^2$, $Q_x = 2x$. Using polar coordinates,

$$\oint_C xy\,dx + x^2\,dy = \iint_R (2x - x)\,dA = \int_{-\pi/2}^{\pi/2} \int_0^1 r\cos\theta\, r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{3}r^3\cos\theta\right)\Big|_0^1\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{3}\cos\theta\,d\theta = \frac{1}{3}\sin\theta\,\Big|_{-\pi/2}^{\pi/2} = \frac{2}{3}$$

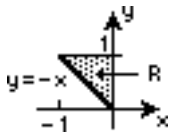


12. $P = e^{x^2}$, $P_y = 0$, $Q = 2\tan^{-1}x$, $Q_x = \dfrac{2}{1 + x^2}$

$$\oint_C e^{x^2}\,dx + 2\tan^{-1}x\,dy = \iint_R \frac{2}{1 + x^2}\,dA = \int_{-1}^0 \int_{-x}^1 \frac{2}{1 + x^2}\,dy\,dx$$

$$= \int_{-1}^0 \left(\frac{2y}{1 + x^2}\right)\Big|_{-x}^1\,dx = \int_{-1}^0 \left(\frac{2}{1 + x^2} + \frac{2x}{1 + x^2}\right)\,dx$$

$$= [2\tan^{-1}x + \ln(1 + x^2)]\,\Big|_{-1}^0 = 0 - \left(-\frac{\pi}{2} + \ln 2\right) = \frac{\pi}{2} - \ln 2$$

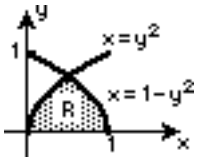


13. $P = \dfrac{1}{3}y^3$, $P_y = y^2$, $Q = xy + xy^2$, $Q_x = y + y^2$

$$\oint_C \frac{1}{3}y^3\,dx + (xy + xy^2)\,dy = \iint_R y\,dA = \int_0^{1/\sqrt{2}} \int_{y^2}^{1-y^2} y\,dx\,dy$$

$$= \int_0^{1/\sqrt{2}} (xy)\,\Big|_{y^2}^{1-y^2}\,dy = \int_0^{1/\sqrt{2}} (y - y^3 - y^3)\,dy$$

$$= \left(\frac{1}{2}y^2 - \frac{1}{2}y^4\right)\Big|_0^{1/\sqrt{2}} = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$



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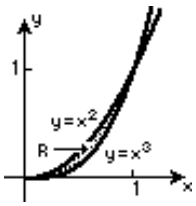
14. $P = xy^2$, $P_y = 2xy$, $Q = 3\cos y$, $Q_x = 0$



$$\oint_C xy^2\,dx + 3\cos y\,dy = \iint_R (-2xy)\,dA = -\int_0^1 \int_{x^3}^{x^2} 2xy\,dy\,dx$$
$$= -\int_0^1 (xy)\Big|_{x^3}^{x^2} dx = -\int_0^1 (x^3 - x^4)\,dx = \left(\frac{1}{4}x^4 - \frac{1}{5}x^5\right)\Big|_0^1 = -\frac{1}{20}$$

15. $P = ay$, $P_y = a$, $Q = bx$, $Q_x = b$. $\displaystyle\oint_C ay\,dx + bx\,dy = \iint_R (b-a)\,dA = (b-a) \times$ (area bounded by C)

16. $P = P(x)$, $P_y = 0$, $Q = Q(y)$, $Q_x = 0$. $\displaystyle\oint_C P(x)\,dx + Q(y)\,dy = \iint_R 0\,dA = 0$

17. For the first integral: $P = 0$, $P_y = 0$, $Q = x$, $Q_x = 1$; $\displaystyle\oint_C x\,dy = \iint_R 1\,dA =$ area of R.

For the second integral: $P = y$, $P_y = 1$, $Q = 0$, $Q_x = 0$; $\displaystyle -\oint_C y\,dx = -\iint_R -1\,dA =$ area of R.

Thus, $\displaystyle\oint_C x\,dy = -\oint_C y\,dx$.

18. $P = -y$, $P_y = -1$, $Q = x$, $Q_x = 1$. $\displaystyle\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\iint_R 2\,dA = \iint_R dA =$ area of R

19. $\displaystyle A = \iint_R dA = \oint_C x\,dy = \int_0^{2\pi} a\cos^3 t(3a\sin^2 t\cos t\,dt) = 3a^2\int_0^{2\pi} \sin^2 t\cos^4 t\,dt$

$$= 3a^2\left(\frac{1}{16}t - \frac{1}{64}\sin 4t + \frac{1}{48}\sin^3 2t\right)\Big|_0^{2\pi} = \frac{3}{8}\pi a^2$$

20. $\displaystyle A = \iint_R dA = \oint_C x\,dy = \int_0^{2\pi} a\cos t(b\cos t\,dt) = ab\int_0^{2\pi} \cos^2 t\,dt = ab\left(\frac{1}{2}t + \frac{1}{4}\sin 2t\right)\Big|_0^{2\pi} = \pi ab$

21. **(a)** Parameterize C by $x = x_1 + (x_2 - x_1)t$ and $y = y_1 + (y_2 - y_1)t$ for $0 \le t \le 1$. Then

$$\int_C -y\,dx + x\,dy = \int_0^1 -[y_1 + (y_2 - y_1)t](x_2 - x_1)\,dt + \int_0^1 [x_1 + (x_2 - x_1)t](y_2 - y_1)\,dt$$
$$= -(x_2 - x_1)\left[y_1 t + \frac{1}{2}(y_2 - y_1)t^2\right]\Big|_0^1 + (y_2 - y_1)\left[x_1 t + \frac{1}{2}(x_2 - x_1)t^2\right]\Big|_0^1$$
$$= -(x_2 - x_1)\left[y_1 + \frac{1}{2}(y_2 - y_1)\right] + (y_2 - y_1)\left[x_1 + \frac{1}{2}(x_2 - x_1)\right] = x_1y_2 - x_2y_1.$$

(b) Let C_i be the line segment from (x_i, y_i) to (x_{i+1}, y_{i+1}) for $i = 1, 2, \ldots, n-1$, and C_2 the line segment from (x_n, y_n) to (x_1, y_1). Then

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy \qquad \boxed{\text{Using Problem 18}}$$
$$= \frac{1}{2}\left[\int_{C_1} -y\,dx + x\,dy + \int_{C_2} -y\,dx + x\,dy + \cdots + \int_{C_{n-1}} -y\,dx + x\,dy + \int_{C_n} -y\,dx + x\,dy\right]$$
$$= \frac{1}{2}(x_1y_2 - x_2y_1) + \frac{1}{2}(x_2y_3 - x_3y_2) + \frac{1}{2}(x_{n-1}y_n - x_ny_{n-1}) + \frac{1}{2}(x_ny_1 - x_1y_n).$$

22. From part **(b)** of Problem 21

$$A = \frac{1}{2}[(-1)(1) - (1)(3)] + \frac{1}{2}[(1)(2) - (4)(1)] + \frac{1}{2}[(4)(5) - (3)(2)] + \frac{1}{2}[(3)(3) - (-1)(5)]$$
$$= \frac{1}{2}(-4 - 2 + 14 + 14) = 11.$$

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23. $P = 4x^2 - y^3$, $P_y = -3y^2$; $Q = x^3 + y^2$, $Q_x = 3x^2$.

$$\oint_C (4x^2 - y^3)\,dx + (x^3 + y^2)\,dy = \iint_R (3x^2 + 3y^2)\,dA = \int_0^{2\pi}\int_1^2 3r^2(r\,dr\,d\theta) = \int_0^{2\pi}\left(\frac{3}{4}r^4\right)\Big|_1^2 d\theta$$
$$= \int_0^{2\pi}\frac{45}{4}\,d\theta = \frac{45\pi}{2}$$

24. $P = \cos x^2 - y$, $P_y = -1$; $Q = \sqrt{y^3+1}$, $Q_x = 0$

$$\oint_C (\cos x^2 - y)\,dx + \sqrt{y^3+1}\,dy = \iint_R (0+1)\,dA \iint_R dA = (6\sqrt{2})^2 - \pi(2)(4) = 72 - 8\pi$$

25. We first observe that $P_y = (y^4 - 3x^2y^2)/(x^2 = y^2)^3 = Q_x$. Letting C' be the circle $x^2 + y^2 = \frac{1}{4}$ we have

$$\oint_C \frac{-y^3\,dx + xy^2\,dy}{(x^2+y^2)^2} = \oint_{C'} \frac{-y^3\,dx + xy^2\,dy}{(x^2+y^2)^2}$$
$$\boxed{x = \tfrac{1}{4}\cos t,\ dx = -\tfrac{1}{4}\sin t\,dt,\ y = \tfrac{1}{4}\sin t,\ dy = \tfrac{1}{4}\cos t\,dt}$$
$$= \int_0^{2\pi} \frac{-\frac{1}{64}\sin^3 t(-\frac{1}{4}\sin t\,dt) + \frac{1}{4}\cos t(\frac{1}{16}\sin^2 t)(\frac{1}{4}\cos t\,dt)}{1/256}$$
$$= \int_0^{2\pi}(\sin^4 t + \sin^2 t\cos^2 t)\,dt = \int_0^{2\pi}(\sin^4 t + (\sin^2 t - \sin^4 t)\,dt$$
$$= \int_0^{2\pi}\sin^2 t\,dt = \left(\frac{1}{2}t - \frac{1}{4}\sin 2t\right)\Big|_0^{2\pi} = \pi$$

26. We first observe that $P_y = [4y^2 - (x+1)^2]/[(x+1)^2 + 4y^2]^2 = Q_x$. Letting C' be the ellipse $(x+1)^2 + 4y^2 = 4$ we have

$$\oint_C \frac{-y}{(x+1)^2+4y^2}\,dx + \frac{x+1}{(x+1)^2+4y^2}\,dy = \oint_{C'} \frac{-y}{(x+1)^2+4y^2}\,dx + \frac{x+1}{(x+1)^2+4y^2}\,dy$$
$$\boxed{x+1 = 2\cos t,\ \ dx = -2\sin t\,dt,\ y = \sin t,\ dy = \cos t\,dt}$$
$$= \int_0^{2\pi}\left[\frac{-\sin t}{4}(-2\sin t) + \frac{2\cos t}{4}\cos t\right]\,dt = \frac{1}{2}\int_0^{2\pi}(\sin^2 t + \cos^2 t)\,dt = \pi.$$

27. Writing $\iint_R x^2\,dA = \iint_R (Q_x - P_y)dA$ we identify $Q = 0$ and $P = -x^2y$. Then, with C: $x = 3\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$, we have

$$\iint_R x^2\,dA = \oint_C P\,dx + Q\,dy = \oint_C -x^2y\,dx = -\int_0^{2\pi} 9\cos^2 t(2\sin t)(-3\sin t)\,dt$$
$$= \frac{54}{4}\int_0^{2\pi} 4\sin^2 t\cos^2 t\,dt = \frac{27}{2}\int_0^{2\pi}\sin^2 2t\,dt = \frac{27}{4}\int_0^{2\pi}(1-\cos 4t)\,dt$$
$$= \frac{27}{4}\left(t - \frac{1}{4}\sin 4t\right)\Big|_0^{2\pi} = \frac{27\pi}{2}.$$

28. Writing $\iint_R [1 - 2(y-1)]\,dA = \iint_R (Q_x - P_y)\,dA$ we identify $Q = x$ and $P = (y-1)^2$. Then, with

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C_1: $x = \cos t$, $y - 1 = \sin t$, $-\pi/2 \le t \le \pi/2$, and C_2: $x = 0$, $2 \ge y \ge 0$,

$$\begin{aligned}\iint_R [1 - 2(y-1)]\,dA &= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_{C_1} (y-1)^2\,dx + x\,dy + \int_{C_2} 0\,dy \\ &= \int_{-\pi/2}^{\pi/2} [\sin^2 t(-\sin t) + \cos t \cos t]\,dt = \int_{-\pi/2}^{\pi/2} [\cos^2 t - (1 - \cos^2 t)\sin t]\,dt \\ &= \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2}(1 + \cos 2t) - \sin t + \cos^2 t \sin t\right] dt \\ &= \left(\frac{1}{2}t + \frac{1}{4}\sin 2t + \cos t - \frac{1}{3}\cos^3 t\right)\Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.\end{aligned}$$

29. $P = x - y$, $P_y = -1$, $Q = x + y$, $Q_x = 1$; $W = \oint_C \mathbf{F} \cdot d\,r = \iint_R 2\,dA = 2 \times \text{area} = 2\left(\frac{3\pi}{4}\right) = \frac{3}{2}\pi$

30. $P = -xy^2$, $P_y = -2xy$, $Q = x^2 y$, $Q_x = 2xy$. Using polar coordinates,

$$\begin{aligned}W = \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R 4xy\,dA = \int_0^{\pi/2} \int_1^2 4(r\cos\theta)(r\sin\theta) r\,dr\,d\theta = \int_0^{\pi/2} (r^4 \cos\theta \sin\theta)\,\Big|_1^2\,d\theta \\ &= 15\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = \frac{15}{2}\sin^2\theta\,\Big|_0^{\pi/2} = \frac{15}{2}.\end{aligned}$$

31. Since $\int_A^B P\,dx + Q\,dy$ is independent of path, $P_y = Q_x$ by Theorem 9.9. Then, by Green's Theorem

$$\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA = \iint_R 0\,dA = 0.$$

32. Let $P = 0$ and $Q = x^2$. Then $Q_x - P_y = 2x$ and

$$\frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_R 2x\,dA = \frac{\iint_R x\,dA}{A} = \bar{x}.$$

Let $P = y^2$ and $Q = 0$. Then $Q_x - P_y = -2y$ and

$$-\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\iint_R -2y\,dA = \frac{\iint_R y\,dA}{A} = \bar{y}.$$

33. Using Green's Theorem,

$$\begin{aligned}W = \oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C -y\,dx + x\,dy = \iint_R 2\,dA = 2\int_0^{2\pi} \int_0^{1+\cos\theta} r\,dr\,d\theta \\ &= 2\int_0^{2\pi} \left(\frac{1}{2}r^2\right)\Big|_0^{1+\cos\theta} d\theta = \int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)\,d\theta \\ &= \left(\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = 3\pi.\end{aligned}$$

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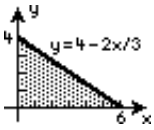
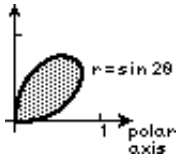
Exercises 9.13

1. Letting $z = 0$, we have $2x + 3y = 12$. Using $f(x, y) = z = 3 - \frac{1}{2}x - \frac{3}{4}y$ we have $f_x = -\frac{1}{2}$, $f_y = -\frac{3}{4}$, $1 + f_x^2 + f_y^2 = \frac{29}{16}$. Then



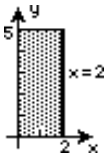
$$A = \int_0^6 \int_0^{4-2x/3} \sqrt{29/16}\, dy\, dx = \frac{\sqrt{29}}{4} \int_0^6 \left(4 - \frac{2}{3}x\right) dx = \frac{\sqrt{29}}{4} \left(4x - \frac{1}{3}x^2\right) \Big|_0^6$$
$$= \frac{\sqrt{29}}{4}(24 - 12) = 3\sqrt{29}.$$

2. We see from the graph in Problem 1 that the plane is entirely above the region bounded by $r = \sin 2\theta$ in the first octant. Using $f(x, y) = z = 3 - \frac{1}{2}x - \frac{3}{4}y$ we have $f_x = -\frac{1}{2}$, $f_y = -\frac{3}{4}$, $1 + f_x^2 + f_y^2 = \frac{29}{16}$. Then



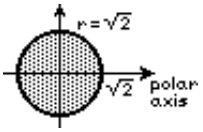
$$A = \int_0^{\pi/2} \int_0^{\sin 2\theta} \sqrt{29/16}\, r\, dr\, d\theta = \frac{\sqrt{29}}{4} \int_0^{\pi/2} \frac{1}{2}r^2 \Big|_0^{\sin 2\theta} d\theta = \frac{\sqrt{29}}{8} \int_0^{\pi/2} \sin^2 2\theta\, d\theta$$
$$= \frac{\sqrt{29}}{8} \left(\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right) \Big|_0^{\pi/2} = \frac{\sqrt{29}\,\pi}{32}.$$

3. Using $f(x, y) = z = \sqrt{16 - x^2}$ we see that for $0 \le x \le 2$ and $0 \le y \le 5$, $z > 0$. Thus, the surface is entirely above the region. Now $f_x = -\frac{x}{\sqrt{16 - x^2}}$, $f_y = 0$, $1 + f_x^2 + f_y^2 = 1 + \frac{x^2}{16 - x^2} = \frac{16}{16 - x^2}$ and



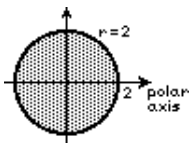
$$A = \int_0^5 \int_0^2 \frac{4}{\sqrt{16 - x^2}}\, dx\, dy = 4 \int_0^5 \sin^{-1}\frac{x}{4} \Big|_0^2 dy = 4 \int_0^5 \frac{\pi}{6}\, dy = \frac{10\pi}{3}.$$

4. The region in the xy-plane beneath the surface is bounded by the graph of $x^2 + y^2 = 2$. Using $f(x, y) = z = x^2 + y^2$ we have $f_x = 2x$, $f_y = 2y$, $1 + f_x^2 + f_y^2 = 1 + 4(x^2 + y^2)$. Then,



$$A = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{12}(1 + 4r^2)^{3/2} \Big|_0^{\sqrt{2}} d\theta = \frac{1}{12} \int_0^{2\pi} (27 - 1) d\theta = \frac{13\pi}{3}.$$

5. Letting $z = 0$ we have $x^2 + y^2 = 4$. Using $f(x, y) = z = 4 - (x^2 + y^2)$ we have $f_x = -2x$, $f_y = -2y$, $1 + f_x^2 + f_y^2 = 1 + 4(x^2 + y^2)$. Then



$$A = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{3}(1 + 4r^2)^{3/2} \Big|_0^2 d\theta$$
$$= \frac{1}{12} \int_0^{2\pi} (17^{3/2} - 1) d\theta = \frac{\pi}{6}(17^{3/2} - 1).$$

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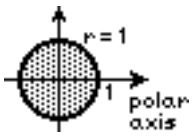
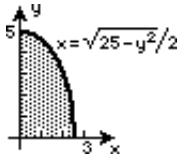
6. The surfaces $x^2+y^2+z^2=2$ and $z^2=x^2+y^2$ intersect on the cylinder $2x^2+2y^2=2$ or $x^2+y^2=1$. There are portions of the sphere within the cone both above and below the xy-plane. Using $f(x,y)=\sqrt{2-x^2-y^2}$ we have $f_x=-\dfrac{x}{\sqrt{2-x^2-y^2}}$, $f_y=-\dfrac{y}{\sqrt{2-x^2-y^2}}$, $1+f_x^2+f_y^2=\dfrac{2}{2-x^2-y^2}$. Then



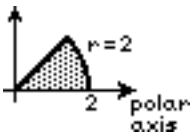
$$A=2\left[\int_0^{2\pi}\int_0^1 \frac{\sqrt{2}}{\sqrt{2-r^2}}\,r\,dr\,d\theta\right]=2\sqrt{2}\int_0^{2\pi}-\sqrt{2-r^2}\,\Big|_0^1\,d\theta$$
$$=2\sqrt{2}\int_0^{2\pi}(\sqrt{2}-1)d\theta=4\pi\sqrt{2}(\sqrt{2}-1).$$

7. Using $f(x,y)=z=\sqrt{25-x^2-y^2}$ we have $f_x=-\dfrac{x}{\sqrt{25-x^2-y^2}}$, $f_y=-\dfrac{y}{\sqrt{25-x^2-y^2}}$, $1+f_x^2+f_y^2=\dfrac{25}{25-x^2-y^2}$. Then



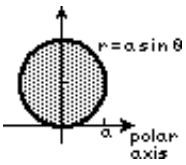
$$A=\int_0^5\int_0^{\sqrt{25-y^2}/2}\frac{5}{\sqrt{25-x^2-y^2}}\,dx\,dy=5\int_0^5 \sin^{-1}\frac{x}{\sqrt{25-y^2}}\,\Big|_0^{\sqrt{25-y^2}/2}\,dy$$
$$=5\int_0^5\frac{\pi}{6}\,dy=\frac{25\pi}{6}.$$

8. In the first octant, the graph of $z=x^2-y^2$ intersects the xy-plane in the line $y=x$. The surface is in the firt octant for $x>y$. Using $f(x,y)=z=x^2-y^2$ we have $f_x=2x$, $f_y=-2y$, $1+f_x^2+f_y^2=1+4x^2+4y^2$. Then



$$A=\int_0^{\pi/4}\int_0^2\sqrt{1+4r^2}\,r\,dr\,d\theta=\int_0^{\pi/4}\frac{1}{12}(1+4r^2)^{3/2}\,\Big|_0^2\,d\theta$$
$$=\frac{1}{12}\int_0^{\pi/4}(17^{3/2}-1)d\theta=\frac{\pi}{48}(17^{3/2}-1).$$

9. There are portions of the sphere within the cylinder both above and below the xy-plane. Using $f(x,y)=z=\sqrt{a^2-x^2-y^2}$ we have $f_x=-\dfrac{x}{\sqrt{1^2-x^2-y^2}}$, $f_y=-\dfrac{y}{\sqrt{a^2-x^2-y^2}}$, $1+f_x^2+f_y^2=\dfrac{a^2}{a^2-x^2-y^2}$. Then, using symmetry,



$$A=2\left[2\int_0^{\pi/2}\int_0^{a\sin\theta}\frac{a}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta\right]=4a\int_0^{\pi/2}-\sqrt{a^2-r^2}\,\Big|_0^{a\sin\theta}\,d\theta$$
$$=4a\int_0^{\pi/2}(a-a\sqrt{1-\sin^2\theta}\,)d\theta=4a^2\int_0^{\pi/2}(1-\cos\theta)d\theta$$
$$=4a^2(\theta-\sin\theta)\,\Big|_0^{\pi/2}=4a^2\left(\frac{\pi}{2}-1\right)=2a^2(\pi-2).$$

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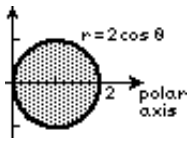
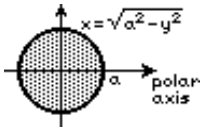
10. There are portions of the cone within the cylinder both above and below the xy-plane. Using $f(x,y) = \frac{1}{2}\sqrt{x^2+y^2}$, we have $f_x = \dfrac{x}{2\sqrt{x^2+y^2}}$, $f_y = \dfrac{y}{2\sqrt{x^2+y^2}}$, $1 + f_x^2 + f_y^2 = \frac{5}{4}$. Then, using symmetry,



$$A = 2\left[2\int_0^{\pi/2}\int_0^{2\cos\theta}\sqrt{\frac{5}{4}}\,r\,dr\,d\theta\right] = 2\sqrt{5}\int_0^{\pi/2}\frac{1}{2}r^2\Big|_0^{2\cos\theta}\,d\theta$$

$$= 4\sqrt{5}\int_0^{\pi/2}\cos^2\theta\,d\theta = 4\sqrt{5}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2} = \sqrt{5}\,\pi.$$

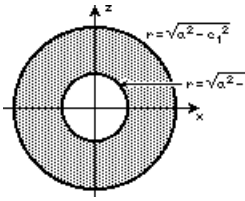
11. There are portions of the surface in each octant with areas equal to the area of the portion in the first octant. Using $f(x,y) = z = \sqrt{a^2-y^2}$ we have $f_x = 0$, $f_y = \dfrac{y}{\sqrt{a^2-y^2}}$, $1 + f_x^2 + f_y^2 = \dfrac{a^2}{a^2-y^2}$. Then



$$A = 8\int_0^a\int_0^{\sqrt{a^2-y^2}}\frac{a}{\sqrt{a^2-y^2}}\,dx\,dy = 8a\int_0^a\frac{x}{\sqrt{a^2-y^2}}\Big|_0^{\sqrt{a^2-y^2}}\,dy = 8a\int_0^a dy = 8a^2.$$

12. From Example 1, the area of the portion of the hemisphere within $x^2+y^2=b^2$ is $2\pi a(a-\sqrt{a^2-b^2}\,)$. Thus, the area of the sphere is $A = 2\lim\limits_{b\to a} 2\pi a(a-\sqrt{a^2-b^2}\,) = 2(2\pi a^2) = 4\pi a^2$.

13. The projection of the surface onto the xz-plane is shown in the graph. Using $f(x,z) = y = \sqrt{a^2-x^2-z^2}$ we have $f_x = -\dfrac{x}{\sqrt{a^2-x^2-z^2}}$,



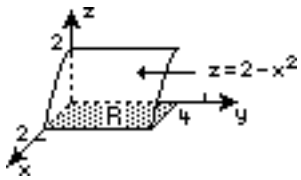
$f_z = -\dfrac{z}{\sqrt{a^2-x^2-z^2}}$, $1 + f_x^2 + f_z^2 = \dfrac{a^2}{a^2-x^2-z^2}$. Then

$$A = \int_0^{2\pi}\int_{\sqrt{a^2-c_2^2}}^{\sqrt{a^2-c_1^2}}\frac{a}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta = a\int_0^{2\pi} -\sqrt{a^2-r^2}\,\Big|_{\sqrt{a^2-c_2^2}}^{\sqrt{a^2-c_1^2}}\,d\theta$$

$$= a\int_0^{2\pi}(c_2-c_1)\,d\theta = 2\pi a(c_2-c_1).$$

14. The surface area of the cylinder $x^2+z^2=a^2$ from $y=c_1$ to $y=c_2$ is the area of a cylinder of radius a and height c_2-c_1. This is $2\pi a(c_2-c_1)$.

15. $z_x = -2x$, $z_y = 0$; $dS = \sqrt{1+4x^2}\,dA$



$$\iint_S x\,dS = \int_0^4\int_0^{\sqrt{2}} x\sqrt{1+4x^2}\,dx\,dy = \int_0^4\frac{1}{12}(1+4x^2)^{3/2}\Big|_0^{\sqrt{2}}\,dy$$

$$= \int_0^4\frac{13}{6}\,dy = \frac{26}{3}$$

16. See Problem 15.

$$\iint_S xy(9-4z)\,dS = \iint_S xy(1+4x^2)\,dS = \int_0^4\int_0^{\sqrt{2}} xy(1+4x^2)^{3/2}\,dx\,dy$$

$$= \int_0^4\frac{y}{20}(1+4x^2)^{5/2}\Big|_0^{\sqrt{2}}\,dy = \int_0^4\frac{242}{20}\,y\,dy = \frac{121}{10}\int_0^4 y\,dy = \frac{121}{10}\left(\frac{1}{2}y^2\right)\Big|_0^4 = \frac{484}{5}$$

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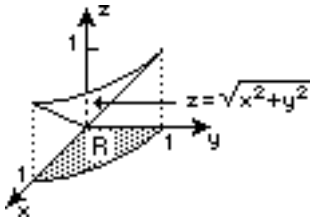
17. $z_x = \dfrac{x}{\sqrt{x^2+y^2}}$, $z_y = \dfrac{y}{\sqrt{x^2+y^2}}$; $dS = \sqrt{2}\,dA$.



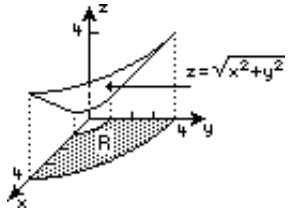
Using polar coordinates,

$$\iint_S xz^3\,dS = \iint_R x(x^2+y^2)^{3/2}\sqrt{2}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^1 (r\cos\theta)r^{3/2}r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 r^{7/2}\cos\theta\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\frac{2}{9}r^{9/2}\cos\theta\,\Big|_0^1\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\frac{2}{9}\cos\theta\,d\theta = \frac{2\sqrt{2}}{9}\sin\theta\,\Big|_0^{2\pi} = 0.$$

18. $z_x = \dfrac{x}{\sqrt{x^2+y^2}}$, $z_y = \dfrac{y}{\sqrt{x^2+y^2}}$; $dS = \sqrt{2}\,dA$.



Using polar coordinates,

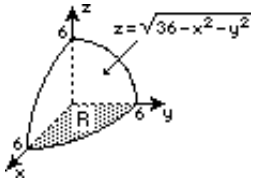
$$\iint_S (x+y+z)\,dS = \iint_R (x+y+\sqrt{x^2+y^2}\,)\sqrt{2}\,dA$$

$$= \sqrt{2}\int_0^{2\pi}\int_1^4 (r\cos\theta + r\sin\theta + r)r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_1^4 r^2(1+\cos\theta+\sin\theta)\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\frac{1}{3}r^3(1+\cos\theta+\sin\theta)\,\Big|_1^4\,d\theta$$

$$= \frac{63\sqrt{2}}{3}\int_0^{2\pi}(1+\cos\theta+\sin\theta)\,d\theta = 21\sqrt{2}(\theta+\sin\theta-\cos\theta)\,\Big|_0^{2\pi} = 42\sqrt{2}\,\pi.$$

19. $z = \sqrt{36-x^2-y^2}$, $z_x = -\dfrac{x}{\sqrt{36-x^2-y^2}}$, $z_y = -\dfrac{y}{\sqrt{36-x^2-y^2}}$;



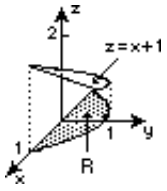
$$dS = \sqrt{1+\frac{x^2}{36-x^2-y^2}+\frac{y^2}{36-x^2-y^2}}\,dA = \frac{6}{\sqrt{36-x^2-y^2}}\,dA.$$

Using polar coordinates,

$$\iint_S (x^2+y^2)z\,dS = \iint_R (x^2+y^2)\sqrt{36-x^2-y^2}\,\frac{6}{\sqrt{36-x^2-y^2}}\,dA$$

$$= 6\int_0^{2\pi}\int_0^6 r^2 r\,dr\,d\theta = 6\int_0^{2\pi}\frac{1}{4}r^4\,\Big|_0^6\,d\theta = 6\int_0^{2\pi}324\,d\theta = 972\pi.$$

20. $z_x = 1$, $z_y = 0$; $dS = \sqrt{2}\,dA$



$$\iint_S z^2\,dS = \int_{-1}^1\int_0^{1-x^2}(x+1)^2\sqrt{2}\,dy\,dx = \sqrt{2}\int_{-1}^1 y(x+1)^2\,\Big|_0^{1-x^2}\,dx$$

$$= \sqrt{2}\int_{-1}^1 (1-x^2)(x+1)^2\,dx = \sqrt{2}\int_{-1}^1 (1+2x-2x^3-x^4)\,dx$$

$$= \sqrt{2}\left(x+x^2-\frac{1}{2}x^4-\frac{1}{5}x^5\right)\Big|_{-1}^1 = \frac{8\sqrt{2}}{5}$$

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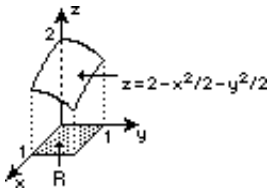
21. $z_x = -x,\ z_y = -y;\ dS = \sqrt{1+x^2+y^2}\ dA$

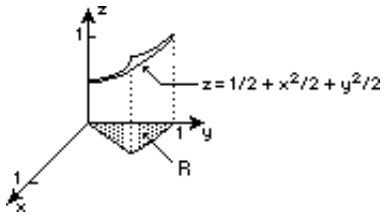


$$\iint_S xy\,dS = \int_0^1 \int_0^1 xy\sqrt{1+x^2+y^2}\ dx\,dy = \int_0^1 \frac{1}{3}y(1+x^2+y^2)^{3/2}\,\Big|_0^1\ dy$$

$$= \int_0^1 \left[\frac{1}{3}y(2+y^2)^{3/2} - \frac{1}{3}y(1+y^2)^{3/2}\right] dy$$

$$= \left[\frac{1}{15}(2+y^2)^{5/2} - \frac{1}{15}(1+y^2)^{5/2}\right]\Big|_0^1 = \frac{1}{15}(3^{5/2} - 2^{7/2} + 1)$$

22. $z = \frac{1}{2} + \frac{1}{2}x^2 + \frac{1}{2}y^2,\ z_x = x,\ z_y = y;\ dS = \sqrt{1+x^2+y^2}\ dA.$



Using polar coordinates,

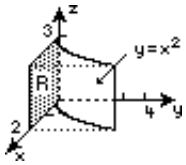
$$\iint_S 2z\,dS = \iint_R (1+x^2+y^2)\sqrt{1+x^2+y^2}\ dA$$

$$= \int_{\pi/3}^{\pi/2} \int_0^1 (1+r^2)\sqrt{1+r^2}\,r\,dr\,d\theta$$

$$= \int_{\pi/3}^{\pi/2} \int_0^1 (1+r^2)^{3/2} r\,dr\,d\theta = \int_{\pi/3}^{\pi/2} \frac{1}{5}(1+r^2)^{5/2}\,\Big|_0^1\ d\theta = \frac{1}{5}\int_{\pi/3}^{\pi/2} (2^{5/2}-1)\,d\theta$$

$$= \frac{4\sqrt{2}-1}{5}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{(4\sqrt{2}-1)\pi}{30}\,.$$

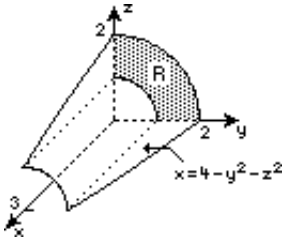
23. $y_x = 2x,\ y_z = 0;\ dS = \sqrt{1+4x^2}\ dA$



$$\iint_S 24\sqrt{y}\,z\,dS = \int_0^3 \int_0^2 24xz\sqrt{1+4x^2}\ dx\,dz = \int_0^3 2z(1+4x^2)^{3/2}\,\Big|_0^2\ dz$$

$$= 2(17^{3/2}-1)\int_0^3 z\,dz = 2(17^{3/2}-1)\left(\frac{1}{2}z^2\right)\Big|_0^3 = 9(17^{3/2}-1)$$

24. $x_y = -2y,\ x_z = -2z;\ \ dS = \sqrt{1+4y^2+4z^2}\ dA.$



Using polar coordinates,

$$\iint_S (1+4y^2+4z^2)^{1/2}\,dS = \int_0^{\pi/2} \int_1^2 (1+4r^2)r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{16}(1+4r^2)^2\,\Big|_1^2\ d\theta = \frac{1}{16}\int_0^{\pi/2} 12\,d\theta = \frac{3\pi}{8}\,.$$

25. Write the equation of the surface as $y = \frac{1}{2}(6 - x - 3z)$.

$y_x = -\frac{1}{2}\,,\ y_z = -\frac{3}{2}\,;\ \ dS = \sqrt{1+1/4+9/4} = \frac{\sqrt{14}}{2}\,.$

$$\iint_S (3z^2+4yz)\,dS = \int_0^2 \int_0^{6-3z} \left[3z^2 + 4z\,\frac{1}{2}(6-x-3z)\right]\frac{\sqrt{14}}{2}\ dx\,dz$$

$$= \frac{\sqrt{14}}{2}\int_0^2 [3z^2x - z(6-x-3z)^2]\,\Big|_0^{6-3z}\ dz$$

$$= \frac{\sqrt{14}}{2}\int_0^2 \left([3z^2(6-3z) - 0] - [0 - z(6-3z)^2]\right)\ dz$$

$$= \frac{\sqrt{14}}{2}\int_0^2 (36z - 18z^2)\,dz = \frac{\sqrt{14}}{2}(18z^2 - 6z^3)\,\Big|_0^2 = \frac{\sqrt{14}}{2}(72-48) = 12\sqrt{14}$$

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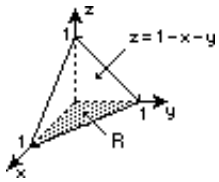
26. Write the equation of the surface as $x = 6 - 2y - 3z$. Then $x_y = -2$, $x_z = -3$; $dS = \sqrt{1+4+9} = \sqrt{14}$.

$$\iint_S (3z^2 + 4yz)\,dS = \int_0^2 \int_0^{3-3z/2} (3z^2 + 4yz)\sqrt{14}\,dy\,dz = \sqrt{14}\int_0^2 (3yz + 2y^2 z)\,\Big|_0^{3-3z/2}\,dz$$

$$= \sqrt{14}\int_0^2 \left[9z\left(1 - \frac{z}{2}\right) + 18z\left(1 - \frac{z}{2}\right)^2\right] dz = \sqrt{14}\int_0^2 \left(27z - \frac{45}{2}z^2 + \frac{9}{2}z^3\right) dz$$

$$= \sqrt{14}\left(\frac{27}{2}z^2 - \frac{15}{2}z^3 + \frac{9}{8}z^4\right)\Big|_0^2 = \sqrt{14}(54 - 60 + 18) = 2\sqrt{14}$$

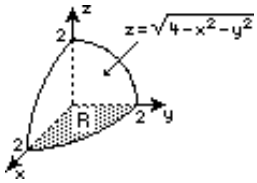
27. The density is $\rho = kx^2$. The surface is $z = 1 - x - y$. Then $z_x = -1$, $z_y = -1$; $dS = \sqrt{3}\,dA$.



$$m = \iint_S kx^2\,dS = k\int_0^1 \int_0^{1-x} x^2\sqrt{3}\,dy\,dx = \sqrt{3}\,k\int_0^1 \frac{1}{3}x^3\,\Big|_0^{1-x}\,dx$$

$$= \frac{\sqrt{3}}{3}\,k\int_0^1 (1-x)^3\,dx = \frac{\sqrt{3}}{3}\,k\left[-\frac{1}{4}(1-x)^4\right]\Big|_0^1 = \frac{\sqrt{3}}{12}\,k$$

28. $z_x = -\dfrac{x}{\sqrt{4 - x^2 - y^2}}$, $z_y = -\dfrac{y}{\sqrt{4 - x^2 - y^2}}$;



$$dS = \sqrt{1 + \frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2}}\,dA = \frac{2}{\sqrt{4 - x^2 - y^2}}\,dA.$$

Using symmetry and polar coordinates,

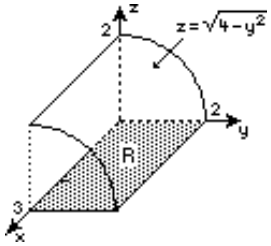
$$m = 4\iint_S |xy|\,dS = 4\int_0^{\pi/2}\int_0^2 (r^2\cos\theta\sin\theta)\frac{2}{\sqrt{4 - r^2}}\,r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^2 r^2(4 - r^2)^{-1/2}\sin 2\theta(r\,dr)\,d\theta \qquad \boxed{u = 4 - r^2,\ du = -2r\,dr,\ r^2 = 4 - u}$$

$$= 4\int_0^{\pi/2}\int_4^0 (4 - u)u^{-1/2}\sin 2\theta\left(-\frac{1}{2}\,du\right)d\theta = -2\int_0^{\pi/2}\int_4^0 (4u^{-1/2} - u^{1/2})\sin 2\theta\,du\,d\theta$$

$$= -2\int_0^{\pi/2}\left(8u^{1/2} - \frac{2}{3}u^{3/2}\right)\Big|_4^0 \sin 2\theta\,d\theta = -2\int_0^{\pi/2}\left(-\frac{32}{3}\sin 2\theta\right)d\theta = \frac{64}{3}\left(-\frac{1}{2}\cos 2\theta\right)\Big|_0^{\pi/2} = \frac{64}{3}.$$

29. The surface is $g(x, y, z) = y^2 + z^2 - 4 = 0$. $\nabla g = 2y\mathbf{j} + 2z\mathbf{k}$,



$|\nabla g| = 2\sqrt{y^2 + z^2}$; $\mathbf{n} = \dfrac{y\mathbf{j} + z\mathbf{k}}{\sqrt{y^2 + z^2}}$;

$\mathbf{F}\cdot\mathbf{n} = \dfrac{2yz}{\sqrt{y^2 + z^2}} + \dfrac{yz}{\sqrt{y^2 + z^2}} = \dfrac{3yz}{\sqrt{y^2 + z^2}}$; $z = \sqrt{4 - y^2}$, $z_x = 0$,

$z_y = -\dfrac{y}{\sqrt{4 - y^2}}$; $dS = \sqrt{1 + \dfrac{y^2}{4 - y^2}}\,dA = \dfrac{2}{\sqrt{4 - y^2}}\,dA$

$$\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \frac{3yz}{\sqrt{y^2 + z^2}}\,\frac{2}{\sqrt{4 - y^2}}\,dA = \iint_R \frac{3y\sqrt{4 - y^2}}{\sqrt{y^2 + 4 - y^2}}\,\frac{2}{\sqrt{4 - y^2}}\,dA$$

$$= \int_0^3\int_0^2 3y\,dy\,dx = \int_0^3 \frac{3}{2}y^2\,\Big|_0^2\,dx = \int_0^3 6\,dx = 18$$

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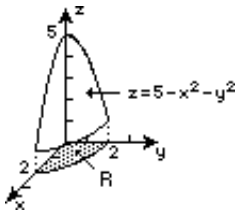
30. The surface is $g(x,y,z) = x^2 + y^2 + z - 5 = 0$. $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$,



$|\nabla g| = \sqrt{1 + 4x^2 + 4y^2}$; $\mathbf{n} = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4x^2 + 4y^2}}$; $\mathbf{F} \cdot \mathbf{n} = \dfrac{z}{\sqrt{1 + 4x^2 + 4y^2}}$;

$z_x = -2x$, $z_y = -2y$, $dS = \sqrt{1 + 4x^2 + 4y^2}\, dA$. Using polar coordinates,

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \frac{z}{\sqrt{1 + 4x^2 + 4y^2}} \sqrt{1 + 4x^2 + 4y^2}\, dA = \iint_R (5 - x^2 - y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^2 (5 - r^2) r\, dr\, d\theta = \int_0^{2\pi} \left(\frac{5}{2} r^2 - \frac{1}{4} r^4 \right) \Big|_0^2 d\theta = \int_0^{2\pi} 6\, d\theta = 12\pi.$$

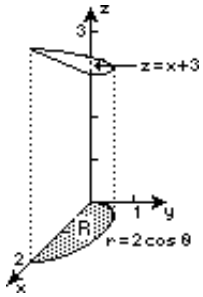
31. From Problem 30, $\mathbf{n} = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4x^2 + 4y^2}}$. Then $\mathbf{F} \cdot \mathbf{n} = \dfrac{2x^2 + 2y^2 + z}{\sqrt{1 + 4x^2 + 4y^2}}$. Also, from Problem 30,

$dS = \sqrt{1 + 4x^2 + 4y^2}\, dA$. Using polar coordinates,

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \frac{2x^2 + 2y^2 + z}{\sqrt{1 + 4x^2 + 4y^2}} \sqrt{1 + 4x^2 + 4y^2}\, dA = \iint_R (2x^2 + 2y^2 + 5 - x^2 - y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^2 (r^2 + 5) r\, dr\, d\theta = \int_0^{2\pi} \left(\frac{1}{4} r^4 + \frac{5}{2} r^2 \right) \Big|_0^2 d\theta = \int_0^{2\pi} 14\, d\theta = 28\pi.$$

32. The surface is $g(x,y,z) = z - x - 3 = 0$. $\nabla g = -\mathbf{i} + \mathbf{k}$, $|\nabla g| = \sqrt{2}$; $\mathbf{n} = \dfrac{-\mathbf{i} + \mathbf{k}}{\sqrt{2}}$;



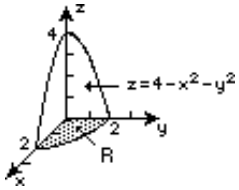
$\mathbf{F} \cdot \mathbf{n} = \dfrac{1}{\sqrt{2}} x^3 y + \dfrac{1}{\sqrt{2}} xy^3$; $z_x = 1$, $z_y = 0$, $dS = \sqrt{2}\, dA$. Using polar coordinates,

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \frac{1}{\sqrt{2}} (x^3 y + xy^3) \sqrt{2}\, dA = \iint_R xy(x^2 + y^2)\, dA$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} (r^2 \cos\theta \sin\theta) r^2 r\, dr\, d\theta = \int_0^{\pi/2} \int_0^{2\cos\theta} r^5 \cos\theta \sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{6} r^6 \cos\theta \sin\theta \Big|_0^{2\cos\theta} d\theta = \frac{1}{6} \int_0^{\pi/2} 64 \cos^7\theta \sin\theta\, d\theta = \frac{32}{3} \left(-\frac{1}{8} \cos^8\theta \right) \Big|_0^{\pi/2} = \frac{4}{3}.$$

33. The surface is $g(x,y,z) = x^2 + y^2 + z - 4$. $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$,



$|\nabla g| = \sqrt{4x^2 + 4y^2 + 1}$; $\mathbf{n} = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}$; $\mathbf{F} \cdot \mathbf{n} = \dfrac{x^3 + y^3 + z}{\sqrt{4x^2 + 4y^2 + 1}}$;

$z_x = -2x$, $z_y = -2y$, $dS = \sqrt{1 + 4x^2 + 4y^2}\, dA$. Using polar coordinates,

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R (x^3 + y^3 + z)\, dA = \iint_R (4 - x^2 - y^2 + x^3 + y^3)\, dA$$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2 + r^3 \cos^3\theta + r^3 \sin^3\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(2r^2 - \frac{1}{4} r^4 + \frac{1}{5} r^5 \cos^3\theta + \frac{1}{5} r^5 \sin^3\theta \right) \Big|_0^2 d\theta$$

$$= \int_0^{2\pi} \left(4 + \frac{32}{5} \cos^3\theta + \frac{32}{5} \sin^3\theta \right) d\theta = 4\theta \Big|_0^{2\pi} + 0 + 0 = 8\pi.$$

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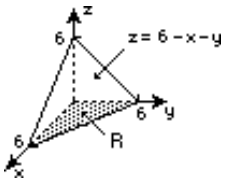
34. The surface is $g(x,y,z) = x+y+z-6$. $\nabla g = \mathbf{i}+\mathbf{j}+\mathbf{k}$, $|\nabla g| = \sqrt{3}$; $\mathbf{n} = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$; $\mathbf{F}\cdot\mathbf{n} = (e^y+e^x+18y)/\sqrt{3}$; $z_x = -1$, $z_y = -1$, $dS = \sqrt{1+1+1}\,dA = \sqrt{3}\,dA$.



$$\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_r (e^y+e^x+18y)\,dA = \int_0^6\int_0^{6-x}(e^y+e^x+18y)\,dy\,dx$$

$$= \int_0^6 (e^y+ye^x+9y^2)\Big|_0^{6-x}\,dx = \int_0^6 [e^{6-x}+(6-x)e^x+9(6-x)^2-1]\,dx$$

$$= [-e^{6-x}+6e^x-xe^x+e^x-3(6-x)^3-x]\Big|_0^6$$

$$= (-1+6e^6-6e^6+e^6-6)-(-e^6+6+1-648) = 2e^6+634 \approx 1440.86$$

35. For S_1: $g(x,y,z) = x^2+y^2-z$, $\nabla g = 2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}$, $|\nabla g| = \sqrt{4x^2+4y^2+1}$; $\mathbf{n}_1 = \dfrac{2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}}{\sqrt{4x^2+4y^2+1}}$; $\mathbf{F}\cdot\mathbf{n}_1 = \dfrac{2xy^2+2x^2y-5z}{\sqrt{4x^2+4y^2+1}}$; $z_x = 2x$, $z_y = 2y$, $dS_1 = \sqrt{1+4x^2+4y^2}\,dA$. For S_2: $g(x,y,z) = z-1$, $\nabla g = \mathbf{k}$, $|\nabla g| = 1$; $\mathbf{n}_2 = \mathbf{k}$; $\mathbf{F}\cdot\mathbf{n}_2 = 5z$; $z_x = 0$, $z_y = 0$, $dS_2 = dA$. Using polar coordinates and R: $x^2+y^2 \le 1$ we have

$$\text{Flux} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}_1\,dS_1 + \iint_{S_2}\mathbf{F}\cdot\mathbf{n}_2\,dS_2 = \iint_R (2xy^2+2x^2y-5z)\,dA + \iint_R 5z\,dA$$

$$= \iint_R [2xy^2+2x^2y-5(x^2+y^2)+5(1)]\,dA$$

$$= \int_0^{2\pi}\int_0^1 (2r^3\cos\theta\sin^2\theta + 2r^3\cos^2\theta\sin\theta - 5r^2+5)r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{2}{5}r^5\cos\theta\sin^2\theta + \frac{2}{5}r^5\cos^2\theta\sin\theta - \frac{5}{4}r^4+\frac{5}{2}r^2\right)\Big|_0^1 d\theta$$

$$= \int_0^{2\pi}\left[\frac{2}{5}(\cos\theta\sin^2\theta+\cos^2\theta\sin\theta)+\frac{5}{4}\right]d\theta = \frac{2}{5}\left(\frac{1}{3}\sin^3\theta - \frac{1}{3}\cos^3\theta\right)\Big|_0^{2\pi} + \frac{5}{4}\theta\Big|_0^{2\pi}$$

$$= \frac{2}{5}\left[-\frac{1}{3}-\left(-\frac{1}{3}\right)\right]+\frac{5}{2}\pi = \frac{5}{2}\pi.$$

36. For S_1: $g(x,y,z) = x^2+y^2+z-4$, $\nabla g = 2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}$, $|\nabla g| = \sqrt{4x^2+4y^2+1}$; $\mathbf{n}_1 = \dfrac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{4x^2+4y^2+1}}$; $\mathbf{F}\cdot\mathbf{n}_1 = 6z^2/\sqrt{4x^2+4y^2+1}$; $z_x = -2x$, $z_y = -2y$, $dS_1 = \sqrt{1+4x^2+4y^2}\,dA$. For S_2: $g(x,y,z) = x^2+y^2-z$, $\nabla g = 2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}$, $|\nabla g| = \sqrt{4x^2+4y^2+1}$; $\mathbf{n}_2 = \dfrac{2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}}{\sqrt{4x^2+y^2+1}}$; $\mathbf{F}\cdot\mathbf{n}_2 = -6z^2/\sqrt{4x^2+4y^2+1}$; $z_x = 2x$, $z_y = 2y$, $dS_2 = \sqrt{1+4x^2+4y^2}\,dA$. Using polar coordinates and R: $x^2+y^2 \le 2$ we have

$$\text{Flux} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}_1\,dS_1 + \iint_{S_1}\mathbf{F}\cdot\mathbf{n}_2\,dS_2 = \iint_R 6z^2\,dA + \iint -6z^2\,dA$$

$$= \iint_R [6(4-x^2-y^2)^2 - 6(x^2+y^2)^2]\,dA = 6\int_0^{2\pi}\int_0^{\sqrt{2}}[(4-r^2)^2-r^4]\,r\,dr\,d\theta$$

$$= 6\int_0^{2\pi}\left[-\frac{1}{6}(4-r^2)^3 - \frac{1}{6}r^6\right]\Big|_0^{\sqrt{2}} d\theta = -\int_0^{2\pi}[(2^3-4^3)+(\sqrt{2}\,)^6]\,d\theta = \int_0^{2\pi} 48\,d\theta = 96\pi.$$

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37. The surface is $g(x,y,z) = x^2+y^2+z^2-a^2 = 0$. $\nabla g = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$,

$|\nabla g| = 2\sqrt{x^2+y^2+z^2}$; $\mathbf{n} = \dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$;

$\mathbf{F}\cdot\mathbf{n} = -(2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k})\cdot\dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = -\dfrac{2x^2+2y^2+2z^2}{\sqrt{x^2+y^2+z^2}} = -2\sqrt{x^2+y^2+z^2} = -2a.$

$\text{Flux} = \displaystyle\iint_S -2a\,dS = -2a\times\text{area} = -2a(4\pi a^2) = -8\pi a^3$

38. $\mathbf{n}_1=\mathbf{k}$, $\mathbf{n}_2=-\mathbf{i}$, $\mathbf{n}_3=\mathbf{j}$, $\mathbf{n}_4=-\mathbf{k}$, $\mathbf{n}_5=\mathbf{i}$, $\mathbf{n}_6=-\mathbf{j}$; $\mathbf{F}\cdot\mathbf{n}_1 = z = 1$, $\mathbf{F}\cdot\mathbf{n}_2 = -x = 0$, $\mathbf{F}\cdot\mathbf{n}_3 = y = 1$,

$\mathbf{F}\cdot\mathbf{n}_4 = -z = 0$, $\mathbf{F}\cdot\mathbf{n}_5 = x = 1$, $\mathbf{F}\cdot\mathbf{n}_6 = -y = 0$; $\text{Flux} = \displaystyle\iint_{S_1} 1\,dS + \iint_{S_3} 1\,dS + \iint_{S_5} 1\,dS = 3$

39. Refering to the solution to Problem 37, we find $\mathbf{n} = \dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$ and $dS = \dfrac{a}{\sqrt{a^2-x^2-y^2}}\,dA$.

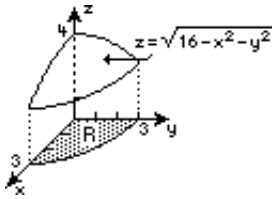
Now

$$\mathbf{F}\cdot\mathbf{n} = kq\frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{kq}{|\mathbf{r}|^4}|\mathbf{r}|^2 = \frac{kq}{|\mathbf{r}|^2} = \frac{kq}{x^2+y^2+z^2} = \frac{kq}{a^2}$$

and

$$\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \frac{kq}{a^2}\,dS = \frac{kq}{a^2}\times\text{area} = \frac{kq}{a^2}(4\pi a^2) = 4\pi kq.$$

40. We are given $\sigma = kz$. Now $z_x = -\dfrac{x}{\sqrt{16-x^2-y^2}}$, $z_y = -\dfrac{y}{\sqrt{16-x^2-y^2}}$;



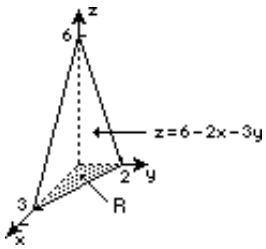
$dS = \sqrt{1+\dfrac{x^2}{16-x^2-y^2}+\dfrac{y^2}{16-x^2-y^2}}\,dA = \dfrac{4}{\sqrt{16-x^2-y^2}}\,dA$

Using polar coordinates,

$$Q = \iint_S kz\,dS = k\iint_R \sqrt{16-x^2-y^2}\,\frac{4}{\sqrt{16-x^2-y^2}}\,dA = 4k\int_0^{2\pi}\int_0^3 r\,dr\,d\theta$$

$$= 4k\int_0^{2\pi}\frac{1}{2}r^2\Big|_0^3 d\theta = 4k\int_0^{2\pi}\frac{9}{2}\,d\theta = 36\pi k.$$

41. The surface is $z = 6-2x-3y$. Then $z_x=-2$, $z_y=-3$, $dS = \sqrt{1+4+9} = \sqrt{14}\,dA$.



The area of the surface is

$$A(s) = \iint_S dS = \int_0^3\int_0^{2-2x/3}\sqrt{14}\,dy\,dx = \sqrt{14}\int_0^3\left(2-\frac{2}{3}x\right)dx$$

$$= \sqrt{14}\left(2x-\frac{1}{3}x^2\right)\Big|_0^3 = 3\sqrt{14}.$$

$$\bar{x} = \frac{1}{3\sqrt{14}}\iint_S x\,dS = \frac{1}{3\sqrt{14}}\int_0^3\int_0^{2-2x/3}\sqrt{14}\,x\,dy\,dx = \frac{1}{3}\int_0^3 xy\Big|_0^{2-2x/3}dx$$

$$= \frac{1}{3}\int_0^3\left(2x-\frac{2}{3}x^2\right)dx = \frac{1}{3}\left(x^2-\frac{2}{9}x^3\right)\Big|_0^3 = 1$$

$$\bar{y} = \frac{1}{3\sqrt{14}}\iint_S y\,dS = \frac{1}{3\sqrt{14}}\int_0^3\int_0^{2-2x/3}\sqrt{14}\,y\,dy\,dx = \frac{1}{3}\int_0^3\frac{1}{2}y^2\Big|_0^{2-2x/3}dx$$

$$= \frac{1}{6}\int_0^3\left(2-\frac{2}{3}x\right)^2dx = \frac{1}{6}\left[-\frac{1}{2}\left(2-\frac{2}{3}x\right)^3\right]\Big|_0^3 = \frac{2}{3}$$

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$$\bar{z} = \frac{1}{3\sqrt{14}} \iint_S z\, dS = \frac{1}{3\sqrt{14}} \int_0^3 \int_0^{2-2x/3} (6 - 2x - 3y)\sqrt{14}\, dy\, dx$$

$$= \frac{1}{3} \int_0^3 \left(6y - 2xy - \frac{3}{2}y^2\right) \Big|_0^{2-2x/3} dx = \frac{1}{3} \int_0^3 \left(6 - 4x + \frac{2}{3}x^2\right) dx = \frac{1}{3} \left(6x - 2x^2 + \frac{2}{9}x^3\right) \Big|_0^3 = 2$$

The centroid is $(1, 2/3, 2)$.

42. The area of the hemisphere is $A(s) = 2\pi a^2$. By symmetry, $\bar{x} = \bar{y} = 0$.

$$z_x = -\frac{x}{\sqrt{a^2 - x^2 - y^2}},\ z_y = -\frac{y}{\sqrt{a^2 - x^2 - y^2}};$$

$$dS = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}}\ dA = \frac{a}{\sqrt{a^2 - x^2 - y^2}}\ dA$$

Using polar coordinates,

$$z = \iint_S \frac{z\, dS}{2\pi a^2} = \frac{1}{2\pi a^2} \iint_R \sqrt{a^2 - x^2 - y^2}\ \frac{a}{\sqrt{a^2 - x^2 - y^2}}\ dA = \frac{1}{2\pi a} \int_0^{2\pi} \int_0^a r\, dr\, d\theta$$

$$= \frac{1}{2\pi a} \int_0^{2\pi} \frac{1}{2} r^2 \Big|_0^a d\theta = \frac{1}{2\pi a} \int_0^{2\pi} \frac{1}{2} s^2\, d\theta = \frac{a}{2}.$$

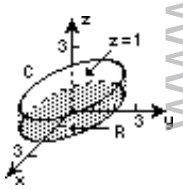
The centroid is $(0, 0, a/2)$.

43. The surface is $g(x, y, z) = z - f(x, y) = 0$. $\nabla g = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}$, $|\nabla g| = \sqrt{f_x^2 + f_y^2 + 1}$;

$$\mathbf{n} = \frac{-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}};\ \mathbf{F} \cdot \mathbf{n} = \frac{-Pf_x - Qf_y + R}{\sqrt{1 + f_x^2 + f_y^2}};\ dS = \sqrt{1 + f_x^2 + f_y^2}\ dA$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \frac{-Pf_x - Qf_y + R}{\sqrt{1 + f_x^2 + f_y^2}}\ \sqrt{1 + f_x^2 + f_y^2}\ dA = \iint_R (-Pf_x - Qf_y + R)\, dA$$

Exercises 9.14



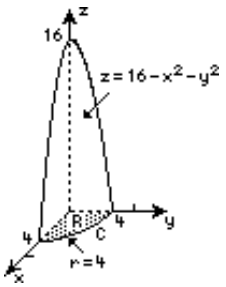
1. Surface Integral: curl $\mathbf{F} = -10\mathbf{k}$. Letting $g(x, y, z) = z - 1$, we have $\nabla g = \mathbf{k}$ and $\mathbf{n} = \mathbf{k}$. Then

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S (-10)\, dS = -10 \times (\text{area of } S) = -10(4\pi) = -40\pi.$$

Line Integral: Parameterize the curve C by $x = 2\cos t$, $y = 2\sin t$, $z = 1$, for $0 \le t \le 2\pi$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C 5y\, dx - 5x\, dy + 3\, dz = \int_0^{2\pi} [10\sin t(-2\sin t) - 10\cos t(2\cos t)]\, dt$$

$$= \int_0^{2\pi} (-20\sin^2 t - 20\cos^2 t)\, dt = \int_0^{2\pi} -20\, dt = -40\pi.$$



2. Surface Integral: curl $\mathbf{F} = 4\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$. Letting $g(x, y, z) = x^2 + y^2 + z - 16$, $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$, and $\mathbf{n} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})/\sqrt{4x^2 + 4y^2 + 1}$. Thus,

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S \frac{8x - 4y - 3}{\sqrt{4x^2 + 4y^2 + 1}}\ dS.$$

Letting the surface be $z = 16 - x^2 - y^2$, we have $z_x = -2x$, $z_y = -2y$, and $dS = \sqrt{1 + 4x^2 + 4y^2}\ dA$. Then, using polar coordinates,

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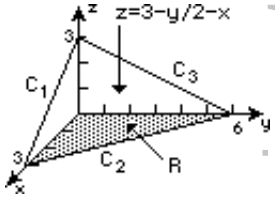
$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (8x - 4y - 3)\, dA = \int_0^{2\pi} \int_0^4 (8r\cos\theta - 4r\sin\theta - 3)\, r\, dr\, d\theta$$
$$= \int_0^{2\pi} \left(\frac{8}{3} r^3 \cos\theta - \frac{4}{3} r^3 \sin\theta - \frac{3}{2} r^2 \right) \Big|_0^4 d\theta = \int_0^{2\pi} \left(\frac{512}{3} \cos\theta - \frac{256}{3} \sin\theta - 24 \right) d\theta$$
$$= \left(\frac{512}{3} \sin\theta + \frac{256}{3} \cos\theta - 24\theta \right) \Big|_0^{2\pi} = -48\pi.$$

Line Integral: Parameterize the curve C by $x = 4\cos t$, $y = 4\sin t$, $z = 0$, for $0 \le t \le 2\pi$. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C 2z\, dx - 3x\, dy + 4y\, dz = \int_0^{2\pi} [-12\cos t(4\cos t)]\, dt$$
$$= \int_0^{2\pi} -48\cos^2 t\, dt = (-24t - 12\sin 2t) \Big|_0^{2\pi} = -48\pi.$$

3. **Surface Integral**: curl $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Letting $g(x, y, z) = 2x + y + 2z - 6$, we have $\nabla g = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{n} = (2\mathbf{i} + \mathbf{j} + 2\mathbf{k})/3$. Then $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S \frac{5}{3}\, dS$. Letting the surface be $z = 3 - \frac{1}{2}y - x$ we have $z_x = -1$, $z_y = -\frac{1}{2}$, and $dS = \sqrt{1 + (-1)^2 + (-\frac{1}{2})^2}\, dA = \frac{3}{2}\, dA$. Then



$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R \frac{5}{3} \left(\frac{3}{2} \right) dA = \frac{5}{2} \times (\text{area of } R) = \frac{5}{2}(9) = \frac{45}{2}.$$

Line Integral: C_1: $z = 3 - x$, $0 \le x \le 3$, $y = 0$; C_2: $y = 6 - 2x$, $3 \ge x \ge 0$, $z = 0$; C_3: $z = 3 - y/2$, $6 \ge y \ge 0$, $x = 0$.

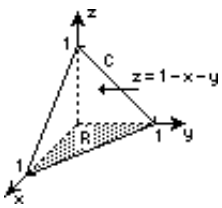
$$\oint_C z\, dx + x\, dy + y\, dz = \iint_{C_1} z\, dx + \int_{C_2} x\, dy + \int_{C_3} y\, dz$$
$$= \int_0^3 (3 - x)\, dx + \int_3^0 x(-2\, dx) + \int_6^0 y(-dy/2)$$
$$= \left(3x - \frac{1}{2}x^2 \right) \Big|_0^3 - x^2 \Big|_3^0 - \frac{1}{4}y^2 \Big|_6^0 = \frac{9}{2} - (0 - 9) - \frac{1}{4}(0 - 36) = \frac{45}{2}$$

4. **Surface Integral**: curl $\mathbf{F} = \mathbf{0}$ and $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = 0$.

Line Integral: the curve is $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t \le 2\pi$.

$$\oint_C x\, dx + y\, dy + z\, dz = \int_0^{2\pi} [\cos t(-\sin t) + \sin t(\cos t)] dt = 0.$$

5. curl $\mathbf{F} = 2\mathbf{i} + \mathbf{j}$. A unit vector normal to the plane is $\mathbf{n} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$. Taking the equation of the plane to be $z = 1 - x - y$, we have $z_x = z_y = -1$. Thus, $dS = \sqrt{1 + 1 + 1}\, dA = \sqrt{3}\, dA$ and



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S \sqrt{3}\, dS = \sqrt{3} \iint_R \sqrt{3}\, dA$$
$$= 3 \times (\text{area of } R) = 3(1/2) = 3/2.$$

6. curl $\mathbf{F} = -2xz\mathbf{i} + z^2\mathbf{k}$. A unit vector normal to the plane is $\mathbf{n} = (\mathbf{j} + \mathbf{k})/\sqrt{2}$. From $z = 1 - y$, we have $z_x = 0$ and $z_y = -1$. Thus, $dS = \sqrt{1 + 1}\, dA = \sqrt{2}\, dA$ and

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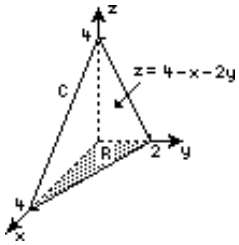
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R \frac{1}{\sqrt{2}}\,z^2\sqrt{2}\;dA = \iint_R (1-y)^2\,dA$$
$$= \int_0^2\int_0^1 (1-y)^2\,dy\,dx = \int_0^2 -\frac{1}{3}(1-y)^3\,\Big|_0^1\,dx = \int_0^2 \frac{1}{3}\,dx = \frac{2}{3}.$$

7. curl $\mathbf{F} = -2y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$. A unit vector normal to the plane is $\mathbf{n} = (\mathbf{j}+\mathbf{k})/\sqrt{2}$. From $z = 1-y$ we have $z_x = 0$ and $z_y = -1$. Then $dS = \sqrt{1+1}\,dA = \sqrt{2}\,dA$ and

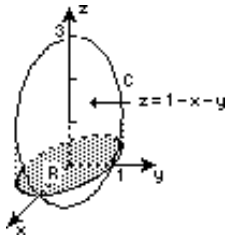
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R \left[-\frac{1}{\sqrt{2}}(z+x)\right]\sqrt{2}\,dA = \iint_R (y-x-1)\,dA$$
$$= \int_0^2\int_0^1 (y-x-1)\,dy\,dx = \int_0^2 \left(\frac{1}{2}y^2 - xy - y\right)\Big|_0^1\,dx = \int_0^2 \left(-x-\frac{1}{2}\right)dx$$
$$= \left(-\frac{1}{2}x^2 - \frac{1}{2}x\right)\Big|_0^2 = -3.$$

8. curl $\mathbf{F} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Letting $g(x,y,z) = x + 2y + z - 4$, we have $\nabla g = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = (\mathbf{i} + 2\mathbf{j} + \mathbf{k})/\sqrt{6}$. From $z = 4 - x - 2y$ we have $z_x = -1$ and $z_y = -2$. Then $dS = \sqrt{6}\,dA$ and



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R \frac{1}{\sqrt{6}}(9)\sqrt{6}\,dA = \iint_R 9\,dA = 9(4) = 36.$$

9. curl $\mathbf{F} = (-3x^2 - 3y^2)\mathbf{k}$. A unit vector normal to the plane is $\mathbf{n} = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$. From $z = 1 - x - y$, we have $z_x = z_y = -1$ and $dS = \sqrt{3}\;dA$. Then, using polar coordinates,



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R (-\sqrt{3}\,x^2 - \sqrt{3}\,y^2)\sqrt{3}\;dA$$
$$= 3\iint_R (-x^2 - y^2)\,dA = 3\int_0^{2\pi}\int_0^1 (-r^2)r\,dr\,d\theta$$
$$= 3\int_0^{2\pi} -\frac{1}{4}r^4\,\Big|_0^1\,d\theta = 3\int_0^{2\pi} -\frac{1}{4}\,d\theta = -\frac{3\pi}{2}.$$

10. curl $\mathbf{F} = 2xyz\mathbf{i} - y^2z\mathbf{j} + (1-x^2)\mathbf{k}$. A unit vector normal to the surface is $\mathbf{n} = \dfrac{2y\mathbf{j}+\mathbf{k}}{\sqrt{4y^2+1}}$. From $z = 9 - y^2$ we have $z_x = 0$, $z_y = -2y$ and $dS = \sqrt{1+4y^2}\,dA$. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R (-2y^3z + 1 - x^2)\,dA = \int_0^3\int_0^{y/2} [-2y^3(9-y^2) + 1 - x^2]\,dx\,dy$$
$$= \int_0^3 \left(-18y^3x + 2y^5x + x - \frac{1}{3}x^3\right)\Big|_0^{y/2}\,dy = \int_0^3 \left(-9y^4 + y^6 + \frac{1}{2}y - \frac{1}{24}y^3\right)dy$$
$$= \left(-\frac{9}{5}y^5 + \frac{1}{7}y^7 + \frac{1}{4}y^2 - \frac{1}{96}y^4\right)\Big|_0^3 \approx 123.57.$$

11. curl $\mathbf{F} = 3x^2y^2\mathbf{k}$. A unit vector normal to the surface is

$$\mathbf{n} = \frac{8x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{64x^2 + 4y^2 + 4z^2}} = \frac{4x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{16x^2 + y^2 + z^2}}.$$

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From $z_x = -\dfrac{4x}{\sqrt{4-4x^2-y^2}}$, $z_y = -\dfrac{y}{\sqrt{4-4x^2-y^2}}$ we obtain $dS = 2\sqrt{\dfrac{1+3x^2}{4-4x^2-y^2}}\,dA$. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R \frac{3x^2y^2z}{\sqrt{16x^2+y^2+z^2}}\left(2\sqrt{\frac{1+3x^2}{4-4x^2-y^2}}\right)dA$$

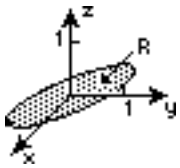
$$= \iint_R 3x^2y^2\,dA \qquad \boxed{\text{Using symmetry}}$$

$$= 12\int_0^1\int_0^{2\sqrt{1-x^2}} x^2y^2\,dy\,dx = 12\int_0^1 \left(\frac{1}{3}x^2y^3\right)\Big|_0^{2\sqrt{1-x^2}}\,dx$$

$$= 32\int_0^1 x^2(1-x^2)^{3/2}dx \qquad \boxed{x=\sin t,\ dx=\cos t\,dt}$$

$$= 32\int_0^{\pi/2}\sin^2 t\cos^4 t\,dt = \pi.$$

12. curl $\mathbf{F} = \mathbf{i}+\mathbf{j}+\mathbf{k}$. Taking the surface S bounded by C to be the portion of the plane $x+y+z=0$ inside C, we have $\mathbf{n} = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$ and $dS = \sqrt{3}\,dA$.



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \iint_S \sqrt{3}\,dS = \sqrt{3}\iint_R \sqrt{3}\,dA = 3\times(\text{area of } R)$$

The region R is obtained by eliminating z from the equations of the plane and the sphere. This gives $x^2+xy+y^2 = \frac{1}{2}$. Rotating axes, we see that R is enclosed by the ellipse $X^2/(1/3)+Y^2/1=1$ in a rotated coordinate system. Thus,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = 3\times(\text{area of } R) = 3\left(\pi\,\frac{1}{\sqrt{3}}\,1\right) = \sqrt{3}\,\pi.$$

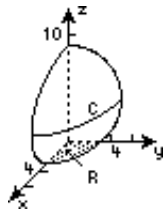
13. Parameterize C by $x = 4\cos t$, $y = 2\sin t$, $z = 4$, for $0\le t\le 2\pi$. Then

$$\iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C 6yz\,dx + 5x\,dy + yze^{x^2}\,dz$$

$$= \int_0^{2\pi}[6(2\sin t)(4)(-4\sin t) + 5(4\cos t)(2\cos t) + 0]\,dt$$

$$= 8\int_0^{2\pi}(-24\sin^2 t + 5\cos^2 t)\,dt = 8\int_0^{2\pi}(5-29\sin^2 t)\,dt = -152\pi.$$

14. Parameterize C by $x = 5\cos t$, $y = 5\sin t$, $z = 4$, for $0\le t\le 2\pi$. Then,



$$\iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \oint_C \mathbf{F}\cdot\mathbf{r} = \oint_C y\,dx + (y-x)\,dy + z^2\,dz$$

$$= \int_0^{2\pi}[(5\sin t)(-5\sin t) + (5\sin t - 5\cos t)(5\cos t)]\,dt$$

$$= \int_0^{2\pi}(25\sin t\cos t - 25)\,dt = \left(\frac{25}{2}\sin^2 t - 25t\right)\Big|_0^{2\pi} = -50\pi.$$

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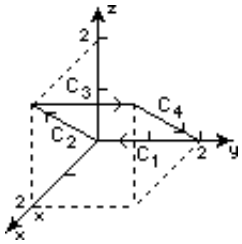
15. Parameterize C by C_1: $x = 0$, $z = 0$, $2 \ge y \ge 0$; C_2: $z = x$, $y = 0$, $0 \le x \le 2$; C_3: $x = 2$, $z = 2$, $0 \le y \le 2$; C_4: $z = x$, $y = 2$, $2 \ge x \ge 0$. Then



$$\begin{aligned}\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS &= \oint_C \mathbf{F} \cdot \mathbf{r} = \oint_C 3x^2\, dx + 8x^3 y\, dy + 3x^2 y\, dz \\ &= \int_{C_1} 0\, dx + 0\, dy + 0\, dz + \int_{C_2} 3x^2\, dx + \int_{C_3} 64\, dy + \int_{C_4} 3x^2\, dx + 6x^2\, dx \\ &= \int_0^2 3x^2\, dx + \int_0^2 64\, dy + \int_2^0 9x^2\, dx = x^3 \Big|_0^2 + 64y \Big|_0^2 + 3x^3 \Big|_2^0 = 112.\end{aligned}$$

16. Parameterize C by $x = \cos t$, $y = \sin t$, $z = \sin t$, $0 \le t \le 2\pi$. Then

$$\begin{aligned}\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS &= \oint_C \mathbf{F} \cdot \mathbf{r} = \oint_C 2xy^2 z\, dx + 2x^2 yz\, dy + (x^2 y^2 - 6x)\, dz \\ &= \int_0^{2\pi} [2\cos t \sin^2 t \sin t(-\sin t) + 2\cos^2 t \sin t \sin t \cos t \\ &\qquad + (\cos^2 t \sin^2 t - 6\cos t)\cos t]\, dt \\ &= \int_0^{2\pi} (-2\cos t \sin^4 t + 3\cos^3 t \sin^2 t - 6\cos^2 t)\, dt = -6\pi.\end{aligned}$$

17. We take the surface to be $z = 0$. Then $\mathbf{n} = \mathbf{k}$ and $dS = dA$. Since curl $\mathbf{F} = \dfrac{1}{1+y^2}\,\mathbf{i} + 2ze^{x^2}\mathbf{j} + y^2\mathbf{k}$,

$$\begin{aligned}\oint_C z^2 e^{x^2}\, dx + xy\, dy + \tan^{-1} y\, dz &= \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S y^2\, dS = \iint_R y^2\, dA \\ &= \int_0^{2\pi} \int_0^3 r^2 \sin^2\theta\, r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{4} r^4 \sin^2\theta \Big|_0^3 d\theta \\ &= \frac{81}{4} \int_0^{2\pi} \sin^2\theta\, d\theta = \frac{81\pi}{4}.\end{aligned}$$

18. **(a)** curl $\mathbf{F} = xz\mathbf{i} - yz\mathbf{j}$. A unit vector normal to the surface is $\mathbf{n} = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}$ and $dS = \sqrt{1 + 4x^2 + 4y^2}\, dA$. Then, using $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$, we have

$$\begin{aligned}\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS &= \iint_R (2x^2 z - 2y^2 z)\, dA = \iint_R (2x^2 - 2y^2)(1 - x^2 - y^2)\, dA \\ &= \iint_R (2x^2 - 2y^2 - 2x^4 + 2y^4)\, dA \\ &= \int_0^{2\pi} \int_0^1 (2r^2\cos^2\theta - 2r^2\sin^2\theta - 2r^4\cos^4\theta + 2r^4\cos^4\theta)\, r\, dr\, d\theta \\ &= 2\int_0^{2\pi} \int_0^1 [r^3\cos 2\theta - r^5(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)]\, dr\, d\theta \\ &= 2\int_0^{2\pi} \int_0^1 (r^3\cos 2\theta - r^5\cos 2\theta)\, dr\, d\theta = 2\int_0^{2\pi} \cos 2\theta \left(\frac{1}{4}r^4 - \frac{1}{6}r^6\right) \Big|_0^1 d\theta \\ &= \frac{1}{6}\int_0^{2\pi} \cos 2\theta\, d\theta = 0.\end{aligned}$$

(b) We take the surface to be $z = 0$. Then $\mathbf{n} = \mathbf{k}$, curl $\mathbf{F} \cdot \mathbf{n} = \text{curl } \mathbf{F} \cdot \mathbf{k} = 0$ and $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = 0$.

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(c) By Stoke's Theorem, using $z = 0$, we have

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C xyz\, dz = \oint_C xy(0)\, dz = 0.$$

Exercises 9.15

1. $\displaystyle\int_2^4 \int_{-2}^2 \int_{-1}^1 (x+y+z)dx\, dy\, dz = \int_2^4 \int_{-2}^2 \left(\frac{1}{2}x^2 + xy + xz\right)\Big|_{-1}^1 dy\, dz$

$$= \int_2^4 \int_{-2}^2 (2y + 2z)\, dy\, dz = \int_2^4 (y^2 + 2yz)\Big|_{-2}^2 dz = \int_2^4 8z\, dz = 4z^2\Big|_2^4 = 48$$

2. $\displaystyle\int_1^3 \int_1^x \int_2^{xy} 24xy\, dz\, dy\, dx = \int_1^3 \int_1^x 24xyz\Big|_2^{xy} dy\, dx = \int_1^3 \int_1^x (24x^2y^2 - 48xy)dy\, dx$

$$= \int_1^3 (8x^2y^3 - 24xy^2)\Big|_1^x dx = \int_1^3 (8x^5 - 24x^3 - 8x^2 + 24x)\, dx$$

$$= \left(\frac{4}{3}x^6 - 6x^4 - \frac{8}{3}x^3 + 12x^2\right)\Big|_1^3 = 522 - \frac{14}{3} = \frac{1552}{3}$$

3. $\displaystyle\int_0^6 \int_0^{6-x} \int_0^{6-x-z} dy\, dz\, dx = \int_0^6 \int_0^{6-x} (6-x-z)dz\, dx = \int_0^6 \left(6z - xz - \frac{1}{2}z^2\right)\Big|_0^{6-x} dx$

$$= \int_0^6 \left[6(6-x) - x(6-x) - \frac{1}{2}(6-x)^2\right] dx = \int_0^6 \left(18 - 6x + \frac{1}{2}x^2\right) dx$$

$$= \left(18x - 3x^2 + \frac{1}{6}x^3\right)\Big|_0^6 = 36$$

4. $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{\sqrt{y}} 4x^2z^3\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} x^2z^4\Big|_0^{\sqrt{y}} dy\, dx = \int_0^1 \int_0^{1-x} x^2y^2\, dy\, dx$

$$= \int_0^1 \frac{1}{3}x^2y^3\Big|_0^{1-x} dx = \frac{1}{3}\int_0^1 x^2(1-x)^3\, dx = \frac{1}{3}\int_0^1 (x^2 - 3x^3 + 3x^4 - x^5)dx$$

$$= \frac{1}{3}\left(\frac{1}{3}x^3 - \frac{3}{4}x^4 + \frac{3}{5}x^5 - \frac{1}{6}x^6\right)\Big|_0^1 = \frac{1}{180}$$

5. $\displaystyle\int_0^{\pi/2} \int_0^{y^2} \int_0^y \cos\frac{x}{y}\, dz\, dx\, dy = \int_0^{\pi/2} \int_0^{y^2} y\cos\frac{x}{y}\, dx\, dy = \int_0^{\pi/2} y^2 \sin\frac{x}{y}\Big|_0^{y^2} dy$

$$= \int_0^{\pi/2} y^2 \sin y\, dy \qquad \boxed{\text{Integration by parts}}$$

$$= (-y^2\cos y + 2\cos y + 2y\sin y)\Big|_0^{\pi/2} = \pi - 2$$

6. $\displaystyle\int_0^{\sqrt{2}} \int_{\sqrt{y}}^2 \int_0^{e^{x^2}} x\, dz\, dx\, dy = \int_0^{\sqrt{2}} \int_{\sqrt{y}}^2 xe^{x^2} dx\, dy = \int_0^{\sqrt{2}} \frac{1}{2}e^{x^2}\Big|_{\sqrt{y}}^2 dy = \frac{1}{2}\int_0^{\sqrt{2}} (e^4 - e^y)dy$

$$= \frac{1}{2}(ye^4 - e^y)\Big|_0^{\sqrt{2}} = \frac{1}{2}[(e^4\sqrt{2} - e^{\sqrt{2}}) - (-1)] = \frac{1}{2}(1 + e^4\sqrt{2} - e^{\sqrt{2}})$$

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7. $\displaystyle\int_0^1\int_0^1\int_0^{2-x^2-y^2} xye^z\,dz\,dx\,dy = \int_0^1\int_0^1 xye^z\,\Big|_0^{2-x^2-y^2}\,dx\,dy = \int_0^1\int_0^1 (xye^{2-x^2-y^2} - xy)dx\,dy$

$$= \int_0^1 \left(-\frac{1}{2}ye^{2-x^2-y^2} - \frac{1}{2}x^2y\right)\Big|_0^1\,dy = \int_0^1 \left(-\frac{1}{2}ye^{1-y^2} - \frac{1}{2}y + \frac{1}{2}ye^{2-y^2}\right)dy$$

$$= \left(\frac{1}{4}e^{1-y^2} - \frac{1}{4}y^2 - \frac{1}{4}e^{2-y^2}\right)\Big|_0^1 = \left(\frac{1}{4} - \frac{1}{4} - \frac{1}{4}e\right) - \left(\frac{1}{4}e - \frac{1}{4}e^2\right) = \frac{1}{4}e^2 - \frac{1}{2}e$$

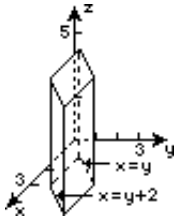
8. $\displaystyle\int_0^4\int_0^{1/2}\int_0^{x^2} \frac{1}{\sqrt{x^2-y^2}}\,dy\,dx\,dz = \int_0^4\int_0^{1/2} \sin^{-1}\frac{y}{x}\,\Big|_0^{x^2}\,dx\,dz = \int_0^4\int_0^{1/2} \sin^{-1}x\,dx\,dz$

Integration by parts

$$= \int_0^4 (x\sin^{-1}x + \sqrt{1-x^2}\,)\,\Big|_0^{1/2}\,dz = \int_0^4 \left(\frac{1}{2}\,\frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1\right)dz = \frac{\pi}{3} + 2\sqrt{3} - 4$$

9. $\displaystyle\iiint_D z\,dV = \int_0^5\int_1^3\int_y^{y+2} z\,dx\,dy\,dz = \int_0^5\int_1^3 xz\,\Big|_y^{y+2}\,dy\,dz = \int_0^5\int_1^3 2z\,dy\,dz$

$$= \int_0^5 2yz\,\Big|_1^3\,dz = \int_0^5 4z\,dz = 2z^2\,\Big|_0^5 = 50$$



10. Using symmetry,

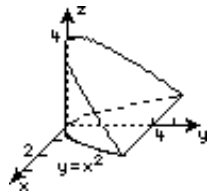
$$\iiint_D (x^2+y^2)\,dV = 2\int_0^2\int_{x^2}^4\int_0^{4-y} (x^2+y^2)\,dz\,dy\,dx = 2\int_0^2\int_{x^2}^4 (x^2+y^2)z\,\Big|_0^{4-y}\,dy\,dx$$

$$= 2\int_0^2\int_{x^2}^4 (4x^2 - x^2y + 4y^2 - y^3)\,dy\,dx$$

$$= 2\int_0^2 \left(4x^2y - \frac{1}{2}x^2y^2 + \frac{4}{3}y^3 - \frac{1}{4}y^4\right)\Big|_{x^2}^4\,dx$$

$$= 2\int_0^2 \left[\left(8x^2 + \frac{64}{3}\right) - \left(4x^4 + \frac{5}{6}x^6 - \frac{1}{4}x^8\right)\right]dx$$

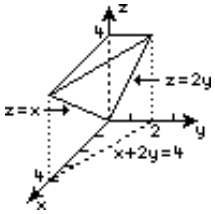
$$= 2\left(\frac{8}{3}x^3 + \frac{64}{3}x - \frac{4}{5}x^5 - \frac{5}{42}x^7 + \frac{1}{36}x^9\right)\Big|_0^2 = \frac{23{,}552}{315}.$$



11. The other five integrals are $\displaystyle\int_0^4\int_0^{2-x/2}\int_{x+2y}^4 F(x,y,z)\,dz\,dy\,dx$,

$$\int_0^4\int_0^z\int_0^{(z-x)/2} F(x,y,z)\,dy\,dx\,dz,\quad \int_0^4\int_x^4\int_0^{(z-x)/2} F(x,y,z)\,dy\,dz\,dx,$$

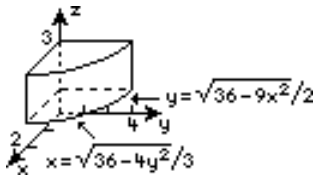
$$\int_0^4\int_0^{z/2}\int_0^{z-2y} F(x,y,z)\,dx\,dy\,dz,\quad \int_0^2\int_{2y}^4\int_0^{z-2y} F(x,y,z)\,dx\,dz\,dy.$$



12. The other five integrals are $\displaystyle\int_0^3\int_0^{\sqrt{36-4y^2}/3}\int_1^3 F(x,y,z)\,dz\,dx\,dy$,

$$\int_1^3\int_0^2\int_0^{\sqrt{36-9x^2}/2} F(x,y,z)\,dy\,dx\,dz,\quad \int_1^3\int_0^3\int_0^{\sqrt{36-4y^2}/3} F(x,y,z)\,dx\,dy\,dz,$$

$$\int_0^3\int_1^3\int_0^{\sqrt{36-4y^2}/3} F(x,y,z)\,dx\,dz\,dy,\quad \int_0^2\int_1^3\int_0^{\sqrt{36-9x^2}/2} F(x,y,z)\,dy\,dz\,dx.$$



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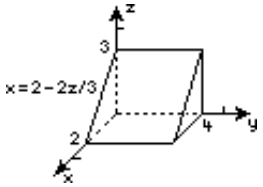
13. **(a)** $V = \displaystyle\int_0^2 \int_{x^3}^8 \int_0^4 dz\,dy\,dx$ **(b)** $V = \displaystyle\int_0^8 \int_0^4 \int_0^{y^{1/3}} dx\,dz\,dy$ **(c)** $V = \displaystyle\int_0^4 \int_0^2 \int_{x^2}^8 dy\,dx\,dz$

14. Solving $z = \sqrt{x}$ and $x + z = 2$, we obtain $x = 1$, $z = 1$. **(a)** $V = \displaystyle\int_0^3 \int_0^1 \int_{z^2}^{2-z} dx\,dz\,dy$

(b) $V = \displaystyle\int_0^1 \int_{z^2}^{2-z} \int_0^3 dy\,dx\,dz$ **(c)** $V = \displaystyle\int_0^3 \int_0^1 \int_0^{\sqrt{x}} dz\,dx\,dy + \int_0^3 \int_1^2 \int_0^{2-x} dz\,dx\,dy$

15.



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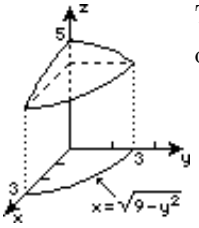
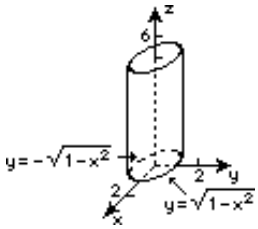


The region in the first octant is shown.

17.



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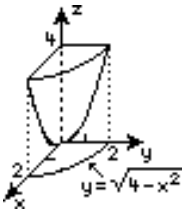

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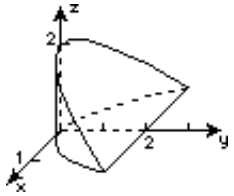

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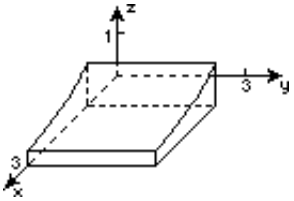
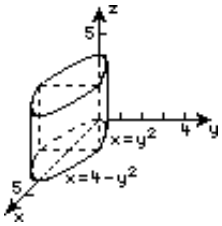


21. Solving $x = y^2$ and $4 - x = y^2$, we obtain $x = 2$, $y = \pm\sqrt{2}$. Using symmetry,

$$V = 2\int_0^3 \int_0^{\sqrt{2}} \int_{y^2}^{4-y^2} dx\,dy\,dz = 2\int_0^3 \int_0^{\sqrt{2}} (4 - 2y^2)\,dy\,dz$$

$$= 2\int_0^3 \left(4y - \frac{2}{3}y^3\right)\Big|_0^{\sqrt{2}} dz = 2\int_0^3 \frac{8\sqrt{2}}{3}\,dz = 16\sqrt{2}\,.$$

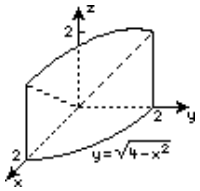


22. $V = \displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{x+y} dz\,dy\,dx = \int_0^2 \int_0^{\sqrt{4-x^2}} z\,\Big|_0^{x+y}\,dy\,dx$

$$= \int_0^2 \int_0^{\sqrt{4-x^2}} (x + y)\,dy\,dx = \int_0^2 \left(xy + \frac{1}{2}y^2\right)\Big|_0^{\sqrt{4-x^2}}\,dx$$

$$= \int_0^2 \left[x\sqrt{4 - x^2} + \frac{1}{2}(4 - x^2)\right]dx = \left[-\frac{1}{3}(4 - x^2)^{3/2} + 2x - \frac{1}{6}x^3\right]\Big|_0^2$$

$$= \left(4 - \frac{4}{3}\right) - \left(-\frac{8}{3}\right) = \frac{16}{3}$$



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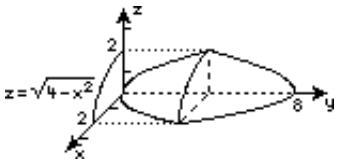
23. Adding the two equations, we obtain $2y = 8$. Thus, the paraboloids intersect in the plane $y = 4$. Their intersection is a circle of radius 2. Using symmetry,

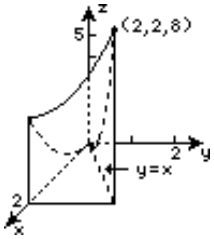


$$V = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+z^2}^{8-x^2-z^2} dy\,dz\,dx = 4\int_0^2 \int_0^{\sqrt{4-x^2}} (8-2x^2-2z^2)\,dz\,dx$$

$$= 4\int_0^2 \left[2(4-x^2)z - \frac{2}{3}z^3\right]\Big|_0^{\sqrt{4-x^2}} dx = 4\int_0^2 \frac{4}{3}(4-x^2)^{3/2}dx \qquad \boxed{\text{Trig substitution}}$$

$$= \frac{16}{3}\left[-\frac{x}{8}(2x^2-20)\sqrt{4-x^2} + 6\sin^{-1}\frac{x}{2}\right]\Big|_0^2 = 16\pi.$$

24. Solving $x = 2$, $y = x$, and $z = x^2 + y^2$, we obtain the point $(2, 2, 8)$.



$$V = \int_0^2 \int_0^x \int_0^{x^2+y^2} dz\,dy\,dx = \int_0^2 \int_0^x (x^2+y^2)\,dy\,dx = \int_0^2 \left(x^2y + \frac{1}{3}y^3\right)\Big|_0^x dx$$

$$= \int_0^2 \frac{4}{3}x^3\,dx = \frac{1}{3}x^4\Big|_0^2 = \frac{16}{3}.$$

25. We are given $\rho(x, y, z) = kz$.

$$m = \int_0^8 \int_0^4 \int_0^{y^{1/3}} kz\,dx\,dz\,dy = k\int_0^8 \int_0^4 xz\Big|_0^{y^{1/3}} dz\,dy = k\int_0^8 \int_0^4 y^{1/3}z\,dz\,dy$$

$$= k\int_0^8 \frac{1}{2}y^{1/3}z^2\Big|_0^4 dy = 8k\int_0^8 y^{1/3}dy = 8k\left(\frac{3}{4}y^{4/3}\right)\Big|_0^8 = 96k$$

$$M_{xy} = \int_0^8 \int_0^4 \int_0^{y^{1/3}} kz^2\,dx\,dz\,dy = k\int_0^8 \int_0^4 xz^2\Big|_0^{y^{1/3}} dz\,dy = k\int_0^8 \int_0^4 y^{1/3}z^2\,dz\,dy$$

$$= k\int_0^8 \frac{1}{3}y^{1/3}z^3\Big|_0^4 dy = \frac{64}{3}k\int_0^8 y^{1/3}dy = \frac{64}{3}k\left(\frac{3}{4}y^{4/3}\right)\Big|_0^8 = 256k$$

$$M_{xz} = \int_0^8 \int_0^4 \int_0^{y^{1/3}} kyz\,dx\,dz\,dy = k\int_0^8 \int_0^4 xyz\Big|_0^{y^{1/3}} dz\,dy = k\int_0^8 \int_0^4 y^{4/3}z\,dz\,dy$$

$$= k\int_0^8 \frac{1}{2}y^{4/3}z^2\Big|_0^4 dy = 8k\int_0^8 y^{4/3}dy = 8k\left(\frac{3}{7}y^{7/3}\right)\Big|_0^8 = \frac{3072}{7}k$$

$$M_{yz} = \int_0^8 \int_0^4 \int_0^{y^{1/3}} kxz\,dx\,dz\,dy = k\int_0^8 \int_0^4 \frac{1}{2}x^2z\Big|_0^{y^{1/3}} dz\,dy = \frac{1}{2}k\int_0^8 \int_0^4 y^{2/3}z\,dz\,dy$$

$$= \frac{1}{2}k\int_0^8 \frac{1}{2}y^{2/3}z^2\Big|_0^4 dy = 4k\int_0^8 y^{2/3}dy = 4k\left(\frac{3}{5}y^{5/3}\right)\Big|_0^8 = \frac{384}{5}k$$

$$\bar{x} = M_{yz}/m = \frac{384k/5}{96k} = 4/5;\quad \bar{y} = M_{xz}/m = \frac{3072k/7}{96k} = 32/7;\quad \bar{z} = M_{xy}/m = \frac{256k}{96k} = 8/3$$

The center of mass is $(4/5, 32/7, 8/3)$.

26. We use the form of the integral in Problem 14**(b)** of this section. Without loss of generality, we take $\rho = 1$.

$$m = \int_0^1 \int_{z^2}^{2-z} \int_0^3 dy\,dx\,dz = \int_0^1 \int_{z^2}^{2-z} 3\,dx\,dz = 3\int_0^1 (2-z-z^2)\,dz$$

$$= 3\left(2z - \frac{1}{2}z^2 - \frac{1}{3}z^3\right)\Big|_0^1 = \frac{7}{2}$$

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$$M_{xy} = \int_0^1 \int_{z^2}^{2-z} \int_0^3 z\,dy\,dx\,dz = \int_0^1 \int_{z^2}^{2-z} yz \,\Big|_0^3 dx\,dz = \int_0^1 \int_{z^2}^{2-z} 3z\,dx\,dz$$

$$= 3\int_0^1 xz \,\Big|_{z^2}^{2-z} dz = 3\int_0^1 (2z - z^2 - z^3)\,dz = 3\left(z^2 - \frac{1}{3}z^3 - \frac{1}{4}z^4\right)\Big|_0^1 = \frac{5}{4}$$

$$M_{xz} = \int_0^1 \int_{z^2}^{2-z} \int_0^3 y\,dy\,dx\,dz = \int_0^1 \int_{z^2}^{2-z} \frac{1}{2}y^2 \,\Big|_0^3 dx\,dz = \frac{9}{2}\int_0^1 \int_{z^2}^{2-z} dx\,dz$$

$$= \frac{9}{2}\int_0^1 (2 - z - z^2)\,dz = \frac{9}{2}\left(2z - \frac{1}{2}z^2 - \frac{1}{3}z^3\right)\Big|_0^1 = \frac{21}{4}$$

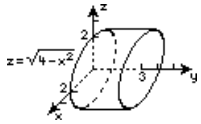
$$M_{yz} = \int_0^1 \int_{z^2}^{2-z} \int_0^3 x\,dy\,dx\,dz = \int_0^1 \int_{z^2}^{2-z} xy \,\Big|_0^3 dx\,dz = \int_0^1 \int_{z^2}^{2-x} 3x\,dx\,dz$$

$$= 3\int_0^1 \frac{1}{2}x^2 \,\Big|_{z^2}^{2-z} dz = \frac{3}{2}\int_0^1 (4 - 4z + z^2 - z^4)\,dz = \frac{3}{2}\left(4z - 2z^2 + \frac{1}{3}z^3 - \frac{1}{5}z^5\right)\Big|_0^1 = \frac{16}{5}$$

$\bar{x} = M_{yz}/m = \dfrac{16/5}{7/2} = 32/35$, $\bar{y} = M_{xz}/m = \dfrac{21/4}{7/2} = 3/2$, $\bar{z} = M_{xy}/m = \dfrac{5/4}{7/2} = 5/14$.

The centroid is $(32/35, 3/2, 5/14)$.

27. The density is $\rho(x, y, z) = ky$. Since both the region and the density function are symmetric with respect to the xy-and yz-planes, $\bar{x} = \bar{z} = 0$. Using symmetry,



$$m = 4\int_0^3 \int_0^2 \int_0^{\sqrt{4-x^2}} ky\,dz\,dx\,dy = 4k\int_0^3 \int_0^2 yz \,\Big|_0^{\sqrt{4-x^2}} dx\,dy = 4k\int_0^3 \int_0^2 y\sqrt{4-x^2}\,dx\,dy$$

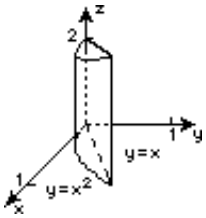
$$= 4k\int_0^3 y\left(\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right)\Big|_0^2 dy = 4k\int_0^3 \pi y\,dy = 4\pi k\left(\frac{1}{2}y^2\right)\Big|_0^3 = 18\pi k$$

$$M_{xz} = 4\int_0^3 \int_0^2 \int_0^{\sqrt{4-x^2}} ky^2\,dz\,dx\,dy = 4k\int_0^3 \int_0^2 y^2 z \,\Big|_0^{\sqrt{4-x^2}} dx\,dy = 4k\int_0^3 \int_0^2 y^2\sqrt{4-x^2}\,dx\,dy$$

$$= 4k\int_0^3 y^2\left(\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right)\Big|_0^2 dy = 4k\int_0^3 \pi y^2\,dy = 4\pi k\left(\frac{1}{3}y^3\right)\Big|_0^3 = 36\pi k.$$

$\bar{y} = M_{xz}/m = \dfrac{36\pi k}{18\pi k} = 2$. The center of mass is $(0, 2, 0)$.

28. The density is $\rho(x, y, z) = kz$.



$$m = \int_0^1 \int_{x^2}^{x} \int_0^{y+2} kz\,dz\,dy\,dx = k\int_0^1 \int_{x^2}^{x} \frac{1}{2}z^2 \,\Big|_0^{y+2} dy\,dx$$

$$= \frac{1}{2}k\int_0^1 \int_{x^2}^{x} (y+2)^2 dy\,dx = \frac{1}{2}k\int_0^1 \frac{1}{3}(y+2)^3 \,\Big|_{x^2}^{x} dx$$

$$= \frac{1}{6}k\int_0^1 [(x+2)^3 - (x^2+2)^3]\,dx = \frac{1}{6}k\int_0^1 [(x+2)^3 - (x^6 + 6x^4 + 12x^2 + 8)]dx$$

$$= \frac{1}{6}k\left[\frac{1}{4}(x+2)^4 - \frac{1}{7}x^7 - \frac{6}{5}x^5 - 4x^3 - 8x\right]\Big|_0^1 = \frac{407}{840}k$$

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$$M_{xy} = \int_0^1 \int_{x^2}^x \int_0^{y+2} kz^2\,dz\,dy\,dx = k\int_0^1 \int_{x^2}^x \frac{1}{3}z^3\Big|_0^{y+2} dy\,dx = \frac{1}{3}k\int_0^1 \int_{x^2}^x (y+2)^3 dy\,dx$$
$$= \frac{1}{3}k\int_0^1 \frac{1}{4}(y+2)^4\Big|_{x^2}^x dx = \frac{1}{12}k\int_0^1 [(x+2)^4 - (x^2+2)^4]\,dx$$
$$= \frac{1}{12}k\int_0^1 [(x+2)^4 - (x^8+8x^6+24x^4+32x^2+16)]\,dx$$
$$= \frac{1}{12}k\left[\frac{1}{5}(x+2)^5 - \frac{1}{9}x^9 - \frac{8}{7}x^7 - \frac{24}{5} - \frac{32}{3}x^3 - 16x\right]\Big|_0^1 = \frac{1493}{1890}k$$

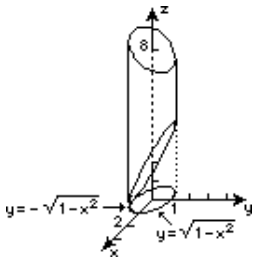
$$M_{xz} = \int_0^1 \int_{x^2}^x \int_0^{y+2} kyz\,dz\,dy\,dx = k\int_0^1 \int_{x^2}^x \frac{1}{2}yz^2\Big|_0^{y+2} dy\,dx = \frac{1}{2}k\int_0^1 \int_{x^2}^x y(y+2)^2\,dy\,dx$$
$$= \frac{1}{2}k\int_0^1 \int_{x^2}^x (y^3+4y^2+4y)\,dy\,dx = \frac{1}{2}k\int_0^1 \left(\frac{1}{4}y^4 + \frac{4}{3}y^3 + 2y^2\right)\Big|_{x^2}^x dx$$
$$= \frac{1}{2}k\int_0^1 \left(-\frac{1}{4}x^8 - \frac{4}{3}x^6 - 74x^4 + \frac{4}{3}x^3 + 2x^2\right) dx$$
$$= \frac{1}{2}k\left(-\frac{1}{36}x^9 - \frac{4}{21}x^7 - \frac{7}{20}x^5 + \frac{1}{3}x^4 + \frac{2}{3}x^3\right)\Big|_0^1 = \frac{68}{315}k$$

$$M_{yz} = \int_0^1 \int_{x^2}^x \int_0^{y+2} kxz\,dz\,dy\,dx = k\int_0^1 \int_{x^2}^x \frac{1}{2}xz^2\Big|_0^{y+2} dy\,dx = \frac{1}{2}k\int_0^1 \int_{x^2}^x x(y+2)^2\,dy\,dx$$
$$= \frac{1}{2}k\int_0^1 \frac{1}{3}x(y+2)^3\Big|_{x^2}^x dx = \frac{1}{6}k\int_0^1 [x(x+2)^3 - x(x^2+2)^3]\,dx$$
$$= \frac{1}{6}k\int_0^1 [x^4+6x^3+12x^2+8x - x(x^2+2)^3]\,dx$$
$$= \frac{1}{6}k\left[\frac{1}{5}x^5 + \frac{3}{2}x^4 + 4x^3 + 4x^2 - \frac{1}{8}(x^2+2)^4\right]\Big|_0^1 = \frac{21}{80}k$$

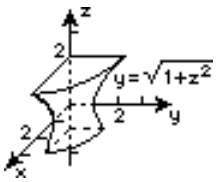
$\bar{x} = M_{yz}/m = \dfrac{21k/80}{407k/840} = 441/814$, $\bar{y} = M_{xz}/m = \dfrac{68k/315}{407k/840} = 544/1221$,

$\bar{z} = M_{xy}/m = \dfrac{1493k/1890}{407k/840} = 5972/3663$. The center of mass is $(441/814, 544/1221, 5972/3663)$.

29. $m = \displaystyle\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{2+2y}^{8-y} (x+y+4)\,dz\,dy\,dx$



30. Both the region and the density function are symmetric with respect to the xz- and yz-planes. Thus, $m = 4\displaystyle\int_{-1}^2 \int_0^{\sqrt{1+z^2}} \int_0^{\sqrt{1+z^2-y^2}} z^2\,dx\,dy\,dz$.



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31. We are given $\rho(x,y,z) = kz$.

$$I_y = \int_0^8 \int_0^4 \int_0^{y^{1/3}} kz(x^2+z^2)dx\,dz\,dy = k\int_0^8 \int_0^4 \left(\frac{1}{3}x^3z + xz^3\right)\Big|_0^{y^{1/3}} dz\,dy$$

$$= k\int_0^8 \int_0^4 \left(\frac{1}{3}yz + y^{1/3}z^3\right) dz\,dy = k\int_0^8 \left(\frac{1}{6}yz^2 + \frac{1}{4}y^{1/3}z^4\right)\Big|_0^4 dy$$

$$= k\int_0^8 \left(\frac{8}{3}y + 64y^{1/3}\right) dy = k\left(\frac{4}{3}y^2 + 48y^{4/3}\right)\Big|_0^8 = \frac{2560}{3}k$$

From Problem 25, $m = 96k$. Thus, $R_g = \sqrt{I_y/m} = \sqrt{\dfrac{2560k/3}{96k}} = \dfrac{4\sqrt{5}}{3}$.

32. We are given $\rho(x,y,z) = k$.

$$I_x = \int_0^1 \int_{z^2}^{2-z} \int_0^3 k(y^2+z^2)dy\,dx\,dz = k\int_0^1 \int_{z^2}^{2-z} \left(\frac{1}{3}y^3 + yz^2\right)\Big|_0^3 dx\,dz = k\int_0^1 \int_{z^2}^{2-z} (9+3z^2)\,dx\,dz$$

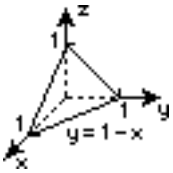
$$= k\int_0^1 (9x + 3xz^2)\Big|_{z^2}^{2-z} dz = k\int_0^1 (18 - 9z - 3z^2 - 3z^3 - 3z^4)\,dz$$

$$= k\left(18z - \frac{9}{2}z^2 - z^3 - \frac{3}{4}z^4 - \frac{3}{5}z^5\right)\Big|_0^1 = \frac{223}{20}k$$

$$m = \int_0^1 \int_{z^2}^{2-z} \int_0^3 k\,dy\,dx\,dz = k\int_0^1 \int_{z^2}^{2-z} 3\,dx\,dz = 3k\int_0^1 (2 - z - z^2)\,dz = 3k\left(2z - \frac{1}{2}z^2 - \frac{1}{3}z^3\right)\Big|_0^1 = \frac{7}{2}k$$

$$R_g = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{223k/20}{7k/2}} = \sqrt{\frac{223}{70}}$$

33. $$I_z = k\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x^2+y^2)\,dz\,dy\,dx = k\int_0^1 \int_0^{1-x} (x^2+y^2)(1-x-y)\,dy\,dx$$

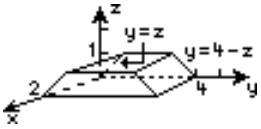


$$= k\int_0^1 \int_0^{1-x} (x^2 - x^3 - x^2y + y^2 - xy^2 - y^3)\,dy\,dx$$

$$= k\int_0^1 \left[(x^2-x^3)y - \frac{1}{2}x^2y^2 + \frac{1}{3}(1-x)y^3 - \frac{1}{4}y^4\right]\Big|_0^{1-x} dx$$

$$= k\int_0^1 \left[\frac{1}{2}x^2 - x^3 + \frac{1}{2}x^4 + \frac{1}{12}(1-x)^4\right] dx = k\left[\frac{1}{6}x^6 - \frac{1}{4}x^4 + \frac{1}{10}x^5 - \frac{1}{60}(1-x)^5\right]\Big|_0^1 = \frac{k}{30}$$

34. We are given $\rho(x,y,z) = kx$.



$$I_y = \int_0^1 \int_0^2 \int_z^{4-z} kx(x^2+z^2)\,dy\,dx\,dz = k\int_0^1 \int_0^2 (x^3+xz^2)y\,\Big|_z^{4-z} dx\,dz$$

$$= k\int_0^1 \int_0^2 (x^3+xz^2)(4-2z)\,dx\,dz = k\int_0^1 \left(\frac{1}{4}x^4 + \frac{1}{2}x^2z^2\right)(4-2z)\,\Big|_0^2 dz$$

$$= k\int_0^1 (4+2z^2)(4-2z)\,dz = 4k\int_0^1 (4 - 2z + 2z^2 - z^3)\,dz = 4k\left(4z - z^2 + \frac{2}{3}z^3 - \frac{1}{4}z^4\right)\Big|_0^1 = \frac{41}{3}k$$

35. $x = 10\cos 3\pi/4 = -5\sqrt{2}$; $y = 10\sin 3\pi/4 = 5\sqrt{2}$; $(-5\sqrt{2}, 5\sqrt{2}, 5)$

36. $x = 2\cos 5\pi/6 = -\sqrt{3}$; $y = 2\sin 5\pi/6 = 1$; $(-\sqrt{3}, 1, -3)$

37. $x = \sqrt{3}\cos\pi/3 = \sqrt{3}/2$; $y = \sqrt{3}\sin\pi/3 = 3/2$; $(\sqrt{3}/2, 3/2, -4)$

38. $x = 4\cos 7\pi/4 = 2\sqrt{2}$; $y = 4\sin 7\pi/4 = -2\sqrt{2}$; $(2\sqrt{2}, -2\sqrt{2}, 0)$

39. With $x = 1$ and $y = -1$ we have $r^2 = 2$ and $\tan\theta = -1$. The point is $(\sqrt{2}, -\pi/4, -9)$.

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40. With $x = 2\sqrt{3}$ and $y = 2$ we have $r^2 = 16$ and $\tan\theta = 1/\sqrt{3}$. The point is $(4, \pi/6, 17)$.

41. With $x = -\sqrt{2}$ and $y = \sqrt{6}$ we have $r^2 = 8$ and $\tan\theta = -\sqrt{3}$. The point is $(2\sqrt{2}, 2\pi/3, 2)$.

42. With $x = 1$ and $y = 2$ we have $r^2 = 5$ and $\tan\theta = 2$. The point is $(\sqrt{5}, \tan^{-1} 2, 7)$.

43. $r^2 + z^2 = 25$

44. $r\cos\theta + r\sin\theta - z = 1$

45. $r^2 - z^2 = 1$

46. $r^2\cos^2\theta + z^2 = 16$

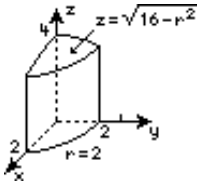
47. $z = x^2 + y^2$

48. $z = 2y$

49. $r\cos\theta = 5, \;\; x = 5$

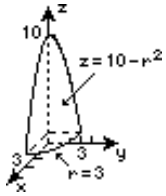
50. $\tan\theta = 1/\sqrt{3}$, $y/x = 1/\sqrt{3}$, $x = \sqrt{3}\,y$, $x > 0$

51. The equations are $r^2 = 4$, $r^2 + z^2 = 16$, and $z = 0$.



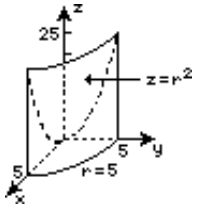
$$V = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r\sqrt{16-r^2}\,dr\,d\theta$$
$$= \int_0^{2\pi} -\frac{1}{3}(16-r^2)^{3/2}\,\Big|_0^2\,d\theta = \int_0^{2\pi}\frac{1}{3}(64-24\sqrt{3}\,)\,d\theta = \frac{2\pi}{3}(64-24\sqrt{3}\,)$$

52. The equation is $z = 10 - r^2$.



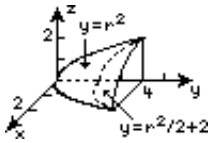
$$V = \int_0^{2\pi}\int_0^3\int_1^{10-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 r(9-r^2)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{9}{2}r^2 - \frac{1}{4}r^4\right)\Big|_0^3\,d\theta$$
$$= \int_0^{2\pi}\frac{81}{4}\,d\theta = \frac{81\pi}{2}\,.$$

53. The equations are $z = r^2$, $r = 5$, and $z = 0$.



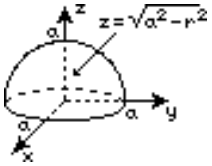
$$V = \int_0^{2\pi}\int_0^5\int_0^{r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^5 r^3\,dr\,d\theta = \int_0^{2\pi}\frac{1}{4}r^4\,\Big|_0^5\,d\theta$$
$$= \int_0^{2\pi}\frac{625}{4}\,d\theta = \frac{625\pi}{2}$$

54. Substituting the first equation into the second, we see that the surfaces intersect in the plane $y = 4$. Using polar coordinates in the xz-plane, the equations of the surfaces become $y = r^2$ and $y = \frac{1}{2}r^2 + 2$.



$$V = \int_0^{2\pi}\int_0^2\int_{r^2}^{r^2/2+2} r\,dy\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r\left(\frac{r^2}{2} + 2 - r^2\right)dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2\left(2r - \frac{1}{2}r^3\right)dr\,d\theta = \int_0^{2\pi}\left(r^2 - \frac{1}{8}r^4\right)\Big|_0^2\,d\theta = \int_0^{2\pi} 2\,d\theta = 4\pi$$

55. The equation is $z = \sqrt{a^2 - r^2}$. By symmetry, $\bar{x} = \bar{y} = 0$.



$$m = \int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,\theta = \int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}\;dr\,d\theta$$
$$= \int_0^{2\pi} -\frac{1}{3}(a^2-r^2)^{3/2}\,\Big|_0^a\,d\theta = \int_0^{2\pi}\frac{1}{3}a^3\,d\theta = \frac{2}{3}\pi a^3$$

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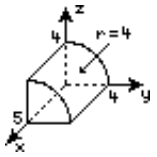
$$M_{xy} = \int_0^{2\pi}\int_0^a \int_0^{\sqrt{a^2-r^2}} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^a \frac{1}{2}rz^2\Big|_0^{\sqrt{a^2-r^2}}\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^a r(a^2-r^2)\,dr\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{1}{2}a^2r^2 - \frac{1}{4}r^4\right)\Big|_0^a d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1}{4}a^4\,d\theta = \frac{1}{4}\pi a^4$$

$\bar{z} = M_{xy}/m = \dfrac{\pi a^4/4}{2\pi a^3/3} = 3a/8$. The centroid is $(0, 0, 3a/8)$.

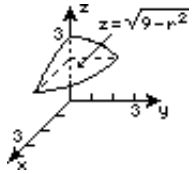
56. We use polar coordinates in the yz-plane. The density is $\rho(x, y, z) = kz$. By symmetry, $\bar{y} = \bar{z} = 0$.



$$m = \int_0^{2\pi}\int_0^4 \int_0^5 kxr\,dx\,dr\,d\theta = k\int_0^{2\pi}\int_0^4 \frac{1}{2}rz^2\Big|_0^5\,dr\,d\theta = \frac{k}{2}\int_0^{2\pi}\int_0^4 25r\,dr\,d\theta$$
$$= \frac{25k}{2}\int_0^{2\pi}\frac{1}{2}r^2\Big|_0^4\,d\theta = \frac{25k}{2}\int_0^{2\pi}8\,d\theta = 200k\pi$$
$$M_{yz} = \int_0^{2\pi}\int_0^4 \int_0^5 kx^2r\,dx\,dr\,d\theta = k\int_0^{2\pi}\int_0^4 \frac{1}{3}rx^3\Big|_0^5\,dr\,d\theta = \frac{1}{3}k\int_0^{2\pi}\int_0^4 125r\,dr\,d\theta$$
$$= \frac{1}{3}k\int_0^{2\pi}\frac{125}{2}r^2\Big|_0^4\,d\theta = \frac{1}{3}k\int_0^{2\pi}1000\,d\theta = \frac{2000}{3}\,k\pi$$

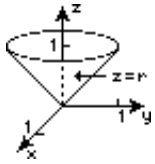
$\bar{x} = M_{yz}/m = \dfrac{2000k\pi/3}{200k\pi} = 10/3$. The center of mass of the given solid is $(10/3, 0, 0)$.

57. The equation is $z = \sqrt{9-r^2}$ and the density is $\rho = k/r^2$. When $z = 2$, $r = \sqrt{5}$.



$$I_z = \int_0^{2\pi}\int_0^{\sqrt{5}}\int_2^{\sqrt{9-r^2}} r^2(k/r^2)r\,dz\,dr\,d\theta = k\int_0^{2\pi}\int_0^{\sqrt{5}} rz\Big|_2^{\sqrt{9-r^2}}\,dr\,d\theta$$
$$= k\int_0^{2\pi}\int_0^{\sqrt{5}}(r\sqrt{9-r^2} - 2r)\,dr\,d\theta = k\int_0^{2\pi}\left[-\frac{1}{3}(9-r^2)^{3/2} - r^2\right]\Big|_0^{\sqrt{5}}\,d\theta$$
$$= k\int_0^{2\pi}\frac{4}{3}\,d\theta = \frac{8}{3}\,\pi k$$

58. The equation is $z = r$ and the density is $\rho = kr$.



$$I_x = \int_0^{2\pi}\int_0^1\int_r^1 (y^2+z^2)(kr)r\,dz\,dr\,d\theta = k\int_0^{2\pi}\int_0^1\int_r^1 (r^4\sin^2\theta + r^2z^2)\,dz\,dr\,d\theta$$
$$= k\int_0^{2\pi}\int_0^1\left[(r^4\sin^2\theta)z + \frac{1}{3}r^2z^3\right]\Big|_r^1\,dr\,d\theta$$
$$= k\int_0^{2\pi}\int_0^1\left(r^4\sin^2\theta + \frac{1}{3}r^2 - r^5\sin^2\theta - \frac{1}{3}r^5\right)dr\,d\theta$$
$$= k\int_0^{2\pi}\left(\frac{1}{5}r^5\sin^2\theta + \frac{1}{9}r^3 - \frac{1}{6}r^6\sin^2\theta - \frac{1}{18}r^6\right)\Big|_0^1\,d\theta = k\int_0^{2\pi}\left(\frac{1}{30}\sin^2\theta + \frac{1}{18}\right)d\theta$$
$$= k\left(\frac{1}{60}\theta - \frac{1}{120}\sin 2\theta + \frac{1}{18}\theta\right)\Big|_0^{2\pi} = \frac{13}{90}\pi k$$

59. **(a)** $x = (2/3)\sin(\pi/2)\cos(\pi/6) = \sqrt{3}/3$; $y = (2/3)\sin(\pi/2)\sin(\pi/6) = 1/3$;
$z = (2/3)\cos(\pi/2) = 0$; $(\sqrt{3}/3, 1/3, 0)$

(b) With $x = \sqrt{3}/3$ and $y = 1/3$ we have $r^2 = 4/9$ and $\tan\theta = \sqrt{3}/3$. The point is $(2/3, \pi/6, 0)$.

60. **(a)** $x = 5\sin(5\pi/4)\cos(2\pi/3) = 5\sqrt{2}/4$; $y = 5\sin(5\pi/4)\sin(2\pi/3) = -5\sqrt{6}/4$;
$z = 5\cos(5\pi/4) = -5\sqrt{2}/2$; $(5\sqrt{2}/4, -5\sqrt{6}/4, -5\sqrt{2}/2)$

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(b) With $x = 5\sqrt{2}/4$ and $y = -5\sqrt{6}/4$ we have $r^2 = 25/2$ and $\tan\theta = -\sqrt{3}$.
The point is $(5/\sqrt{2}, 2\pi/3, -5\sqrt{2}/2)$.

61. (a) $x = 8\sin(\pi/4)\cos(3\pi/4) = -4$; $y = 8\sin(\pi/4)\sin(3\pi/4) = 4$; $z = 8\cos(\pi/4) = 4\sqrt{2}$;
$(-4, 4, 4\sqrt{2})$

(b) With $x = -4$ and $y = 4$ we have $r^2 = 32$ and $\tan\theta = -1$. The point is $(4\sqrt{2}, 3\pi/4, 4\sqrt{2})$.

62. (a) $x = (1/3)\sin(5\pi/3)\cos(\pi/6) = -1/4$; $y = (1/3)\sin(5\pi/3)\sin(\pi/6) = -\sqrt{3}/12$;
$z = (1/3)\cos(5\pi/3) = 1/6$; $(-1/4, -\sqrt{3}/12, 1/6)$

(b) With $x = -1/4$ and $y = -\sqrt{3}/12$ we have $r^2 = 1/12$ and $\tan\theta = \sqrt{3}/3$.
The point is $(1/2\sqrt{3}, \pi/6, 1/6)$.

63. With $x = -5$, $y = -5$, and $z = 0$, we have $\rho^2 = 50$, $\tan\theta = 1$, and $\cos\phi = 0$. The point is $(5\sqrt{2}, \pi/2, 5\pi/4)$.

64. With $x = 1$, $y = -\sqrt{3}$, and $z = 1$, we have $\rho^2 = 5$, $\tan\theta = -\sqrt{3}$, and $\cos\phi = 1/\sqrt{5}$.
The point is $(\sqrt{5}, \cos^{-1} 1/\sqrt{5}, -\pi/3)$.

65. With $x = \sqrt{3}/2$, $y = 1/2$, and $z = 1$, we have $\rho^2 = 2$, $\tan\theta = 1/\sqrt{3}$, and $\cos\phi = 1/\sqrt{2}$.
The point is $(\sqrt{2}, \pi/4, \pi/6)$.

66. With $x = -\sqrt{3}/2$, $y = 0$, and $z = -1/2$, we have $\rho^2 = 1$, $\tan\theta = 0$, and $\cos\phi = -1/2$.
The point is $(1, 2\pi/3, 0)$.

67. $\rho = 8$

68. $\rho^2 = 4\rho\cos\phi$; $\rho = 4\cos\phi$

69. $4z^2 = 3x^2 + 3y^2 + 3z^2$; $4\rho^2\cos^2\phi = 3\rho^2$; $\cos\phi = \pm\sqrt{3}/2$; $\phi = \pi/6,\ 5\pi/6$

70. $-x^2 - y^2 - z^2 = 1 - 2z^2$; $-\rho^2 = 1 - 2\rho^2\cos^2\phi$; $\rho^2(2\cos^2\phi - 1) = 1$

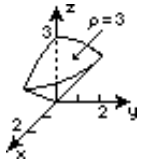
71. $x^2 + y^2 + z^2 = 100$

72. $\cos\phi = 1/2$; $\rho^2\cos^2\phi = \rho^2/4$; $4z^2 = x^2 + y^2 + z^2$; $x^2 + y^2 = 3z^2$

73. $\rho\cos\phi = 2$; $z = 2$

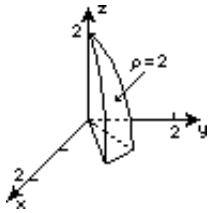
74. $\rho(1 - \cos^2\phi) = \cos\phi$; $\rho^2 - \rho^2\cos^2\phi = \rho\cos\phi$; $x^2 + y^2 + z^2 - z^2 = z$; $z = x^2 + y^2$

75. The equations are $\phi = \pi/4$ and $\rho = 3$.



$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\rho^3\sin\phi\,\Big|_0^3\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4} 9\sin\phi\,d\phi\,d\theta$$
$$= \int_0^{2\pi} -9\cos\phi\,\Big|_0^{\pi/4}\,d\theta = -9\int_0^{2\pi}\left(\frac{\sqrt{2}}{2} - 1\right)d\theta = 9\pi(2 - \sqrt{2})$$

76. The equations are $\rho = 2$, $\theta = \pi/4$, and $\theta = \pi/3$.



$$\int_{\pi/4}^{\pi/3}\int_0^{\pi/2}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_{\pi/4}^{\pi/3}\int_0^{\pi/2} \frac{1}{3}\rho^3\sin\phi\,\Big|_0^2\,d\phi\,d\theta$$
$$= \int_{\pi/4}^{\pi/3}\int_0^{\pi/2} \frac{8}{3}\sin\phi\,d\phi\,d\theta = \frac{8}{3}\int_{\pi/4}^{\pi/3} -\cos\phi\,\Big|_0^{\pi/2}\,d\theta$$
$$= \frac{8}{3}\int_{\pi/4}^{\pi/3}(0 + 1)\,d\theta = \frac{2\pi}{9}$$

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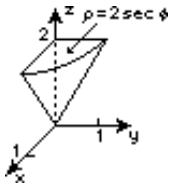
77. Using Problem 69, the equations are $\phi = \pi/6$, $\theta = \pi/2$, $\theta = 0$, and $\rho\cos\phi = 2$.

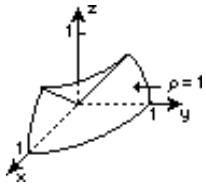


$$V = \int_0^{\pi/2}\int_0^{\pi/6}\int_0^{2\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/6} \frac{1}{3}\rho^3\sin\phi\,\Big|_0^{2\sec\phi}\,d\phi\,d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2}\int_0^{\pi/6} \sec^3\phi\sin\phi\,d\phi\,d\theta = \frac{8}{3}\int_0^{\pi/2}\int_0^{\pi/6} \sec^2\phi\tan\phi\,d\phi\,d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \frac{1}{2}\tan^2\phi\,\Big|_0^{\pi/6}\,d\theta = \frac{4}{3}\int_0^{\pi/2}\frac{1}{3}\,d\theta = \frac{2}{9}\pi$$

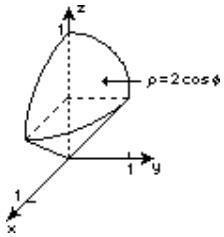
78. The equations are $\rho = 1$ and $\phi = \pi/4$. We find the volume above the xy-plane and double.



$$V = 2\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\frac{1}{3}\rho^3\sin\phi\,\Big|_0^1\,d\phi\,d\theta$$

$$= \frac{2}{3}\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\sin\phi\,d\phi\,d\theta = \frac{2}{3}\int_0^{2\pi} -\cos\phi\,\Big|_{\pi/4}^{\pi/2}\,d\theta = \frac{2}{3}\int_0^{2\pi}\frac{\sqrt{2}}{2}\,d\theta = \frac{2\pi\sqrt{2}}{3}$$

79. By symmetry, $\bar{x} = \bar{y} = 0$. The equations are $\phi = \pi/4$ and $\rho = 2\cos\phi$.



$$m = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{1}{3}\rho^3\sin\phi\,\Big|_0^{2\cos\phi}\,d\phi\,d\theta$$

$$= \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\cos^3\phi\,d\phi\,d\theta = \frac{8}{3}\int_0^{2\pi} -\frac{1}{4}\cos^4\phi\,\Big|_0^{\pi/4}\,d\theta$$

$$= -\frac{2}{3}\int_0^{2\pi}\left(\frac{1}{4} - 1\right)d\theta = \pi$$

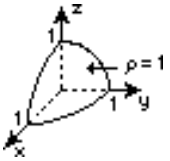
$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} z\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi}\rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\frac{1}{4}\rho^4\sin\phi\cos\phi\,\Big|_0^{2\cos\phi}\,d\phi\,d\theta = 4\int_0^{2\pi}\int_0^{\pi/4}\cos^5\phi\sin\phi\,d\phi\,d\theta$$

$$= 4\int_0^{2\pi} -\frac{1}{6}\cos^6\phi\,\Big|_0^{\pi/4}\,d\theta = -\frac{2}{3}\int_0^{2\pi}\left(\frac{1}{8} - 1\right)d\theta = \frac{7}{6}\pi$$

$\bar{z} = M_{xy}/m = \dfrac{7\pi/6}{\pi} = 7/6$. The centroid is $(0, 0, 7/6)$.

80. We are given density $= kz$. By symmetry, $\bar{x} = \bar{y} = 0$. The equation is $\rho = 1$.



$$m = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 kz\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = k\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta$$

$$= k\int_0^{2\pi}\int_0^{\pi/2}\frac{1}{4}\rho^4\sin\phi\cos\phi\,\Big|_0^1\,d\phi\,d\theta = \frac{1}{4}k\int_0^{2\pi}\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi\,d\theta$$

$$= \frac{1}{4}k\int_0^{2\pi}\frac{1}{2}\sin^2\phi\,\Big|_0^{\pi/2}\,d\theta = \frac{1}{8}k\int_0^{2\pi}d\theta = \frac{k\pi}{4}$$

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 kz^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = k\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^4\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= k\int_0^{2\pi}\int_0^{\pi/2}\frac{1}{5}\rho^5\cos^2\phi\sin\phi\,\Big|_0^1\,d\phi\,d\theta = \frac{1}{5}k\int_0^{2\pi}\int_0^{\pi/2}\cos^2\phi\sin\phi\,d\phi\,d\theta$$

$$= \frac{1}{5}k\int_0^{2\pi} -\frac{1}{3}\cos^3\phi\,\Big|_0^{\pi/2}\,d\theta = -\frac{1}{15}k\int_0^{2\pi}(0-1)\,d\theta = \frac{2}{15}\,k\pi$$

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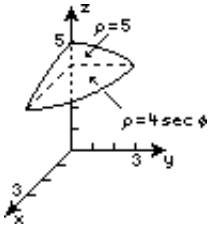
$\bar{z} = M_{xy}/m = \dfrac{2k\pi/15}{k\pi/4} = 8/15$. The center of mass is $(0, 0, 8/15)$.

81. We are given density $= k/\rho$.



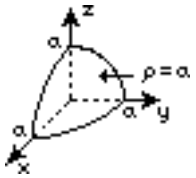
$$m = \int_0^{2\pi}\int_0^{\cos^{-1}4/5}\int_{4\sec\phi}^{5} \frac{k}{\rho}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = k\int_0^{2\pi}\int_0^{\cos^{-1}4/5} \frac{1}{2}\rho^2\sin\phi\,\Big|_{4\sec\phi}^{5} d\phi\,d\theta$$

$$= \frac{1}{2}k\int_0^{2\pi}\int_0^{\cos^{-1}4/5} (25\sin\phi - 16\tan\phi\sec\phi)\,d\phi\,d\theta$$

$$= \frac{1}{2}k\int_0^{2\pi} (-25\cos\phi - 16\sec\phi)\,\Big|_0^{\cos^{-1}4/5} d\theta = \frac{1}{2}k\int_0^{2\pi} [-25(4/5) - 16(5/4) - (-25-16)]\,d\theta$$

$$= \frac{1}{2}k\int_0^{2\pi} d\theta = k\pi$$

82. We are given density $= k\rho$.



$$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} (x^2+y^2)(k\rho)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

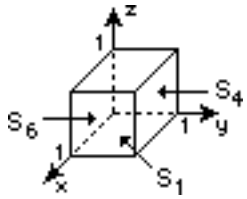
$$= k\int_0^{2\pi}\int_0^{\pi}\int_0^{a} (\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta)\rho^3\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= k\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^5\sin^3\phi\,d\rho\,d\phi\,d\theta = k\int_0^{2\pi}\int_0^{\pi} \frac{1}{6}\rho^6\sin^3\phi\,\Big|_0^{a} d\phi\,d\theta = \frac{1}{6}ka^6\int_0^{2\pi}\int_0^{\pi}\sin^3\phi\,d\phi\,d\theta$$

$$= \frac{1}{6}ka^3\int_0^{2\pi}\int_0^{\pi}(1-\cos^2\phi)\sin\phi\,d\phi\,d\theta = \frac{1}{6}ka^3\int_0^{2\pi}\left(-\cos\phi + \frac{1}{3}\cos^3\phi\right)\Big|_0^{\pi} d\theta = \frac{1}{6}ka^3\int_0^{2\pi}\frac{4}{3}\,d\theta = \frac{4\pi}{9}\,ka^6$$

Exercises 9.16

1. $\operatorname{div}\mathbf{F} = y + z + x$



The Triple Integral:

$$\iiint_D \operatorname{div}\mathbf{F}\,dV = \int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left(\frac{1}{2}x^2 + xy + xz\right)\Big|_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 \left(\frac{1}{2} + y + z\right) dy\,dz = \int_0^1 \left(\frac{1}{2}y + \frac{1}{2}y^2 + yz\right)\Big|_0^1 dz$$

$$= \int_0^1 (1+z)\,dz = \frac{1}{2}(1+z^2)\,\Big|_0^1 = 2 - \frac{1}{2} = \frac{3}{2}$$

The Surface Integral: Let the surfaces be S_1 in $z = 0$, S_2 in $z = 1$, S_3 in $y = 0$, S_4 in $y = 1$, S_5 in $x = 0$, and S_6 in $x = 1$. The unit outward normal vectors are $-\mathbf{k}$, $\mathbf{k}$, $-\mathbf{j}$, $\mathbf{j}$, $-\mathbf{i}$ and $\mathbf{i}$, respectively. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1}\mathbf{F}\cdot(-\mathbf{k})\,dS_1 + \iint_{S_2}\mathbf{F}\cdot\mathbf{k}\,dS_2 + \iint_{S_3}\mathbf{F}\cdot(-\mathbf{j})\,dS_3 + \iint_{S_4}\mathbf{F}\cdot\mathbf{j}\,dS_4$$
$$+ \iint_{S_5}\mathbf{F}\cdot(-\mathbf{i})\,dS_5 + \iint_{S_6}\mathbf{F}\cdot\mathbf{i}\,dS_6$$

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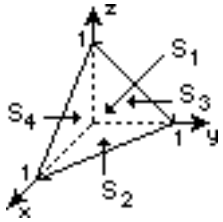
$$= \iint_{S_1} (-xz)\,dS_1 + \iint_{S_2} xz\,dS_2 + \iint_{S_3} (-yz)\,dS_3 + \iint_{S_4} yz\,dS_4 + \iint_{S_5} (-xy)\,dS_5 + \iint_{S_6} xy\,dS_6$$
$$= \iint_{S_2} x\,dS_2 + \iint_{S_4} z\,dS_4 + \iint_{S_6} y\,dS_6$$
$$= \int_0^1 \int_0^1 x\,dx\,dy + \int_0^1 \int_0^1 z\,dz\,dx + \int_0^1 \int_0^1 y\,dy\,dz$$
$$= \int_0^1 \frac{1}{2}\,dy + \int_0^1 \frac{1}{2}\,dx + \int_0^1 \frac{1}{2}\,dz = \frac{3}{2}.$$

2. div $\mathbf{F} = 6y + 4z$



The Triple Integral:

$$\iiint_D \text{div } \mathbf{F}\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (6y + 4z)\,dz\,dy\,dx$$
$$= \int_0^1 \int_0^{1-x} (6yz + 2z^2)\,\Big|_0^{1-x-y}\,dy\,dx$$
$$= \int_0^1 \int_0^{1-x} (-4y^2 + 2y - 2xy + 2x^2 - 4x + 2)\,dy\,dx$$
$$= \int_0^1 \left(-\frac{4}{3}y^3 + y^2 - xy^2 + 2x^2y - 4xy + 2y\right)\Big|_0^{1-x}\,dx$$
$$= \int_0^1 \left(-\frac{5}{3}x^3 + 5x^2 - 5x + \frac{5}{3}\right)dx = \left(-\frac{5}{12}x^4 + \frac{5}{3}x^3 - \frac{5}{2}x^2 + \frac{5}{3}x\right)\Big|_0^1 = \frac{5}{12}$$

The Surface Integral: Let the surfaces be S_1 in the plane $x + y + z = 1$, S_2 in $z = 0$, S_3 in $x = 0$, and S_4 in $y = 0$. The unit outward normal vectors are $\mathbf{n}_1 = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$, $\mathbf{n}_2 = -\mathbf{k}$, $\mathbf{n}_3 = -\mathbf{i}$, and $\mathbf{n}_4 = -\mathbf{j}$, respectively. Now on S_1, $dS_1 = \sqrt{3}\,dA_1$, on S_3, $x = 0$, and on S_4, $y = 0$, so

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}_1\,dS_1 + \iint_{S_2} \mathbf{F}\cdot(-\mathbf{k})\,dS_2 + \iint_{S_3} \mathbf{F}\cdot(-\mathbf{j})\,dS_3 + \iint_{S_4} \mathbf{F}\cdot(-\mathbf{i})\,dS_4$$
$$= \int_0^1 \int_0^{1-x} (6xy + 4y(1-x-y) + xe^{-y})\,dy\,dx + \int_0^1 \int_0^{1-x} (-xe^{-y})\,dy\,dx + \iint_{S_3} (-6xy)\,dS_3 + \iint_{S_4} (-4yz)\,dS_4$$
$$= \int_0^1 \left(xy^2 + 2y^2 - \frac{4}{3}y^3 - xe^{-y}\right)\Big|_0^{1-x} dx + \int_0^1 xe^{-y}\,\Big|_0^{1-x} dx + 0 + 0$$
$$= \int_0^1 [x(1-x)^2 + 2(1-x)^2 - \frac{4}{3}(1-x)^3 - xe^{x-1} + x]\,dx + \int_0^1 (xe^{x-1} - x)\,dx$$
$$= \left[\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 - \frac{2}{3}(1-x)^3 + \frac{1}{3}(1-x)^4\right]\Big|_0^1 = \frac{5}{12}.$$

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3. div $\mathbf{F} = 3x^2 + 3y^2 + 3z^2$. Using spherical coordinates,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D 3(x^2+y^2+z^2)\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 3\rho^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} \frac{3}{5}\rho^5\sin\phi\,\Big|_0^a\,d\phi\,d\theta = \frac{3a^5}{5}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta$$
$$= \frac{3a^5}{5}\int_0^{2\pi} -\cos\phi\,\Big|_0^{\pi}\,d\theta = \frac{6a^5}{5}\int_0^{2\pi} d\theta = \frac{12\pi a^5}{5}.$$

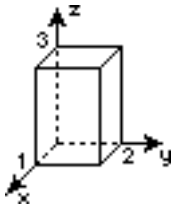
4. div $\mathbf{F} = 4+1+4 = 9$. Using the formula for the volume of a sphere,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D 9\,dV = 9\left(\frac{4}{3}\pi 2^3\right) = 96\pi.$$

5. div $\mathbf{F} = 2(z-1)$. Using cylindrical coordinates,

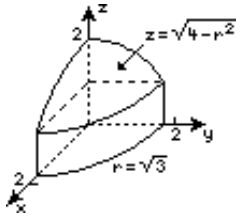
$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D 2(z-1)\,V = \int_0^{2\pi}\int_0^{4}\int_1^{5} 2(z-1)\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{4}(z-1)^2\,\Big|_1^5\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^{4} 16r\,dr\,d\theta = \int_0^{2\pi} 8r^2\,\Big|_0^4\,d\theta = 128\int_0^{2\pi} d\theta = 256\pi.$$

6. div $\mathbf{F} = 2x + 2z + 12z^2$.



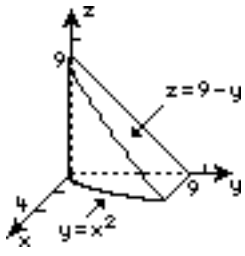
$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \text{div }\mathbf{F}\,dV = \int_0^3\int_0^2\int_0^1 (2x+2z+12z^2)\,dx\,dy\,dz$$
$$= \int_0^3\int_0^2 (x^2+2xz+12xz^2)\,\Big|_0^1\,dy\,dz = \int_0^3\int_0^2 (1+2z+12z^2)\,dy\,dz$$
$$= \int_0^3 2(1+2z+12z^2)\,dz = (2z+2z^2+8z^3)\,\Big|_0^3 = 240$$

7. div $\mathbf{F} = 3z^2$. Using cylindrical coordinates,



$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \text{div }\mathbf{F}\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{\sqrt{4-r^2}} 3z^2 r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\sqrt{3}} rz^3\,\Big|_0^{\sqrt{4-r^2}}\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} r(4-r^2)^{3/2}\,dr\,d\theta$$
$$= \int_0^{2\pi} -\frac{1}{5}(4-r^2)^{5/2}\,\Big|_0^{\sqrt{3}}\,d\theta = \int_0^{2\pi} -\frac{1}{5}(1-32)\,d\theta = \int_0^{2\pi}\frac{31}{5}\,d\theta = \frac{62\pi}{5}.$$

8. div $\mathbf{F} = 2x$.



$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \text{div }\mathbf{F}\,dV = \int_0^3\int_{x^2}^9\int_0^{9-y} 2x\,dz\,dy\,dx$$
$$= \int_0^3\int_{x^2}^9 2x(9-y)\,dy\,dx = \int_0^3 -x(9-y)^2\,\Big|_{x^2}^9\,dx = \int_0^3 x(9-x)^2\,dx$$
$$= \int_0^3 (x^3-18x^2+81x)\,dx = \left(\frac{1}{4}x^4 - 6x^3 + \frac{81}{2}x^2\right)\,\Big|_0^3 = \frac{891}{4}$$

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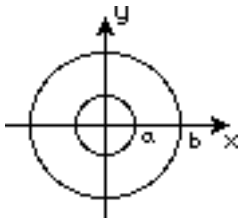
9. div $\mathbf{F} = \dfrac{1}{x^2+y^2+z^2}$. Using spherical coordinates,



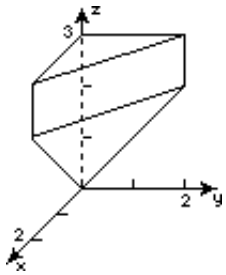
$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_D \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{1}{\rho^2}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}(b-a)\sin\phi\,d\phi\,d\theta = (b-a)\int_0^{2\pi} -\cos\phi\,\Big|_0^{\pi}\,d\theta \\ &= (b-a)\int_0^{2\pi} 2\,d\theta = 4\pi(b-a).\end{aligned}$$

10. Since div $\mathbf{F} = 0$, $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D 0\,dV = 0.$

11. div $\mathbf{F} = 2z + 10y - 2z = 10y$.

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_D 10y\,dV = \int_0^2\int_0^{2-x^2/2}\int_z^{4-z} 10y\,dy\,dz\,dx = \int_0^2\int_0^{2-x^2/2} 5y^2\,\Big|_z^{4-z}\,dz\,dx \\ &= \int_0^2\int_0^{2-x^2/2}(80-40z)\,dz\,dx = \int_0^2 (80z-20z^2)\,\Big|_0^{2-x^2/2}\,dx = \int_0^2 (80-5x^4)\,dx = (80x-x^5)\,\Big|_0^2 = 128\end{aligned}$$

12. div $\mathbf{F} = 30xy$.



$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_D 30xy\,dV = \int_0^2\int_0^{2-x}\int_{x+y}^3 30xy\,dz\,dy\,dx \\ &= \int_0^2\int_0^{2-x} 30xyz\,\Big|_{x+y}^3\,dy\,dx \\ &= \int_0^2\int_0^{2-x}(90xy-30x^2y-30xy^2)\,dy\,dx \\ &= \int_0^2 (45xy^2-15x^2y^2-10xy^3)\,\Big|_0^{2-x}\,dx \\ &= \int_0^2(-5x^4+45x^3-120x^2+100x)\,dx = \left(-x^5+\frac{45}{4}x^4-40x^3+50x^2\right)\Big|_0^2 = 28\end{aligned}$$

13. div $\mathbf{F} = 6xy^2 + 1 - 6xy^2 = 1$. Using cylindrical coordinates,

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_D dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} dz\,r\,dr\,d\theta = \int_0^{\pi}\int_0^{2\sin\theta}(2r\sin\theta - r^2)r\,dr\,d\theta \\ &= \int_0^{\pi}\left(\frac{2}{3}r^3\sin\theta - \frac{1}{4}r^4\right)\Big|_0^{2\sin\theta}\,d\theta = \int_0^{\pi}\left(\frac{16}{3}\sin^4\theta - 4\sin^4\theta\right)d\theta \\ &= \frac{4}{3}\int_0^{\pi}\sin^4\theta\,d\theta = \frac{4}{3}\left(\frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right)\Big|_0^{\pi} = \frac{\pi}{2}\end{aligned}$$

14. div $\mathbf{F} = y^2 + x^2$. Using spherical coordinates, we have $x^2+y^2 = \rho^2\sin^2\phi$ and $z = \rho\cos\phi$ or $\rho = z\sec\phi$. Then

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_D (x^2+y^2)\,dS = \int_0^{2\pi}\int_0^{\pi/4}\int_{2\sec\phi}^{4\sec\phi} \rho^2\sin^2\phi\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/4}\frac{1}{5}\rho^5\sin^3\phi\,\Big|_{2\sec\phi}^{4\sec\phi}\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{992}{5}\sec^5\phi\sin^3\phi\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/4}\frac{992}{5}\tan^3\phi\sec^2\phi\,d\phi\,d\theta = \frac{992}{5}\int_0^{2\pi}\frac{1}{4}\tan^4\phi\,\Big|_0^{\pi/4}\,d\theta = \frac{992}{5}\int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{496\pi}{5}.\end{aligned}$$

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15. **(a)** $\operatorname{div}\mathbf{E} = q\left[\dfrac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} + \dfrac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}} + \dfrac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}\right] = 0$

$$\iint_{S\cup S_a} (\mathbf{E}\cdot\mathbf{n})\,dS = \iiint_D \operatorname{div}\mathbf{E}\,dV = \iiint_D 0\,dV = 0$$

(b) From **(a)**, $\displaystyle\iint_S (\mathbf{E}\cdot\mathbf{n})\,dS + \iint_{S_a} (\mathbf{E}\cdot\mathbf{n})\,dS = 0$ and $\displaystyle\iint_S (\mathbf{E}\cdot\mathbf{n})\,dS = -\iint_{S_a} (\mathbf{E}\cdot\mathbf{n})\,dS$. On S_a, $|\mathbf{r}| = a$, $\mathbf{n} = -(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/a = -\mathbf{r}/a$ and $\mathbf{E}\cdot\mathbf{n} = (q\mathbf{r}/a^3)\cdot(-\mathbf{r}/a) = -qa^2/a^4 = -q/a^2$. Thus

$$\iint_S (\mathbf{E}\cdot\mathbf{n})\,dS = -\iint_{S_a}\left(-\frac{q}{a^2}\right)dS = \frac{q}{a^2}\iint_{S_a} dS = \frac{q}{a^2}\times(\text{area of } S_a) = \frac{q}{a^2}(4\pi a^2) = 4\pi q.$$

16. **(a)** By Gauss' Law $\displaystyle\iint (\mathbf{E}\cdot\mathbf{n})\,dS = \iiint_D 4\pi\rho\,dV$, and by the Divergence Theorem $\displaystyle\iint_S (\mathbf{E}\cdot\mathbf{n})\,dS = \iiint_D \operatorname{div}\mathbf{E}\,dV$. Thus $\displaystyle\iiint_D 4\pi\rho\,dV = \iiint_D \operatorname{div}\mathbf{E}\,dV$ and $\displaystyle\iiint_D (4\pi\rho - \operatorname{div}\mathbf{E})\,dV = 0$. Since this holds for all regions D, $4\pi\rho - \operatorname{div}\mathbf{E} = 0$ and $\operatorname{div}\mathbf{E} = 4\pi\rho$.

(b) Since $\mathbf{E}$ is irrotational, $\mathbf{E} = \nabla\phi$ and $\nabla^2\phi = \nabla\cdot\nabla\phi = \nabla\mathbf{E} = \operatorname{div}\mathbf{E} = 4\pi\rho$.

17. Since $\operatorname{div}\mathbf{a} = 0$, by the Divergence Theorem

$$\iint_S (\mathbf{a}\cdot\mathbf{n})\,dS = \iiint_D \operatorname{div}\mathbf{a}\,dV = \iiint_D 0\,dV = 0.$$

18. By the Divergence Theorem and Problem 30 in Section 9.7,

$$\iint_S (\operatorname{curl}\mathbf{F}\cdot\mathbf{n})\,dS = \iiint_D \operatorname{div}(\operatorname{curl}\mathbf{F})\,dV = \iiint_D 0\,dV = 0.$$

19. By the Divergence Theorem and Problem 27 in Section 9.7,

$$\begin{aligned}\iint_S (f\nabla g)\cdot\mathbf{n}\,dS &= \iiint_D \operatorname{div}(f\nabla g)\,dV = \iiint_D \nabla\cdot(f\nabla g)\,dV = \iiint_D [f(\nabla\cdot\nabla g) + \nabla g\cdot\nabla f]\,dV \\ &= \iiint_D (f\nabla^2 g + \nabla g\cdot\nabla f)\,dV.\end{aligned}$$

20. By the Divergence Theorem and Problems 25 and 27 in Section 8.7,

$$\begin{aligned}\iint_S (f\nabla g - g\nabla f)\cdot\mathbf{n}\,dS &= \iiint_D \operatorname{div}(f\nabla g - g\nabla f)\,dV = \iiint_D \nabla\cdot(f\nabla g - g\nabla f)\,dV \\ &= \iiint_D [f(\nabla\cdot\nabla g) + \nabla g\cdot\nabla f - g(\nabla\cdot\nabla f) - \nabla f\cdot\nabla g]\,dV \\ &= \iiint_D (f\nabla^2 g - g\nabla^2 f)\,dV.\end{aligned}$$

21. If $G(x,y,z)$ is a vector valued function then we define surface integrals and triple integrals of $\mathbf{G}$ component-wise. In this case, if $\mathbf{a}$ is a constant vector it is easily shown that

$$\iint_S \mathbf{a}\cdot\mathbf{G}\,dS = \mathbf{a}\cdot\iint_S \mathbf{G}\,dS \quad\text{and}\quad \iiint_D \mathbf{a}\cdot\mathbf{G}\,dV = \mathbf{a}\cdot\iiint_D \mathbf{G}\,dV.$$

Now let $\mathbf{F} = f\mathbf{a}$. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S (f\mathbf{a})\cdot\mathbf{n}\,dS = \iint_S \mathbf{a}\cdot(f\mathbf{n})\,dS$$

and, using Problem 27 in Section 8.7 and the fact that $\nabla\cdot\mathbf{a} = 0$, we have

$$\iiint_D \operatorname{div}\mathbf{F}\,dV = \iiint_D \nabla\cdot(f\mathbf{a})\,dV = \iiint_D [f(\nabla\cdot\mathbf{a}) + \mathbf{a}\cdot\nabla f]\,dV = \iiint_D \mathbf{a}\cdot\nabla f\,dV.$$

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By the Divergence Theorem,

$$\iint_S \mathbf{a}\cdot(f\mathbf{n})\,dS = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \operatorname{div}\mathbf{F}\,dV = \iiint_D \mathbf{a}\cdot\nabla f\,dV$$

and

$$\mathbf{a}\cdot\left(\iint_S f\mathbf{n}\,dS\right) = \mathbf{a}\cdot\left(\iiint_D \nabla f\,dV\right) \quad \text{or} \quad \mathbf{a}\cdot\left(\iint_S f\mathbf{n}\,dS - \iiint_D \nabla f\,dV\right) = 0.$$

Since $\mathbf{a}$ is arbitrary,

$$\iint_S f\mathbf{n}\,dS - \iiint_D \nabla f\,dV = 0 \quad \text{and} \quad \iint_S f\mathbf{n}\,dS = \iiint_D \nabla f\,dV.$$

22. $\mathbf{B}+\mathbf{W} = -\iint_S p\mathbf{n}\,dS + m\mathbf{g} = m\mathbf{g} - \iiint_D \nabla p\,dV = m\mathbf{g} - \iiint_D \rho\mathbf{g}\,dV$

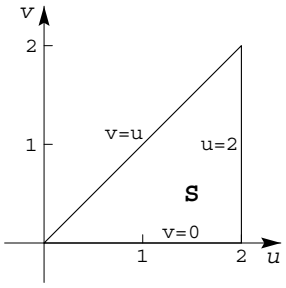
$= m\mathbf{g} - \left(\iiint_D \rho\,dV\right)\mathbf{g} = m\mathbf{g} - m\mathbf{g} = \mathbf{0}$

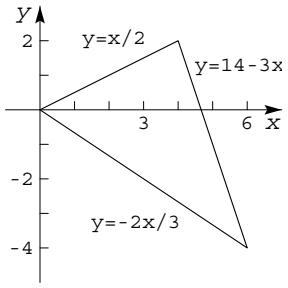
Exercises 9.17

1. T: $(0,0)\to(0,0)$; $(0,2)\to(-2,8)$; $(4,0)\to(16,20)$; $(4,2)\to(14,28)$

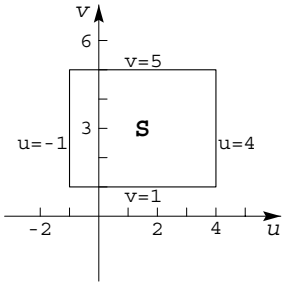
2. Writing $x^2 = v-u$ and $y = v+u$ and solving for u and v, we obtain $u = (y-x^2)/2$ and $v = (x^2+y)/2$. Then the images under T^{-1} are $(1,1)\to(0,1)$; $(1,3)\to(1,2)$; $(\sqrt{2},2)\to(0,2)$.

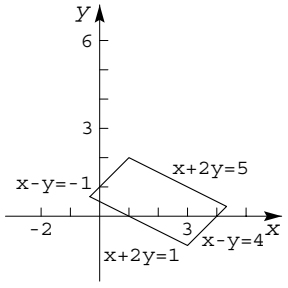
3. The uv-corner points $(0,0)$, $(2,0)$, $(2,2)$ correspond to xy-points $(0,0)$, $(4,2)$, $(6,-4)$.
$v=0$: $x=2u$, $y=u \implies y=x/2$
$u=2$: $x=4+v$, $y=2-3v \implies$
$y = 2-3(x-4) = -3x+14$
$v=u$: $x=3u$, $y=-2u \implies y=-2x/3$





4. Solving for x and y we see that the transformation is $x = 2u/3+v/3$, $y = -u/3+v/3$. The uv-corner points $(-1,1)$, $(4,1)$, $(4,5)$, $(-1,5)$ correspond to the xy-points $(-1/3, 2/3)$, $(3,-1)$, $(13/3, 1/3)$, $(1,2)$.
$v=1$: $x+2y=1$; $\quad v=5$: $x+2y=5$;
$u=-1$: $x-y=-1$; $\quad u=4$: $x-y=4$





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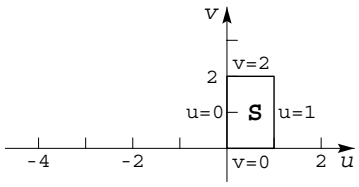
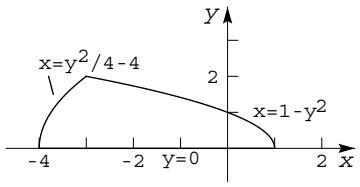
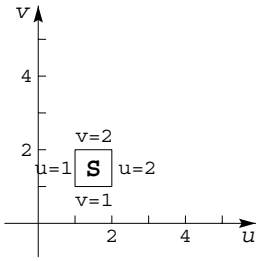
5. The uv-corner points $(0,0)$, $(1,0)$, $(1,2)$, $(0,2)$ correspond to the xy-points $(0,0)$, $(1,0)$, $(-3,2)$, $(-4,0)$.

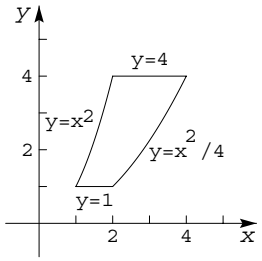




$v = 0$: $x = u^2$, $y = 0 \implies y = 0$ and $0 \le x \le 1$
$u = 1$: $x = 1 - v^2$, $y = v \implies x = 1 - y^2$
$v = 2$: $x = u^2 - 4$, $y = 2u \implies x = y^2/4 - 4$
$u = 0$: $x = -v^2$, $y = 0 \implies y = 0$ and $-4 \le x \le 0$

6. The uv-corner points $(1,1)$, $(2,1)$, $(2,2)$, $(1,2)$ correspond to the xy-points $(1,1)$, $(2,1)$, $(4,4)$, $(2,4)$.
$v = 1$: $x = u$, $y = 1 \implies y = 1$, $1 \le x \le 2$
$u = 2$: $x = 2v$, $y = v^2 \implies y = x^2/4$
$v = 2$: $x = 2u$, $y = 4 \implies y = 4$, $2 \le x \le 4$
$u = 1$: $x = v$, $y = v^2 \implies y = x^2$





7. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -ve^{-u} & e^{-u} \\ ve^{u} & e^{u} \end{vmatrix} = -2v$

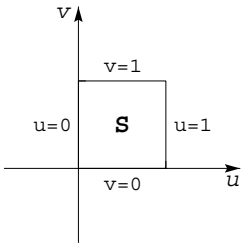
8. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3e^{3u}\sin v & e^{3u}\cos v \\ 3e^{3u}\cos v & -e^{3u}\sin v \end{vmatrix} = -3e^{6u}$

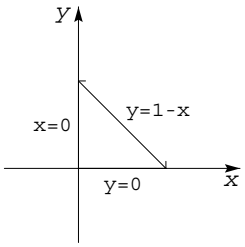
9. $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -2y/x^3 & 1/x^2 \\ -y^2/x^2 & 2y/x \end{vmatrix} = -\dfrac{3y^2}{x^4} = -3\left(\dfrac{y}{x^2}\right)^2 = -3u^2; \quad \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{-3u^2} = -\dfrac{1}{3u^2}$

10. $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{2(y^2-x^2)}{(x^2+y^2)^2} & \dfrac{-4xy}{(x^2+y^2)^2} \\ \dfrac{4xy}{(x^2+y^2)^2} & \dfrac{2(y^2-x^2)}{(x^2+y^2)^2} \end{vmatrix} = \dfrac{4}{(x^2+y^2)^2}$

From $u = 2x/(x^2+y^2)$ and $v = -2y(x^2+y^2)$ we obtain $u^2 + v^2 = 4/(x^2+y^2)$. Then $x^2 + y^2 = 4/(u^2+v^2)$ and $\partial(x,y)/\partial(u,v) = (x^2+y^2)^2/4 = 4/(u^2+v^2)^2$.

11. **(a)** The uv-corner points $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ correspond to the xy-points $(0,0)$, $(1,0)$, $(0,1)$, $(0,0)$.
$v = 0$: $x = u$, $y = 0 \implies y = 0$, $0 \le x \le 1$
$u = 1$: $x = 1 - v$, $y = v \implies y = 1 - x$
$v = 1$: $x = 0$, $y = u \implies x = 0$, $0 \le y \le 1$
$u = 0$: $x = 0$, $y = 0$





(b) Since the segment $u = 0$, $0 \le v \le 1$ in the uv-plane maps to the origin in the xy-plane, the transformation is not one-to-one.

12. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & v \\ -u & u \end{vmatrix} = u$. The transformation is 0 when u is 0, for $0 \le v \le 1$.

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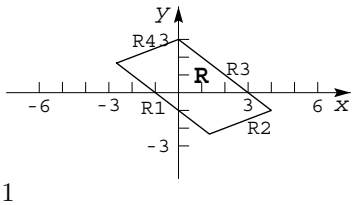
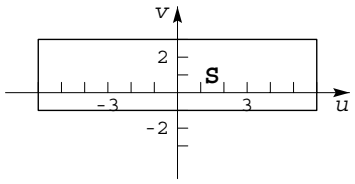
13. $R1$: $x+y=-1 \implies v=-1$
$R2$: $x-2y=6 \implies u=6$
$R3$: $x+y=3 \implies v=3$
$R4$: $x-2y=-6 \implies u=-6$

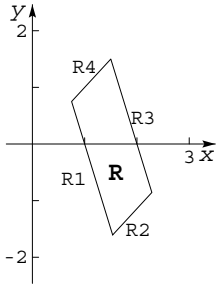


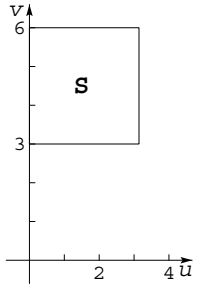


$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}1 & -2\\ 1 & 1\end{vmatrix}=3 \implies \frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{3}$$

$$\iint_R (x+y)\,dA=\iint_S v\left(\frac{1}{3}\right)dA'=\frac{1}{3}\int_{-1}^{3}\int_{-6}^{6} v\,du\,dv=\frac{1}{3}(12)\int_{-1}^{3} v\,dv=4\left(\frac{1}{2}\right)v^2\Big|_{-1}^{3}=16$$

14. $R1$: $y=-3x+3 \implies v=3$
$R2$: $y=x-\pi \implies u=\pi$
$R3$: $y=-3x+6 \implies v=6$
$R4$: $y=x \implies u=0$

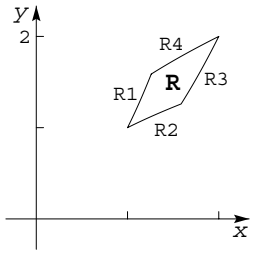


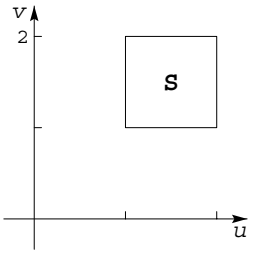


$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}1 & -1\\ 3 & 1\end{vmatrix}=4 \implies \frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{4}$$

$$\iint_R \frac{\cos\frac{1}{2}(x-y)}{3x+y}\,dA=\iint_S \frac{\cos u/2}{v}\left(\frac{1}{4}\right)dA'=\frac{1}{4}\int_3^6\int_0^\pi \frac{\cos u/2}{v}\,du\,dv=\frac{1}{4}\int_3^6 \frac{2\sin u/2}{v}\Big|_0^\pi\,dv$$
$$=\frac{1}{2}\int_3^6 \frac{dv}{v}=\frac{1}{2}\ln v\Big|_3^6=\frac{1}{2}\ln 2$$

15. $R1$: $y=x^2 \implies u=1$
$R2$: $x=y^2 \implies v=1$
$R3$: $y=\frac{1}{2}x^2 \implies u=2$
$R4$: $x=\frac{1}{2}y^2 \implies v=2$

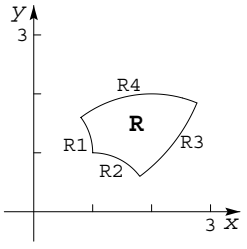


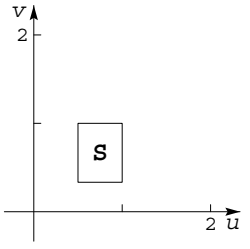


$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}2x/y & -x^2/y^2\\ -y^2/x^2 & 2y/x\end{vmatrix}=3 \implies \frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{3}$$

$$\iint_R \frac{y^2}{x}\,dA=\iint_S v\left(\frac{1}{3}\right)dA'=\frac{1}{3}\int_1^2\int_1^2 v\,du\,dv=\frac{1}{3}\int_1^2 v\,dv=\frac{1}{6}v^2\Big|_1^2=\frac{1}{2}$$

16. $R1$: $x^2+y^2=2y \implies v=1$
$R2$: $x^2+y^2=2x \implies u=1$
$R3$: $x^2+y^2=6y \implies v=1/3$
$R4$: $x^2+y^2=4x \implies u=1/2$





$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\dfrac{2(y^2-x^2)}{(x^2+y^2)^2} & \dfrac{-4xy}{(x^2+y^2)^2}\\[2ex] \dfrac{-4xy}{(x^2+y^2)^2} & \dfrac{2(x^2-y^2)}{(x^2+y^2)^2}\end{vmatrix}=\frac{-4}{(x^2+y^2)^2}$$

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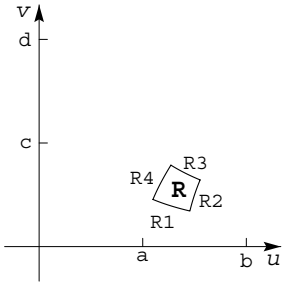
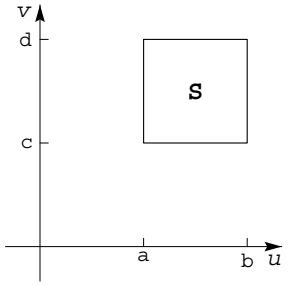
Using $u^2+v^2=4/(x^2+y^2)$ we see that $\partial(x,y)/\partial(u,v)=-4/(u^2+v^2)^2$.

$$\iint_R (x^2+y^2)^{-3}\,dA=\iint_S\left(\frac{4}{u^2+v^2}\right)^{-3}\left|\frac{-4}{(u^2+v^2)^2}\right|dA'=\frac{1}{16}\int_{1/3}^{1}\int_{1/2}^{1}(u^2+v^2)\,du\,dv=\frac{115}{5184}$$

17. $R1$: $2xy=c \implies v=c$
$R2$: $x^2-y^2=b \implies u=b$
$R3$: $2xy=d \implies v=d$
$R4$: $x^2-y^2=a \implies u=a$



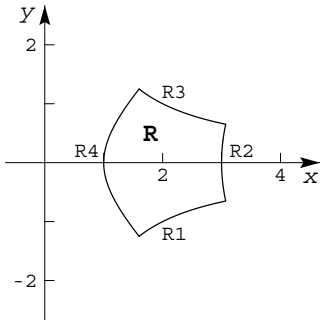


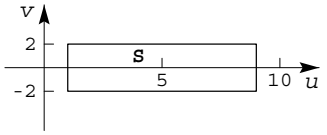
$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}2x & -2y\\ 2y & 2x\end{vmatrix}=4(x^2+y^2)$$

$$\implies \frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{4(x^2+y^2)}$$

$$\iint_R (x^2+y^2)\,dA=\iint_S (x^2+y^2)\frac{1}{4(x^2+y^2)}\,dA'=\frac{1}{4}\int_c^d\int_a^b du\,dv=\frac{1}{4}(b-a)(d-c)$$

18. $R1$: $xy=-2 \implies v=-2$
$R2$: $x^2-y^2=9 \implies u=9$
$R3$: $xy=2 \implies v=2$
$R4$: $x^2-y^2=1 \implies u=1$



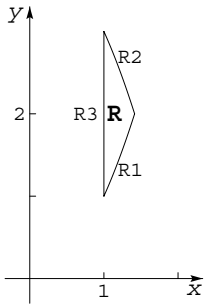


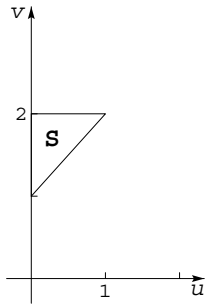
$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}2x & -2y\\ y & x\end{vmatrix}=2(x^2+y^2)$$

$$\implies \frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{2(x^2+y^2)}$$

$$\iint_R (x^2+y^2)\sin xy\,dA=\iint_S (x^2+y^2)\sin v\left(\frac{1}{2(x^2+y^2)}\right)dA'=\frac{1}{2}\int_{-2}^{2}\int_1^9 \sin v\,du\,dv=\frac{1}{2}\int_{-2}^{2}8\sin v\,dv=0$$

19. $R1$: $y=x^2 \implies v+u=v-u \implies u=0$
$R2:\ y=4-x^2 \implies v+u=4-(v-u)$
$\implies v+u=4-v+u \implies v=2$
$R3$: $x=1 \implies v-u=1 \implies v=1+u$





$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}-\dfrac{1}{2\sqrt{v-u}} & \dfrac{1}{2\sqrt{v-u}}\\ 1 & 1\end{vmatrix}=-\frac{1}{\sqrt{v-u}}$$

$$\iint_R \frac{x}{y+x^2}\,dA=\iint_S \frac{\sqrt{v-u}}{2v}\left|-\frac{1}{\sqrt{v-u}}\right|dA'=\frac{1}{2}\int_0^1\int_{1+u}^{2}\frac{1}{v}\,dv\,du=\frac{1}{2}\int_0^1[\ln 2-\ln(1+u)]\,du$$

$$=\frac{1}{2}\ln 2-\frac{1}{2}\left[(1+u)\ln(1+u)-(1+u)\right]\Big|_0^1=\frac{1}{2}\ln 2-\frac{1}{2}[2\ln 2-2-(0-1)]=\frac{1}{2}-\frac{1}{2}\ln 2$$

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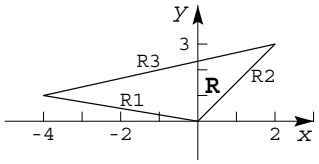
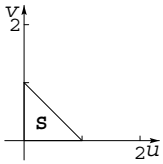
20. Solving $x = 2u - 4v$, $y = 3u + v$ for u and v we obtain $u = \dfrac{1}{14}x + \dfrac{2}{7}y$, $v = -\dfrac{3}{14}x + \dfrac{1}{7}y$. The xy-corner points $(-4, 1)$, $(0, 0)$, $(2, 3)$ correspond to the uv-points $(0, 1)$, $(0, 0)$, $(1, 0)$.



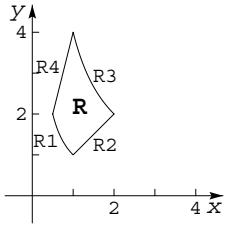


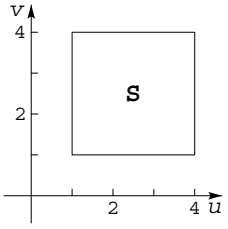
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix} = 14$$

$$\iint_R y\,dA = \iint_S (3u+v)(14)\,dA' = 14\int_0^1 \int_0^{1-u} (3u+v)\,dv\,du = 14\int_0^1 \left(3uv + \frac{1}{2}v^2\right)\Big|_0^{1-u} du$$

$$= 14\int_0^1 \left(\frac{1}{2} + 2u - \frac{5}{2}u^2\right) du = \left(7u + 14u^2 - \frac{35}{3}u^3\right)\Big|_0^1 = \frac{28}{3}$$

21. $R1$: $y = 1/x \implies u = 1$
$R2$: $y = x \implies v = 1$
$R3$: $y = 4/x \implies u = 4$
$R4$: $y = 4x \implies v = 4$





$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = \frac{2y}{x} \implies \frac{\partial(x,y)}{\partial(u,v)} = \frac{x}{2y}$$

$$8\iint_R y^4\,dA = \iint_S u^2v^2\left(\frac{1}{2v}\right) du\,dv = \frac{1}{2}\int_1^4 u^2v\,du\,dv = \frac{1}{2}\int_1^4 \frac{1}{3}u^3v\,\Big|_1^4\,dv = \frac{1}{6}\int_1^4 63v\,dv = \frac{21}{4}v^2\,\Big|_1^4 = \frac{315}{4}$$

22. Under the transformation $u = y+z$, $v = -y+z$, $w = x-y$ the parallelepiped D is mapped to the parallelepiped E: $1 \le u \le 3$, $-1 \le v \le 1$, $0 \le w \le 3$.

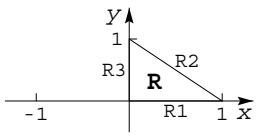
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 2 \implies \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{2}$$

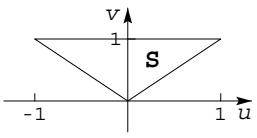
$$\iint_D (4z + 2x - 2y)\,dV = \iint_E (2u+2v+2w)\frac{1}{2}\,dV' = \frac{1}{2}\int_0^3 \int_{-1}^1 \int_1^3 (2u+2v+2w)\,du\,dv\,dw$$

$$= \frac{1}{2}\int_0^3 \int_{-1}^1 (u^2 + 2uv + 2uw)\,\Big|_1^3\,dv\,dw = \frac{1}{2}\int_0^3 \int_{-1}^1 (8 + 4v + 4w)\,dv\,dw$$

$$= \int_0^3 (4v + v^2 + 2vw)\,\Big|_{-1}^1\,dw = \int_0^3 (8 + 4w)\,dw = (8w + 2w^2)\,\Big|_0^3 = 42$$

23. We let $u = y - x$ and $v = y + x$.
$R1$: $y = 0 \implies u = -x$, $v = x \implies v = -u$
$R2$: $x + y = 1 \implies v = 1$
$R3$: $x = 0 \implies u = y$, $v = y \implies v = u$





$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2 \implies \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

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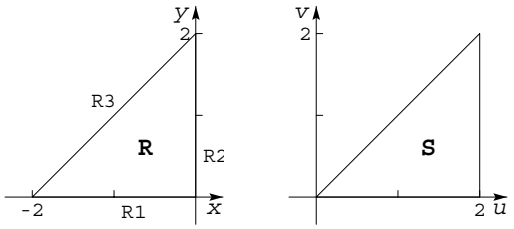
$$\iint_R e^{(y-x)/(y+x)}dA = \iint_S e^{u/v}\left|-\frac{1}{2}\right| dA' = \frac{1}{2}\int_0^1\int_{-v}^{v} e^{u/v}\,du\,dv = \frac{1}{2}\int_0^1 ve^{u/v}\,\Big|_{-v}^{v}\,dv$$
$$= \frac{1}{2}\int_0^1 v(e-e^{-1})\,dv = \frac{1}{2}(e-e^{-1})\frac{1}{2}v^2\,\Big|_0^1 = \frac{1}{4}(e-e^{-1})$$

24. We let $u = y - x$ and $v = y$.



$R1$: $y = 0 \implies v = 0$, $u = -x \implies v = 0$, $0 \le u \le 2$

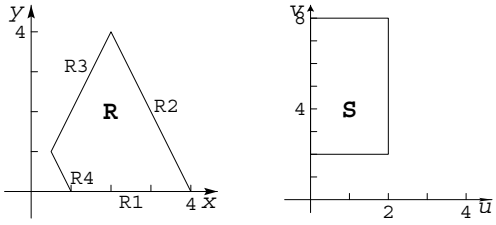
$R2$: $x = 0 \implies v = u$

$R3$: $y = x + 2 \implies u = 2$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \implies \frac{\partial(x,y)}{\partial(u,v)} = -1$$

$$\iint_R e^{y^2-2xy+x^2}dA = \iint_S e^{u^2}|-1|\,dA' = \int_0^2\int_0^u e^{u^2}\,dv\,du = \int_0^2 ue^{u^2}\,du = \frac{1}{2}e^{u^2}\,\Big|_0^2 = \frac{1}{2}(e^4-1)$$

25. Noting that $R2$, $R3$, and $R4$ have equations $y+2x = 8$, $y-2x = 0$, and $y + 2x = 2$, we let $u = y/x$ and $v = y + 2x$.



$R1$: $y = 0 \implies u = 0$, $v = 2x \implies u = 0$, $2 \le v \le 8$

$R2$: $y + 2x = 8 \implies v = 8$

$R3$: $y - 2x = 0 \implies u = 2$

$R4$: $y + 2x = 2 \implies v = 2$

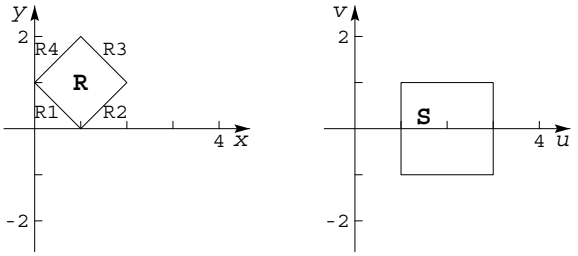
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -y/x^2 & 1/x \\ 2 & 1 \end{vmatrix} = -\frac{y+2x}{x^2} \implies \frac{\partial(x,y)}{\partial(u,v)} = -\frac{x^2}{y+2x}$$

$$\iint_R (6x+3y)\,dA = 3\iint_S (y+2x)\left|-\frac{x^2}{y+2x}\right| dA' = 3\iint_S x^2\,dA'$$

From $y = ux$ we see that $v = ux + 2x$ and $x = v/(u+2)$. Then

$$3\iint_S x^2\,dA' = 3\int_0^2\int_2^8 v^2(u+2)^2\,dv\,du = \int_0^2 \frac{v^3}{(u+2)^2}\,\Big|_2^8\,du = 504\int_0^2 \frac{du}{(u+2)^2} = -\frac{504}{u+2}\,\Big|_0^2 = 126.$$

26. We let $u = x + y$ and $v = x - y$.



$R1$: $x + y = 1 \implies u = 1$

$R2$: $x - y = 1 \implies v = 1$

$R3$: $x + y = 3 \implies u = 3$

$R4$: $x - y = -1 \implies v = -1$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \implies \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

$$\iint_R (x+y)^4e^{x-y}\,dA = \iint_S u^4e^v\left|-\frac{1}{2}\right| dA' = \frac{1}{2}\int_1^3\int_{-1}^1 u^4e^v\,dv\,du = \frac{1}{2}\int_1^3 u^4e^v\,\Big|_{-1}^1\,du$$
$$= \frac{e-e^{-1}}{2}\int_1^3 u^4\,du = \frac{e-e^{-1}}{10}\,u^5\,\Big|_1^3 = \frac{242(e-e^{-1})}{10} = \frac{121}{5}(e-e^{-1})$$

27. Let $u = xy$ and $v = xy^{1.4}$. Then $xy^{1.4} = c \implies v = c$; $xy = b \implies u = b$; $xy^{1.4} = d \implies v = d$; $xy = a \implies u = a$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ y^{1.4} & 1.4xy^{0.4} \end{vmatrix} = 0.4xy^{1.4} = 0.4v \implies \frac{\partial(x,y)}{\partial(u,v)} = \frac{5}{2v}$$

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$$\iint_R dA = \iint_S \frac{5}{2v}\, dA' = \int_c^d \int_a^b \frac{5}{2v}\, du\, dv = \frac{5}{2}(b-a)\int_c^d \frac{dv}{v} = \frac{5}{2}(b-a)(\ln d - \ln c)$$

28. The image of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ under the transformation $u = x/a$, $v = y/b$, $w = z/c$, is the unit sphere $u^2 + v^2 + w^2 = 1$. The volume of this sphere is $\frac{4}{3}\pi$. Now

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

and

$$\iiint_D dV = \iiint_E abc\, dV' = abc \iiint_E dV' = abc\left(\frac{4}{3}\pi\right) = \frac{4}{3}\pi abc.$$

29. The image of the ellipse is the unit circle $x^2 + y^2 = 1$. From $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} = 15$ we obtain

$$\iint_R \left(\frac{x^2}{25} + \frac{y^2}{9}\right) dA = \iint_S (u^2+v^2)15\, dA' = 15\int_0^{2\pi}\int_0^1 r^2 r\, dr\, d\theta = \frac{15}{4}\int_0^{2\pi} r^4 \Big|_0^1 d\theta$$
$$= \frac{15}{4}\int_0^{2\pi} d\theta = \frac{15\pi}{2}.$$

30. $$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \cos\phi(\rho^2\sin\phi\cos\phi\cos^2\theta + \rho^2\sin\phi\cos\phi\sin^2\theta) + \rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta)$$
$$= \rho^2\sin\phi\cos^2\phi(\cos^2\theta + \sin^2\theta) + \rho^2\sin^3\phi(\cos^2\theta + \sin^2\theta) = \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) = \rho^2\sin\phi$$

Chapter 9 Review Exercises

1. True; $|\mathbf{v}(t)| = \sqrt{2}$

2. True; for all t, $y = 4$.

3. True

4. False; consider $\mathbf{r}(t) = t^2\mathbf{i}$. In this case, $\mathbf{v}(t) = 2t\mathbf{i}$ and $\mathbf{a}(t) = 2\mathbf{i}$. Since $\mathbf{v}\cdot\mathbf{a} = 4t$, the velocity and acceleration vectors are not orthogonal for $t \neq 0$.

5. False; ∇f is perpendicular to the level curve $f(x,y) = c$.

6. False; consider $f(x,y) = xy$ at $(0,0)$.

7. True; the value is $4/3$.

8. True; since $2xy\,dx - x^2\,dy$ is not exact.

9. False; $\displaystyle\int_C x\,dx + x^2\,dy = 0$ from $(-1,0)$ to $(1,0)$ along the x-axis and along the semicircle $y = \sqrt{1-x^2}$, but since $x\,dx + x^2\,dy$ is not exact, the integral is not independent of path.

10. True

11. False; unless the first partial derivatives are continuous.

12. True

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13. True

14. True; since curl $\mathbf{F} = \mathbf{0}$ when $\mathbf{F}$ is a conservative vector field.

15. True

16. True

17. True

18. True

19. $\mathbf{F} = \nabla\phi = -x(x^2+y^2)^{-3/2}\mathbf{i} - y(x^2+y^2)^{-3/2}\mathbf{j}$

20. curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = \mathbf{0}$

21. $\mathbf{v}(t) = 6\mathbf{i} + \mathbf{j} + 2t\mathbf{k}$; $\mathbf{a}(t) = 2\mathbf{k}$. To find when the particle passes through the plane, we solve $-6t + t + t^2 = -4$ or $t^2 - 5t + 4 = 0$. This gives $t = 1$ and $t = 4$. $\mathbf{v}(1) = 6\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{a}(1) = 2\mathbf{k}$; $\mathbf{v}(4) = 6\mathbf{i} + \mathbf{j} + 8\mathbf{k}$, $\mathbf{a}(4) = 2\mathbf{k}$

22. We are given $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (-10t\mathbf{i} + (3t^2 - 4t)\mathbf{j} + \mathbf{k})\,dt = -5t^2\mathbf{i} + (t^3 - 2t^2)\mathbf{j} + t\mathbf{k} + \mathbf{c}$$

$$\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \mathbf{r}(0) = \mathbf{c}$$

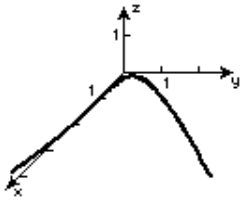
$$\mathbf{r}(t) = (1 - 5t^2)\mathbf{i} + (t^3 - 2t^2 + 2)\mathbf{j} + (t + 3)\mathbf{k}$$

$$\mathbf{r}(2) = -19\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$$

23. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (\sqrt{2}\sin t\mathbf{i} + \sqrt{2}\cos t\mathbf{j})\,dt = -\sqrt{2}\cos t\mathbf{i} + \sqrt{2}\sin t\mathbf{j} + \mathbf{c}$;
$-\mathbf{i} + \mathbf{j} + \mathbf{k} = \mathbf{v}(\pi/4) = -\mathbf{i} + \mathbf{j} + \mathbf{c}$, $\mathbf{c} = \mathbf{k}$; $\mathbf{v}(t) = -\sqrt{2}\cos t\mathbf{i} + \sqrt{2}\sin t\mathbf{j} + \mathbf{k}$;
$\mathbf{r}(t) = -\sqrt{2}\sin t\mathbf{i} - \sqrt{2}\cos t\mathbf{j} + t\mathbf{k} + \mathbf{b}$; $\mathbf{i} + 2\mathbf{j} + (\pi/4)\mathbf{k} = \mathbf{r}(\pi/4) = -\mathbf{i} - \mathbf{j} + (\pi/4)\mathbf{k} + \mathbf{b}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$;
$\mathbf{r}(t) = (2 - \sqrt{2}\sin t)\mathbf{i} + (3 - \sqrt{2}\cos t)\mathbf{j} + t\mathbf{k}$; $\mathbf{r}(3\pi/4) = \mathbf{i} + 4\mathbf{j} + (3\pi/4)\mathbf{k}$

24. $\mathbf{v}(t) = t\mathbf{i} + t^2\mathbf{j} - t\mathbf{k}$; $|\mathbf{v}| = t\sqrt{t^2+2}$, $t > 0$; $\mathbf{a}(t) = \mathbf{i} + 2t\mathbf{j} - \mathbf{k}$; $\mathbf{v}\cdot\mathbf{a} = t + 2t^3 + t = 2t + 2t^3$;
$\mathbf{v}\times\mathbf{a} = t^2\mathbf{i} + t^2\mathbf{k}$, $|\mathbf{v}\times\mathbf{a}| = t^2\sqrt{2}$; $a_T = \dfrac{2t+2t^3}{t\sqrt{t^2+2}} = \dfrac{2+2t^2}{\sqrt{t^2+2}}$, $a_N = \dfrac{t^2\sqrt{2}}{t\sqrt{t^2+2}} = \dfrac{\sqrt{2}\,t}{\sqrt{t^2+2}}$;
$\kappa = \dfrac{t^2\sqrt{2}}{t^3(t^2+2)^{3/2}} = \dfrac{\sqrt{2}}{t(t^2+2)^{3/2}}$

25.



26. $\mathbf{r}'(t) = \sinh t\mathbf{i} + \cosh t\mathbf{j} + \mathbf{k}$, $\mathbf{r}'(1) = \sinh 1\mathbf{i} + \cosh 1\mathbf{j} + \mathbf{k}$;
$|\mathbf{r}'(t)| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t$; $|\mathbf{r}'(1)| = \sqrt{2}\cosh 1$;
$\mathbf{T}(t) = \dfrac{1}{\sqrt{2}}\tanh t\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j} + \dfrac{1}{\sqrt{2}}\operatorname{sech} t\mathbf{k}$, $\mathbf{T}(1) = \dfrac{1}{\sqrt{2}}(\tanh 1\mathbf{i} + \mathbf{j} + \operatorname{sech} 1\mathbf{k})$;
$\dfrac{d\mathbf{T}}{dt} = \dfrac{1}{\sqrt{2}}\operatorname{sech}^2 t\mathbf{i} - \dfrac{1}{\sqrt{2}}\operatorname{sech} t\tanh t\mathbf{k}$; $\dfrac{d}{dt}\mathbf{T}(1) = \dfrac{1}{\sqrt{2}}\operatorname{sech}^2 1\mathbf{i} - \dfrac{1}{\sqrt{2}}\operatorname{sech} 1\tanh 1\mathbf{k}$,
$\left|\dfrac{d}{dt}\mathbf{T}(1)\right| = \dfrac{\operatorname{sech} 1}{\sqrt{2}}\sqrt{\operatorname{sech}^2 1 + \tanh^2 1} = \dfrac{1}{\sqrt{2}}\operatorname{sech} 1$; $\mathbf{N}(1) = \operatorname{sech} 1\mathbf{i} - \tanh 1\mathbf{k}$;

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$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = -\frac{1}{\sqrt{2}} \tanh 1\mathbf{i} + \frac{1}{\sqrt{2}}(\tanh^2 1 + \operatorname{sech}^2 1)\mathbf{j} - \frac{1}{\sqrt{2}} \operatorname{sech} 1\mathbf{k}$$

$$= \frac{1}{\sqrt{2}}(-\tanh 1\mathbf{i} + \mathbf{j} - \operatorname{sech} 1\mathbf{k})$$

$$\kappa = \left| \frac{d}{dt}\mathbf{T}(1) \right| / |\mathbf{r}'(1)| = \frac{(\operatorname{sech} 1)/\sqrt{2}}{\sqrt{2}\cosh 1} = \frac{1}{2}\operatorname{sech}^2 1$$

27. $\nabla f = (2xy - y^2)\mathbf{i} + (x^2 - 2xy)\mathbf{j}$; $\mathbf{u} = \dfrac{2}{\sqrt{40}}\,\mathbf{i} + \dfrac{6}{\sqrt{40}}\,\mathbf{j} = \dfrac{1}{\sqrt{10}}\,(\mathbf{i} + 3\mathbf{j})$;

$$D_{\mathbf{u}} f = \frac{1}{\sqrt{10}}\,(2xy - y^2 + 3x^2 - 6xy) = \frac{1}{\sqrt{10}}\,(3x^2 - 4xy - y^2)$$

28. $\nabla F = \dfrac{2x}{x^2+y^2+z^2}\,\mathbf{i} + \dfrac{2y}{x^2+y^2+z^2}\,\mathbf{j} + \dfrac{2z}{x^2+y^2+z^2}\,\mathbf{k}$; $\mathbf{u} = -\dfrac{2}{3}\mathbf{i} + \dfrac{1}{3}\,\mathbf{j} + \dfrac{2}{3}\,\mathbf{k}$; $D_{\mathbf{u}}F = \dfrac{-4x + 2y + 4z}{3(x^2+y^2+z^2)}$

29. $f_x = 2xy^4$, $f_y = 4x^2y^3$.

(a) $\mathbf{u} = \mathbf{i}$, $D_{\mathbf{u}}(1,1) = f_x(1,1) = 2$

(b) $\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$, $D_{\mathbf{u}}(1,1) = (2-4)/\sqrt{2} = -2/\sqrt{2}$

(c) $\mathbf{u} = \mathbf{j}$, $D_{\mathbf{u}}(1,1) = f_y(1,1) = 4$

30. (a) $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\,6\cos 2t + \frac{y}{\sqrt{x^2+y^2+z^2}}\,(-8\sin 2t) + \frac{z}{\sqrt{x^2+y^2+z^2}}\,15t^2$$

$$= \frac{(6x\cos 2t - 8y\sin 2t + 15zt^2)}{\sqrt{x^2+y^2+z^2}}$$

(b) $$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\,\frac{6}{r}\cos\frac{2t}{r} + \frac{y}{\sqrt{x^2+y^2+z^2}}\left(\frac{8r}{t^2}\sin\frac{2r}{t}\right) + \frac{z}{\sqrt{x^2+y^2+z^2}}\,15t^2r^3$$

$$= \frac{\left(\dfrac{6x}{r}\cos\dfrac{2t}{r} + \dfrac{8yr}{t^2}\,\sin\dfrac{2r}{t} + 15zt^2r^3\right)}{\sqrt{x^2+y^2+z^2}}$$

31. $F(x,y,z) = \sin xy - z$; $\nabla F = y\cos xy\mathbf{i} + x\cos xy\mathbf{j} - \mathbf{k}$; $\nabla F(1/2, 2\pi/3, \sqrt{3}/2) = \dfrac{\pi}{3}\,\mathbf{i} + \dfrac{1}{4}\mathbf{j} - \mathbf{k}$.

The equation of the tangent plane is $\dfrac{\pi}{3}\left(x - \dfrac{1}{2}\right) + \dfrac{1}{4}\left(y - \dfrac{2\pi}{3}\right) - \left(z - \dfrac{\sqrt{3}}{2}\right) = 0$ or $4\pi x + 3y - 12z = 4\pi - 6\sqrt{3}$.

32. We want to find a normal to the surface that is parallel to $\mathbf{k}$. $\nabla F = (y-2)\mathbf{i} + (x-2y)\mathbf{j} + 2z\mathbf{k}$. We need $y - 2 = 0$ and $x - 2y = 0$. The tangent plane is parallel to $z = 2$ when $y = 2$ and $x = 4$. In this case $z^2 = 5$. The points are $(4, 2, \sqrt{5}\,)$ and $(4, 2, -\sqrt{5}\,)$.

33. (a) $$V = \int_0^1 \int_x^{2x} \sqrt{1-x^2}\,dy\,dx = \int_0^1 y\sqrt{1-x^2}\;\Big|_x^{2x} dx = \int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{3}(1-x^2)^{3/2}\;\Big|_0^1 = \frac{1}{3}$$

(b) $$V = \int_0^1 \int_{y/2}^{y} \sqrt{1-x^2}\,dx\,dy + \int_1^2 \int_{y/2}^{1} \sqrt{1-x^2}\,dx\,dy$$

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34. We are given $\rho = k(x^2 + y^2)$.

$$m = \int_0^1 \int_{x^3}^{x^2} k(x^2+y^2)\,dy\,dx = k\int_0^1 \left(x^2y + \frac{1}{3}y^3\right)\Big|_{x^3}^{x^2} dx$$
$$= k\int_0^1 \left(x^4 + \frac{1}{3}x^6 - x^5 - \frac{1}{3}x^9\right)dx = k\left(\frac{1}{5}x^5 + \frac{1}{21}x^7 - \frac{1}{6}x^6 - \frac{1}{30}x^{10}\right)\Big|_0^1 = \frac{k}{21}$$

$$M_y = \int_0^1 \int_{x^3}^{x^2} k(x^3+xy^2)\,dy\,dx = k\int_0^1 \left(x^3y + \frac{1}{3}xy^3\right)\Big|_{x^3}^{x^2} dx = k\int_0^1 \left(x^5 + \frac{1}{3}x^7 - x^6 - \frac{1}{3}x^{10}\right)dx$$
$$= k\left(\frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{7}x^7 - \frac{1}{33}x^{11}\right)\Big|_0^1 = \frac{65k}{1848}$$

$$M_x = \int_0^1 \int_{x^3}^{x^2} k(x^2y+y^3)\,dy\,dx = k\int_0^1 \left(\frac{1}{2}x^2y^2 + \frac{1}{4}y^4\right)\Big|_{x^3}^{x^2} dx = k\int_0^1 \left(\frac{1}{2}x^6 + \frac{1}{4}x^8 - \frac{1}{2}x^8 - \frac{1}{4}x^{12}\right)dx$$
$$= k\left(\frac{1}{14}x^7 - \frac{1}{36}x^9 - \frac{1}{52}x^{13}\right)\Big|_0^1 = \frac{20k}{819}$$

$\bar{x} = M_y/m = \dfrac{65k/1848}{k/21} = 65/88$; $\bar{y} = M_x/m = \dfrac{20k/819}{k/21} = 20/39$ The center of mass is $(65/88, 20/39)$.

35. $$I_y = \int_0^1 \int_{x^3}^{x^2} k(x^4+x^2y^2)\,dy\,dx = k\int_0^1 \left(x^4y + \frac{1}{3}x^2y^3\right)\Big|_{x^3}^{x^2} dx = k\int_0^1 \left(x^6 + \frac{1}{3}x^8 - x^7 - \frac{1}{3}x^{11}\right)dx$$
$$= k\left(\frac{1}{7}x^7 + \frac{1}{27}x^9 - \frac{1}{8}x^8 - \frac{1}{36}x^{12}\right)\Big|_0^1 = \frac{41}{1512}k$$

36. **(a)** Using symmetry,

$$V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

Trig substitution

$$= 8\int_0^a \left(\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]\Big|_0^{\sqrt{a^2-x^2}} dx = 8\int_0^a \frac{\pi}{2}\,\frac{a^2-x^2}{2}\,dx$$
$$= 2\pi\left(a^2x - \frac{1}{3}x^3\right)\Big|_0^a = \frac{4}{3}\pi a^3$$

(b) Using symmetry,

$$V = 2\int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta = 2\int_0^{2\pi} \int_0^a r\sqrt{a^2-r^2}\,dr\,d\theta$$
$$= 2\int_0^{2\pi} -\frac{1}{3}(a^2-r^2)^{3/2}\Big|_0^a\,d\theta = \frac{2}{3}\int_0^{2\pi} a^3\,d\theta = \frac{4}{3}\pi a^3$$

(c) $$V = \int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi} \int_0^{\pi} \frac{1}{3}\rho^3\sin\phi\,\Big|_0^a\,d\phi\,d\theta$$
$$= \frac{1}{3}\int_0^{2\pi} \int_0^{\pi} a^3\sin\phi\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi} -a^3\cos\phi\,\Big|_0^{\pi}\,d\theta = \frac{1}{3}\int_0^{2\pi} 2a^3\,d\theta = \frac{4}{3}\pi a^3$$

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37. We use spherical coordinates.

$$V = \int_0^{2\pi}\int_{\tan^{-1}1/3}^{\pi/4}\int_0^{3\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_{\tan^{-1}1/3}^{\pi/4} \frac{1}{3}\rho^3\sin\phi\,\Big|_0^{3\sec\phi}\,d\phi\,d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_{\tan^{-1}1/3}^{\pi/4} 27\sec^3\phi\sin\phi\,d\phi\,d\theta = 9\int_0^{2\pi}\int_{\tan^{-1}1/3}^{\pi/4}\tan\phi\sec^2\phi\,d\phi\,d\theta$$

$$= 9\int_0^{2\pi}\frac{1}{2}\tan^2\phi\,\Big|_{\tan^{-1}1/3}^{\pi/4}\,d\theta = \frac{9}{2}\int_0^{2\pi}\left(1-\frac{1}{9}\right)d\theta = 8\pi$$

38. $V = \displaystyle\int_0^{2\pi}\int_0^{\pi/6}\int_1^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/6}\frac{1}{3}\rho^3\sin\phi\,\Big|_1^2\,d\phi\,d\theta$

$$= \int_0^{2\pi}\int_0^{\pi/6}\left(\frac{8}{3}\sin\phi - \frac{1}{3}\sin\phi\right)d\phi\,d\theta = \frac{7}{3}\int_0^{2\pi}\int_0^{\pi/6}\sin\phi\,d\phi\,d\theta = \frac{7}{3}\int_0^{2\pi} -\cos\phi\,\Big|_0^{\pi/6}\,d\theta$$

$$= \frac{7}{3}\int_0^{2\pi}\left[-\frac{\sqrt{3}}{2}-(-1)\right]d\theta = \frac{7}{3}\left(1-\frac{\sqrt{3}}{2}\right)2\pi = \frac{7\pi}{3}(2-\sqrt{3}\,)$$

39. $2xy + 2xy + 2xy = 6xy$

40. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & xy^2 & 2xyz \end{vmatrix} = 2xz\mathbf{i} - 2yz\mathbf{j} + (y^2-x^2)\mathbf{k}$

41. $\dfrac{\partial}{\partial x}(2xz) - \dfrac{\partial}{\partial y}(2yz) + \dfrac{\partial}{\partial z}(y^2-x^2) = 0$

42. $\nabla(6xy) = 6y\mathbf{i} + 6x\mathbf{j}$

43. $\displaystyle\int_C \frac{z^2}{x^2+y^2}\,ds = \int_\pi^{2\pi}\frac{4t^2}{\cos^2 2t+\sin^2 2t}\,\sqrt{4\sin^2 2t + 4\cos^2 2t + 4}\,dt = \int_\pi^{2\pi} 8\sqrt{2}\,t^2\,dt = \frac{8\sqrt{2}}{3}\,t^3\,\Big|_\pi^{2\pi} = \frac{56\sqrt{2}\,\pi^3}{3}$

44. $\displaystyle\int_C (xy+4x)\,ds = \int_1^0 [x(2-2x)+4x]\sqrt{1+4}\,dx = \sqrt{5}\int_1^0 (6x-2x^2)\,dx = \sqrt{5}\left(3x^2 - \frac{2}{3}x^3\right)\Big|_1^0 = -\frac{7\sqrt{5}}{3}$

45. Since $P_y = 6x^2y = Q_x$, the integral is independent of path.

$\phi_x = 3x^2y^2,\ \ \phi = x^3y^2 + g(y),\ \ \phi_y = 2x^3y + g'(y) = 2x^3y - 3y^2;\ \ g(y) = -y^3;\ \ \phi = x^3y^2 - y^3;$

$$\int_{(0,0)}^{(1,-2)} 3x^2y^2\,dx + (2x^3y - 3y^2)\,dy = (x^3y^2 - y^3)\,\Big|_{(0,0)}^{(1,-2)} = 12$$

46. Let $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$. Then using $dx = -a\sin t\,dt$, $dy = a\cos t\,dt$, $x^2+y^2 = a^2$ we have

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi}\frac{1}{a^2}[-a\sin t(-a\sin t) + a\cos t(a\cos t)]\,dt = \int_0^{2\pi}(\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} dt = 2\pi.$$

47. $\displaystyle\int_C y\sin\pi z\,dx + x^2e^y\,dy + 3xyz\,dz$

$$= \int_0^1 [t^2\sin\pi t^3 + t^2e^{t^2}(2t) + 3tt^2t^3(3t^2)]\,dt = \int_0^1 (t^2\sin\pi t^3 + 2t^3e^{t^2} + 9t^8)\,dt$$

$$= \left(-\frac{1}{3\pi}\cos\pi t^3 + t^9\right)\Big|_0^1 + 2\int_0^1 t^3e^{t^2}\,dt \qquad \boxed{\text{Integration by parts}}$$

$$= \frac{2}{3\pi} + 1 + (t^2e^{t^2} - e^{t^2})\,\Big|_0^1 = \frac{2}{3\pi} + 2$$

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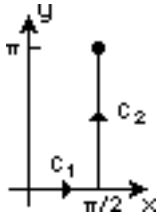
48. Parameterize C by $x = \cos t$, $y = \sin t$; $0 \le t \le 2\pi$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} [4 \sin t(-\sin t\, dt) + 6 \cos t(\cos t)\, dt] = \int_0^{2\pi} (6\cos^2 t - 4\sin^2 t)\, dt$$
$$= \int_0^{2\pi} (10\cos^2 t - 4)\, dt = \left(5t + \frac{5}{2}\sin 2t - 4t\right)\Big|_0^{2\pi} = 2\pi.$$

Using Green's Theorem, $Q_x - P_y = 6 - 4 = 2$ and $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R 2\, dA = 2(\pi \cdot 1^2) = 2\pi$.

49. Let $\mathbf{r}_1 = \dfrac{\pi}{2}\, t\mathbf{i}$ and $\mathbf{r}_2 = \dfrac{\pi}{2}\mathbf{i} + \pi t\mathbf{j}$ for $0 \le t \le 1$. Then $d\mathbf{r}_1 = \dfrac{\pi}{2}\,\mathbf{i}$, $d\mathbf{r}_2 = \pi\mathbf{j}$, $\mathbf{F}_1 = \mathbf{0}$,



$$\mathbf{F}_2 = \frac{\pi}{2}\sin \pi t\mathbf{i} + \pi t \sin\frac{\pi}{2}\mathbf{j} = \frac{\pi}{2}\sin \pi t\mathbf{i} + \pi t\mathbf{j},$$

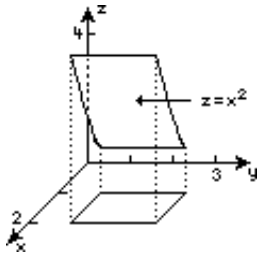
and

$$W = \int_{C_1} \mathbf{F}_1 \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F}_2 \cdot d\mathbf{r}_2 = \int_0^1 \pi^2 t\, dt = \frac{1}{2}\pi^2 t^2 \Big|_0^1 = \frac{\pi^2}{2}\,.$$

50. Parameterize the line segment from $(-1/2, 1/2)$ to $(-1, 1)$ using $y = -x$ as x goes from $-1/2$ to -1. Parameterize the line segment from $(-1, 1)$ to $(1, 1)$ using $y = 1$ as x goes from -1 to 1. Parameterize the line segment from $(1, 1)$ to $(1, \sqrt{3}\,)$ using $x = 1$ as y goes from 1 to $\sqrt{3}$. Then

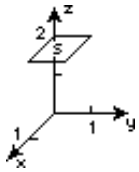
$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-1/2}^{-1} \mathbf{F} \cdot (dx\mathbf{i} - dx\mathbf{j}) + \int_{-1}^{1} \mathbf{F} \cdot (dx\mathbf{i}) + \int_1^{\sqrt{3}} \mathbf{F} \cdot (dy\mathbf{j})$$
$$= \int_{-1/2}^{-1} \left(\frac{2}{x^2 + (-x)^2} - \frac{1}{x^2 + (-x)^2}\right) dx + \int_{-1}^{1} \frac{2}{x^2 + 1}\, dx + \int_1^{\sqrt{3}} \frac{1}{1 + y^2}\, dy$$
$$= \int_{-1/2}^{-1} \frac{1}{2x^2}\, dx + \int_{-1}^{1} \frac{2}{1 + x^2}\, dx + \int_1^{\sqrt{3}} \frac{1}{1 + y^2}\, dy$$
$$= -\frac{1}{2x}\Big|_{-1/2}^{-1} + 2\tan^{-1} x\,\Big|_{-1}^{1} + \tan^{-1} y\,\Big|_1^{\sqrt{3}} = -\frac{1}{2} + 2\left(\frac{\pi}{2}\right) + \frac{\pi}{12} = \frac{13\pi - 6}{12}\,.$$

51. $z_x = 2x$, $z_y = 0$; $dS = \sqrt{1 + 4x^2}\, dA$



$$\iint_S \frac{z}{xy}\, dS = \int_1^3 \int_1^2 \frac{x^2}{xy}\sqrt{1 + 4x^2}\, dx\, dy = \int_1^3 \frac{1}{y}\left[\frac{1}{12}(1 + 4x^2)^{3/2}\right]\Big|_1^2 dy$$
$$= \frac{1}{12}\int_1^3 \frac{17^{3/2} - 5^{3/2}}{y}\, dy = \frac{17\sqrt{17} - 5\sqrt{5}}{12}\, \ln y\,\Big|_1^3$$
$$= \frac{17\sqrt{17} - 5\sqrt{5}}{12}\, \ln 3$$

52. $\mathbf{n} = \mathbf{k}$, $\mathbf{F} \cdot \mathbf{n} = 3$;
flux $= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = 3 \iint_S dS = 3 \times$ (area of S) $= 3(1) = 3$



53. The surface is $g(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$. $\nabla g = 2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 2\mathbf{r}$, $\mathbf{n} = \mathbf{r}/|\mathbf{r}|$,

$$\mathbf{F} = c\nabla(1/|\mathbf{r}|) + c\nabla(x^2 + y^2 + z^2)^{-1/2} = c\,\frac{-x\mathbf{i} - y\mathbf{j} - z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} = -c\mathbf{r}/|\mathbf{r}|^3$$

$$\mathbf{F} \cdot \mathbf{n} = -\frac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = -c\,\frac{\mathbf{r} \cdot \mathbf{r}}{|\mathbf{r}|^4} = -c\,\frac{|\mathbf{r}|^2}{|\mathbf{r}|^4} = -\frac{c}{|\mathbf{r}|^2} = -\frac{c}{a^2}$$

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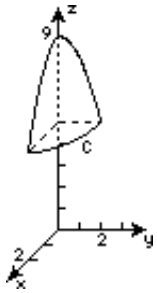
$$\text{flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = -\frac{c}{a^2}\iint_S dS = -\frac{c}{a^2}\times(\text{area of } S) = -\frac{c}{a^2}(4\pi a^2) = -4\pi c$$

54. In Problem 53, $\mathbf{F}$ is not continuous at $(0,0,0)$ which is in any acceptable region containing the sphere.

55. Since $\mathbf{F} = c\nabla(1/r)$, $\text{div }\mathbf{F} = \nabla\cdot(c\nabla(1/r)) = c\nabla^2(1/r) = c\nabla^2[(x^2+y^2+z^2)^{-1/2}] = 0$ by Problem 37 in Section 9.7. Then, by the Divergence Theorem,

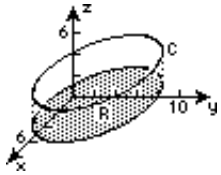
$$\text{flux } \mathbf{F} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \text{div }\mathbf{F}\,dV = \iiint_D 0\,dV = 0.$$

56. Parameterize C by $x = 2\cos t$, $y = 2\sin t$, $z = 5$, for $0 \le t \le 2\pi$. Then



$$\begin{aligned}\iint_S (\text{curl }\mathbf{F}\cdot\mathbf{n})\,dS &= \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C 6x\,dx + 7z\,dy + 8y\,dz\\ &= \int_0^{2\pi}[12\cos t(-2\sin t) + 35(2\cos t)]\,dt\\ &= \int_0^{2\pi}(70\cos t - 24\sin t\cos t)\,dt = (70\sin t - 12\sin^2 t)\,\Big|_0^{2\pi} = 0.\end{aligned}$$

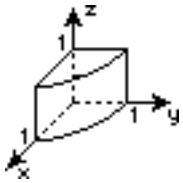
57. Identify $\mathbf{F} = -2y\mathbf{i} + 3x\mathbf{j} + 10z\mathbf{k}$. Then curl $\mathbf{F} = 5\mathbf{k}$. The curve C lies in the plane $z = 3$, so $\mathbf{n} = \mathbf{k}$ and $dS = dA$. Thus,



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R 5\,dA = 5\times(\text{area of } R) = 5(25\pi) = 125\pi.$$

58. Since curl $\mathbf{F} = \mathbf{0}$, $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F}\cdot\mathbf{n})\,dS = \iint_S 0\,dS = 0.$

59. $\text{div }\mathbf{f} = 1 + 1 = 1 = 3$;

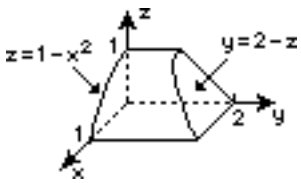


$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \text{div }\mathbf{F}\,dV = \iiint_D 3\,dV = 3\times(\text{volume of } D) = 3\pi$$

60. $\text{div }\mathbf{F} = x^2 + y^2 + z^2$. Using cylindrical coordinates,

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_D \text{div }\mathbf{F}\,dV = \iiint_D (x^2+y^2+z^2)\,dV = \int_0^{2\pi}\int_0^1\int_0^1 (r^2+z^2)r\,dz\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^1 \left(r^3 z + \frac{1}{3}rz^3\right)\Big|_0^1 dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(r^3 + \frac{1}{3}r\right)dr\,d\theta\\ &= \int_0^{2\pi}\left(\frac{1}{4}r^4 + \frac{1}{6}r^2\right)\Big|_0^1 d\theta = \int_0^{2\pi}\frac{5}{12}\,d\theta = \frac{5\pi}{6}.\end{aligned}$$

61. $\text{div }\mathbf{F} = 2x + 2(x+y) - 2y = 4x$



$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_D \text{div }\mathbf{F}\,dV = \iiint_D 4x\,dV = \int_0^1\int_0^{1-x^2}\int_0^{2-z} 4x\,dy\,dz\,dx\\ &= \int_0^1\int_0^{1-x^2} 4x(2-z)\,dz\,dx = \int_0^1\int_0^{1-x^2}(8x - 4xz)\,dz\,dx\\ &= \int_0^1 (8xz - 2xz^2)\,\Big|_0^{1-x^2} dx = \int_0^1 [8x(1-x^2) - 2x(1-x^2)^2]\,dx\\ &= \left[-2(1-x^2)^2 + \frac{1}{3}(1-x^2)^3\right]\Big|_0^1 = \frac{5}{3}\end{aligned}$$

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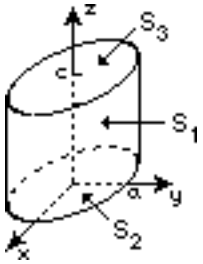
62. For S_1, $\mathbf{n} = (x\mathbf{i} + y\mathbf{j})/\sqrt{x^2+y^2}$; for S_2, $\mathbf{n}_2 = -\mathbf{k}$ and $z = 0$; and for S_3, $\mathbf{n}_3 = \mathbf{k}$ and $z = c$.



Then

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_{S_1} \mathbf{F}\cdot\mathbf{n}_1\,dS_1 + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}_2\,dS_2 + \iint_{S_3} \mathbf{F}\cdot\mathbf{n}_3\,dS_3\\ &= \iint_{S_1} \frac{x^2+y^2}{\sqrt{x^2+y^2}}\,dS_1 + \iint_{S_2} (-z^2-1)\,dS_2 + \iint_{S_3} (z^2+1)\,dS_3\\ &= \iint_{S_1} \sqrt{x^2+y^2}\,dS_1 + \iint_{S_2} (-1)\,dS_2 + \iint_{S_3} (c^2+1)\,dS_3\\ &= a\iint_{S_1} dS_1 - \iint_{S_2} dS_2 + (c^2+1)\iint_{S_3} dS_3\\ &= a(2\pi ac) - \pi a^2 + (c^2+1)\pi a^2 = 2\pi a^2 c + \pi a^2 c^2.\end{aligned}$$

63. $x = 0 \implies u = 0,\ v = -y^2 \implies u = 0,\ -1 \le v \le 0$

$x = 1 \implies u = 2y,\ v = 1 - y^2 = 1 - u^2/4$

$y = 0 \implies u = 0,\ v = x^2 \implies u = 0,\ 0 \le v \le 1$

$y = 1 \implies u = 2x,\ v = x^2 - 1 = u^2/4 - 1$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4(x^2+y^2) \implies \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{4(x^2+y^2)}$$

$$\begin{aligned}\iint_R (x^2+y^2)\sqrt[3]{x^2-y^2}\,dA &= \iint_S (x^2+y^2)\sqrt[3]{v}\left|-\frac{1}{4(x^2+y^2)}\right| dA' = \frac{1}{4}\int_0^2 \int_{u^2/4-1}^{1-u^2/4} v^{1/3}\,dv\,du\\ &= \frac{1}{4}\int_0^2 \frac{3}{4}v^{4/3}\Big|_{u^2/4-1}^{1-u^2/4} du = \frac{3}{16}\int_0^2 \left[(1-u^2/4)^{4/3} - (u^2/4-1)^{4/3}\right] du\\ &= \frac{3}{16}\int_0^2 \left[(1-u^2/4)^{4/3} - (1-u^2/4)^{4/3}\right] du = 0\end{aligned}$$

64. $y = x \implies u + uv = v + uv \implies v = u$

$x = 2 \implies u + uv = 2 \implies v = (2-u)/u$

$y = 0 \implies v + uv = 0 \implies v = 0$ or $u = -1$

(we take $v = 0$)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+w & u \\ v & 1+u \end{vmatrix} = 1 + u + v$$

Using $x = u + uv$ and $y = v + uv$ we find

$$(x-y)^2 = (u + uv - v - uv)^2 = (u-v)^2 = u^2 - 2uv + v^2$$

$$x + y = u + uv + v + uv = u + v + 2uv$$

$$(x-y)^2 + 2(x+y) + 1 = u^2 + 2uv + v^2 + 2(u+v) + 1 = (u+v)^2 + 2(u+v) + 1 = (u+v+1)^2.$$

Then

$$\begin{aligned}\iint_R \frac{1}{\sqrt{(x-y)^2 + 2(x+y) + 1}}\,dA &= \iint_S \frac{1}{u+v+1}(u+v+1)\,dA' = \int_0^1 \int_v^{2/(1+v)} du\,dv\\ &= \int_0^1 \left(\frac{2}{1+v} - v\right) dv = \left[2\ln(1+v) - \frac{1}{2}v^2\right]\Big|_0^1 = 2\ln 2 - \frac{1}{2}.\end{aligned}$$

65. The equations of the spheres are $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + (z-a)^2 = 1$. Subtracting these equations, we

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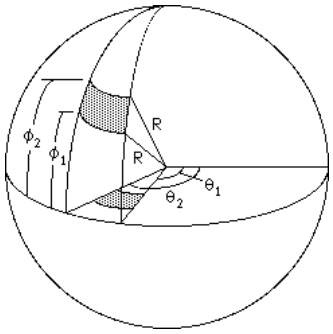
obtain $(z-a)^2 - z^2 = 1 - a^2$ or $-2az + a^2 = 1 - a^2$. Thus, the spheres intersect on the plane $z = a - 1/2a$. The region of integration is $x^2 + y^2 + (a - 1/2a)^2 = a^2$ or $r^2 = 1 - 1/4a^2$. The area is

$$A = a\int_0^{2\pi}\int_0^{\sqrt{1-1/4a^2}} (a^2 - r^2)^{-1/r} r\, dr\, d\theta = 2\pi a[-(a^2 - r^2)^{1/2}]\,\Big|_0^{\sqrt{1-1/4a^2}}$$

$$= 2\pi a\left(a - \left[a^2 - \left(1 - \frac{1}{4a^2}\right)\right]^{1/2}\right) = 2\pi a\left(a - \left[\left(a - \frac{1}{2a}\right)^2\right]^{1/2}\right) = \pi.$$

66. (a) Both states span 7 degrees of longitude and 4 degrees of latitude, but Colorado is larger because it lies to the south of Wyoming. Lines of longitude converge as they go north, so the east-west dimensions of Wyoming are shorter than those of Colorado.

(b) We use the function $f(x, y) = \sqrt{R^2 - x^2 - y^2}$ to describe the northern hemisphere, where $R \approx 3960$ miles is the radius of the Earth. We need to compute the surface area over a polar rectangle P of the form $\theta_1 \le \theta \le \theta_2$, $R\cos\phi_2 \le r \le R\cos\phi_1$. We have



$$f_x = \frac{-x}{\sqrt{R^2 - x^2 - y^2}} \qquad \text{and} \qquad f_y = \frac{-y}{\sqrt{R^2 - x^2 - y^2}}$$

so that

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \frac{x^2 + y^2}{R^2 - x^2 - y^2}} = \frac{R}{\sqrt{R^2 - r^2}}\,.$$

Thus

$$A = \iint_P \sqrt{1 + f_x^2 + f_y^2}\; dA = \int_{\theta_1}^{\theta_2}\int_{R\cos\phi_2}^{R\cos\phi_1} \frac{R}{\sqrt{R^2 - r^2}}\; r\, dr\, d\theta$$

$$= (\theta_2 - \theta_1)R\sqrt{R^2 - r^2}\;\Big|_{R\cos\phi_1}^{R\cos\phi_2} = (\theta_2 - \theta_1)R^2(\sin\phi_2 - \sin\phi_1).$$

The ratio of Wyoming to Colorado is then $\dfrac{\sin 45^\circ - \sin 41^\circ}{\sin 41^\circ - \sin 37^\circ} \approx 0.941$. Thus Wyoming is about 6% smaller

than Colorado.

(c) 97,914/104,247 $\approx$ 0.939, which is close to the theoretical value of 0.941. (Our formula for the area says that the area of Colorado is approximately 103,924 square miles, while the area of Wyoming is approximately 97,801 square miles.)

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10 Systems of Linear Differential Equations

Exercises 10.1

1. Let $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\mathbf{X}' = \begin{pmatrix} 3 & -5 \\ 4 & 8 \end{pmatrix} \mathbf{X}.$$

2. Let $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\mathbf{X}' = \begin{pmatrix} 4 & -7 \\ 5 & 0 \end{pmatrix} \mathbf{X}.$$

3. Let $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$\mathbf{X}' = \begin{pmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{pmatrix} \mathbf{X}.$$

4. Let $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix} \mathbf{X}.$$

5. Let $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ -3t^2 \\ t^2 \end{pmatrix} + \begin{pmatrix} t \\ 0 \\ -t \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

6. Let $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$\mathbf{X}' = \begin{pmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{pmatrix} \mathbf{X} + \begin{pmatrix} e^{-t} \sin 2t \\ 4e^{-t} \cos 2t \\ -e^{-t} \end{pmatrix}$$

7. $\dfrac{dx}{dt} = 4x + 2y + e^t; \quad \dfrac{dy}{dt} = -x + 3y - e^t$

8. $\dfrac{dx}{dt} = 7x + 5y - 9z - 8e^{-2t}; \quad \dfrac{dy}{dt} = 4x + y + z + 2e^{5t}; \quad \dfrac{dz}{dt} = -2y + 3z + e^{5t} - 3e^{-2t}$

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9. $\dfrac{dx}{dt} = x - y + 2z + e^{-t} - 3t; \quad \dfrac{dy}{dt} = 3x - 4y + z + 2e^{-t} + t; \quad \dfrac{dz}{dt} = -2x + 5y + 6z + 2e^{-t} - t$

10. $\dfrac{dx}{dt} = 3x - 7y + 4\sin t + (t-4)e^{4t}; \quad \dfrac{dy}{dt} = x + y + 8\sin t + (2t+1)e^{4t}$

11. Since

$$\mathbf{X}' = \begin{pmatrix} -5 \\ -10 \end{pmatrix} e^{-5t} \quad \text{and} \quad \begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{X} = \begin{pmatrix} -5 \\ -10 \end{pmatrix} e^{-5t}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{X}.$$

12. Since

$$\mathbf{X}' = \begin{pmatrix} 5\cos t - 5\sin t \\ 2\cos t - 4\sin t \end{pmatrix} e^{t} \quad \text{and} \quad \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 5\cos t - 5\sin t \\ 2\cos t - 4\sin t \end{pmatrix} e^{t}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix} \mathbf{X}.$$

13. Since

$$\mathbf{X}' = \begin{pmatrix} 3/2 \\ -3 \end{pmatrix} e^{-3t/2} \quad \text{and} \quad \begin{pmatrix} -1 & 1/4 \\ 1 & -1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 3/2 \\ -3 \end{pmatrix} e^{-3t/2}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} -1 & 1/4 \\ 1 & -1 \end{pmatrix} \mathbf{X}.$$

14. Since

$$\mathbf{X}' = \begin{pmatrix} 5 \\ -1 \end{pmatrix} e^{t} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} te^{t} \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} e^{t} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} te^{t}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{X}.$$

15. Since

$$\mathbf{X}' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \mathbf{X}.$$

16. Since

$$\mathbf{X}' = \begin{pmatrix} \cos t \\ \frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\cos t - \sin t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} \cos t \\ \frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\cos t - \sin t \end{pmatrix}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \mathbf{X}.$$

17. Yes, since $W(\mathbf{X}_1, \mathbf{X}_2) = -2e^{-8t} \neq 0$ and $\mathbf{X}_1$ and $\mathbf{X}_2$ are linearly independent on $-\infty < t < \infty$.

18. Yes, since $W(\mathbf{X}_1, \mathbf{X}_2) = 8e^{2t} \neq 0$ and $\mathbf{X}_1$ and $\mathbf{X}_2$ are linearly independent on $-\infty < t < \infty$.

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19. No, since $W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) = 0$ and $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ are linearly dependent on $-\infty < t < \infty$.

20. Yes, since $W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) = -84e^{-t} \neq 0$ and $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ are linearly independent on $-\infty < t < \infty$.

21. Since

$$\mathbf{X}_p' = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} 2 \\ -4 \end{pmatrix} t + \begin{pmatrix} -7 \\ -18 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

we see that

$$\mathbf{X}_p' = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} 2 \\ -4 \end{pmatrix} t + \begin{pmatrix} -7 \\ -18 \end{pmatrix}.$$

22. Since

$$\mathbf{X}_p' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we see that

$$\mathbf{X}_p' = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} -5 \\ 2 \end{pmatrix}.$$

23. Since

$$\mathbf{X}_p' = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \mathbf{X}_p - \begin{pmatrix} 1 \\ 7 \end{pmatrix} e^t = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t$$

we see that

$$\mathbf{X}_p' = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \mathbf{X}_p - \begin{pmatrix} 1 \\ 7 \end{pmatrix} e^t.$$

24. Since

$$\mathbf{X}_p' = \begin{pmatrix} 3\cos 3t \\ 0 \\ -3\sin 3t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ -6 & 1 & 0 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \sin 3t = \begin{pmatrix} 3\cos 3t \\ 0 \\ -3\sin 3t \end{pmatrix}$$

we see that

$$\mathbf{X}_p' = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ -6 & 1 & 0 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \sin 3t.$$

25. Let

$$\mathbf{X}_1 = \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t}, \quad \mathbf{X}_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t}, \quad \mathbf{X}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t}, \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then

$$\mathbf{X}_1' = \begin{pmatrix} -6 \\ 1 \\ 5 \end{pmatrix} e^{-t} = \mathbf{A}\mathbf{X}_1,$$

$$\mathbf{X}_2' = \begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} e^{-2t} = \mathbf{A}\mathbf{X}_2,$$

$$\mathbf{X}_3' = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} e^{3t} = \mathbf{A}\mathbf{X}_3,$$

and $W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) = 20 \neq 0$ so that $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ form a fundamental set for $\mathbf{X}' = \mathbf{A}\mathbf{X}$ on $-\infty < t < \infty$.

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26. Let

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{\sqrt{2}\,t},$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix} e^{-\sqrt{2}\,t},$$

$$\mathbf{X}_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

$$\mathbf{A} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{X}_1' = \begin{pmatrix} \sqrt{2} \\ -2-\sqrt{2} \end{pmatrix} e^{\sqrt{2}\,t} = \mathbf{A}\mathbf{X}_1,$$

$$\mathbf{X}_2' = \begin{pmatrix} -\sqrt{2} \\ -2-\sqrt{2} \end{pmatrix} e^{-\sqrt{2}\,t} = \mathbf{A}\mathbf{X}_2,$$

$$\mathbf{X}_p' = \begin{pmatrix} 2 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \mathbf{A}\mathbf{X}_p + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 4 \\ -6 \end{pmatrix} t + \begin{pmatrix} -1 \\ 5 \end{pmatrix},$$

and $W(\mathbf{X}_1, \mathbf{X}_2) = 2\sqrt{2} \neq 0$ so that $\mathbf{X}_p$ is a particular solution and $\mathbf{X}_1$ and $\mathbf{X}_2$ form a fundamental set on $-\infty < t < \infty$.

Exercises 10.2

1. The system is

$$\mathbf{X}' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 5)(\lambda + 1) = 0$. For $\lambda_1 = 5$ we obtain

$$\left(\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = -1$ we obtain

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}.$$

2. The system is

$$\mathbf{X}' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 1)(\lambda - 4) = 0$. For $\lambda_1 = 1$ we obtain

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 4$ we obtain

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

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Then

$$\mathbf{X} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.$$

3. The system is

$$\mathbf{X}' = \begin{pmatrix} -4 & 2 \\ -5/2 & 2 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 1)(\lambda + 3) = 0$. For $\lambda_1 = 1$ we obtain

$$\left(\begin{array}{cc|c} -5 & 2 & 0 \\ -5/2 & 1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} -5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

For $\lambda_2 = -3$ we obtain

$$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ -5/2 & 5 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t}.$$

4. The system is

$$\mathbf{X}' = \begin{pmatrix} -5/2 & 2 \\ 3/4 & -2 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda + 1)(2\lambda + 7) = 0$. For $\lambda_1 = -7/2$ we obtain

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3/4 & 3/2 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = -1$ we obtain

$$\left(\begin{array}{cc|c} -3/2 & 2 & 0 \\ 3/4 & -1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} -3 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-7t/2} + c_2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{-t}.$$

5. The system is

$$\mathbf{X}' = \begin{pmatrix} 10 & -5 \\ 8 & -12 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 8)(\lambda + 10) = 0$. For $\lambda_1 = 8$ we obtain

$$\left(\begin{array}{cc|c} 2 & -5 & 0 \\ 8 & -20 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & -5/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = -10$ we obtain

$$\left(\begin{array}{cc|c} 20 & -5 & 0 \\ 8 & -2 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & -1/4 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-10t}.$$

6. The system is

$$\mathbf{X}' = \begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix} \mathbf{X}$$

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and $\det(\mathbf{A}-\lambda\mathbf{I})=\lambda(\lambda+5)=0$. For $\lambda_1=0$ we obtain

$$\left(\begin{array}{cc|c}-6 & 2 & 0\\ -3 & 1 & 0\end{array}\right)\Longrightarrow\left(\begin{array}{cc|c}1 & -1/3 & 0\\ 0 & 0 & 0\end{array}\right)\quad\text{so that}\quad\mathbf{K}_1=\begin{pmatrix}1\\3\end{pmatrix}.$$

For $\lambda_2=-5$ we obtain

$$\left(\begin{array}{cc|c}-1 & 2 & 0\\ -3 & 6 & 0\end{array}\right)\Longrightarrow\left(\begin{array}{cc|c}1 & -2 & 0\\ 0 & 0 & 0\end{array}\right)\quad\text{so that}\quad\mathbf{K}_2=\begin{pmatrix}2\\1\end{pmatrix}.$$

Then

$$\mathbf{X}=c_1\begin{pmatrix}1\\3\end{pmatrix}+c_2\begin{pmatrix}2\\1\end{pmatrix}e^{-5t}.$$

7. The system is

$$\mathbf{X}'=\begin{pmatrix}1 & 1 & -1\\ 0 & 2 & 0\\ 0 & 1 & -1\end{pmatrix}\mathbf{X}$$

and $\det(\mathbf{A}-\lambda\mathbf{I})=(\lambda-1)(2-\lambda)(\lambda+1)=0$. For $\lambda_1=1$, $\lambda_2=2$, and $\lambda_3=-1$ we obtain

$$\mathbf{K}_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad\mathbf{K}_2=\begin{pmatrix}2\\3\\1\end{pmatrix},\quad\text{and}\quad\mathbf{K}_3=\begin{pmatrix}1\\0\\2\end{pmatrix},$$

so that

$$\mathbf{X}=c_1\begin{pmatrix}1\\0\\0\end{pmatrix}e^{t}+c_2\begin{pmatrix}2\\3\\1\end{pmatrix}e^{2t}+c_3\begin{pmatrix}1\\0\\2\end{pmatrix}e^{-t}.$$

8. The system is

$$\mathbf{X}'=\begin{pmatrix}2 & -7 & 0\\ 5 & 10 & 4\\ 0 & 5 & 2\end{pmatrix}\mathbf{X}$$

and $\det(\mathbf{A}-\lambda\mathbf{I})=(2-\lambda)(\lambda-5)(\lambda-7)=0$. For $\lambda_1=2$, $\lambda_2=5$, and $\lambda_3=7$ we obtain

$$\mathbf{K}_1=\begin{pmatrix}4\\0\\-5\end{pmatrix},\quad\mathbf{K}_2=\begin{pmatrix}-7\\3\\5\end{pmatrix},\quad\text{and}\quad\mathbf{K}_3=\begin{pmatrix}-7\\5\\5\end{pmatrix},$$

so that

$$\mathbf{X}=c_1\begin{pmatrix}4\\0\\-5\end{pmatrix}e^{2t}+c_2\begin{pmatrix}-7\\3\\5\end{pmatrix}e^{5t}+c_3\begin{pmatrix}-7\\5\\5\end{pmatrix}e^{7t}.$$

9. We have $\det(\mathbf{A}-\lambda\mathbf{I})=-(\lambda+1)(\lambda-3)(\lambda+2)=0$. For $\lambda_1=-1$, $\lambda_2=3$, and $\lambda_3=-2$ we obtain

$$\mathbf{K}_1=\begin{pmatrix}-1\\0\\1\end{pmatrix},\quad\mathbf{K}_2=\begin{pmatrix}1\\4\\3\end{pmatrix},\quad\text{and}\quad\mathbf{K}_3=\begin{pmatrix}1\\-1\\3\end{pmatrix},$$

so that

$$\mathbf{X}=c_1\begin{pmatrix}-1\\0\\1\end{pmatrix}e^{-t}+c_2\begin{pmatrix}1\\4\\3\end{pmatrix}e^{3t}+c_3\begin{pmatrix}1\\-1\\3\end{pmatrix}e^{-2t}.$$

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10. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda(\lambda - 1)(\lambda - 2) = 0$. For $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

11. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda + 1)(\lambda + 1/2)(\lambda + 3/2) = 0$. For $\lambda_1 = -1$, $\lambda_2 = -1/2$, and $\lambda_3 = -3/2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} -12 \\ 6 \\ 5 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -12 \\ 6 \\ 5 \end{pmatrix} e^{-t/2} + c_3 \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} e^{-3t/2}.$$

12. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 3)(\lambda + 5)(6 - \lambda) = 0$. For $\lambda_1 = 3$, $\lambda_2 = -5$, and $\lambda_3 = 6$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 2 \\ -2 \\ 11 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-5t} + c_3 \begin{pmatrix} 2 \\ -2 \\ 11 \end{pmatrix} e^{6t}.$$

13. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda + 1/2)(\lambda - 1/2) = 0$. For $\lambda_1 = -1/2$ and $\lambda_2 = 1/2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2}.$$

If

$$\mathbf{X}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

then $c_1 = 2$ and $c_2 = 3$.

14. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (2 - \lambda)(\lambda - 3)(\lambda + 1) = 0$. For $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = -1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^{-t}.$$

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If

$$\mathbf{X}(0) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

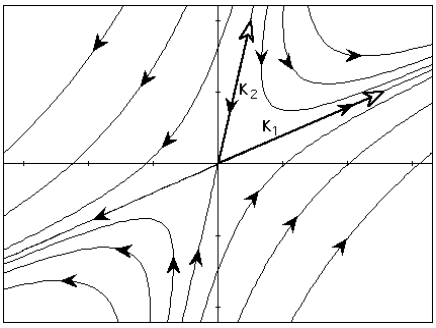
then $c_1 = -1$, $c_2 = 5/2$, and $c_3 = -1/2$.

15. $\mathbf{X} = c_1 \begin{pmatrix} 0.382175 \\ 0.851161 \\ 0.359815 \end{pmatrix} e^{8.58979t} + c_2 \begin{pmatrix} 0.405188 \\ -0.676043 \\ 0.615458 \end{pmatrix} e^{2.25684t} + c_3 \begin{pmatrix} -0.923562 \\ -0.132174 \\ 0.35995 \end{pmatrix} e^{-0.0466321t}$

16. $\mathbf{X} = c_1 \begin{pmatrix} 0.0312209 \\ 0.949058 \\ 0.239535 \\ 0.195825 \\ 0.0508861 \end{pmatrix} e^{5.05452t} + c_2 \begin{pmatrix} -0.280232 \\ -0.836611 \\ -0.275304 \\ 0.176045 \\ 0.338775 \end{pmatrix} e^{4.09561t} + c_3 \begin{pmatrix} 0.262219 \\ -0.162664 \\ -0.826218 \\ -0.346439 \\ 0.31957 \end{pmatrix} e^{-2.92362t}$

$$+c_4 \begin{pmatrix} 0.313235 \\ 0.64181 \\ 0.31754 \\ 0.173787 \\ -0.599108 \end{pmatrix} e^{2.02882t} + c_5 \begin{pmatrix} -0.301294 \\ 0.466599 \\ 0.222136 \\ 0.0534311 \\ -0.799567 \end{pmatrix} e^{-0.155338t}$$

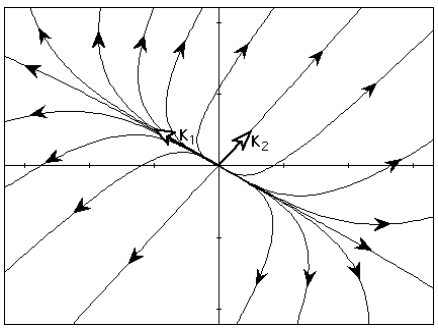
17. (a)



(b) Letting $c_1 = 1$ and $c_2 = 0$ we get $x = 5e^{8t}$, $y = 2e^{8t}$. Eliminating the parameter we find $y = \frac{2}{5}x$, $x > 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = \frac{2}{5}x$, $x < 0$. Letting $c_1 = 0$ and $c_2 = 1$ we get $x = e^{-10t}$, $y = 4e^{-10t}$. Eliminating the parameter we find $y = 4x$, $x > 0$. Letting $c_1 = 0$ and $c_2 = -1$ we find $y = 4x$, $x < 0$.

(c) The eigenvectors $\mathbf{K}_1 = (5, 2)$ and $\mathbf{K}_2 = (1, 4)$ are shown in the figure in part (**a**).

18. (a) Letting $c_1 = 1$ and $c_2 = 0$ we get $x = -2e^t$, $y = e^t$. Eliminating the parameter we find $y = -\frac{1}{2}x$, $x < 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = -\frac{1}{2}x$, $x > 0$. Letting $c_1 = 0$ and $c_2 = 1$ we get $x = e^{4t}$, $y = e^{4t}$. Eliminating the parameter we find $y = x$, $x > 0$. When $c_1 = 0$ and $c_2 = -1$ we find $y = x$, $x < 0$.



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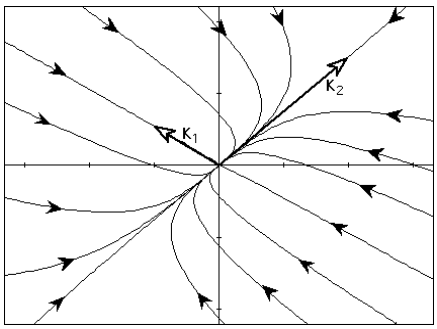
Letting $c_1 = 1$ and $c_2 = 0$ we get $x = -2e^{-7t/2}$, $y = e^{-7t/2}$. Eliminating the parameter we find $y = -\frac{1}{2}x$, $x < 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = -\frac{1}{2}x$, $x > 0$. Letting $c_1 = 0$ and $c_2 = 1$ we get $x = 4e^{-t}$, $y = 3e^{-t}$. Eliminating the parameter we find $y = \frac{3}{4}x$, $x > 0$. When $c_1 = 0$ and $c_2 = -1$ we find $y = \frac{3}{4}x$, $x < 0$.



19. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 = 0$. For $\lambda_1 = 0$ we obtain

$$\mathbf{K} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right].$$

20. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda + 1)^2 = 0$. For $\lambda_1 = -1$ we obtain

$$\mathbf{K} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 0 \\ 1/5 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 1/5 \end{pmatrix} e^{-t} \right].$$

21. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 2)^2 = 0$. For $\lambda_1 = 2$ we obtain

$$\mathbf{K} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} -1/3 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} -1/3 \\ 0 \end{pmatrix} e^{2t} \right].$$

22. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 6)^2 = 0$. For $\lambda_1 = 6$ we obtain

$$\mathbf{K} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

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A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{6t} + c_2 \left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} te^{6t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{6t} \right].$$

23. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (1 - \lambda)(\lambda - 2)^2 = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 2$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}.$$

24. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 8)(\lambda + 1)^2 = 0$. For $\lambda_1 = 8$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = -1$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{-t}.$$

25. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda(5 - \lambda)^2 = 0$. For $\lambda_1 = 0$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = 5$ we obtain

$$\mathbf{K} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 5/2 \\ 1/2 \\ 0 \end{pmatrix}$$

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so that

$$\mathbf{X} = c_1 \begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^{5t} + c_3 \left[\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} te^{5t} + \begin{pmatrix} 5/2 \\ 1/2 \\ 0 \end{pmatrix} e^{5t} \right].$$

26. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (1 - \lambda)(\lambda - 2)^2 = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = 2$ we obtain

$$\mathbf{K} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t} + c_3 \left[\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} e^{2t} \right].$$

27. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda - 1)^3 = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Solutions of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ and $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{Q} = \mathbf{P}$ are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} te^{t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t} \right] + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \frac{t^2}{2} e^{t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^{t} + \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix} e^{t} \right].$$

28. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 4)^3 = 0$. For $\lambda_1 = 4$ we obtain

$$\mathbf{K} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Solutions of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ and $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{Q} = \mathbf{P}$ are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

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so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{4t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{4t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{4t} \right] + c_3 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{t^2}{2} e^{4t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^{4t} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{4t} \right].$$

29. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 4)^2 = 0$. For $\lambda_1 = 4$ we obtain

$$\mathbf{K} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} \right].$$

If

$$\mathbf{X}(0) = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

then $c_1 = -7$ and $c_2 = 13$.

30. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda + 1)(\lambda - 1)^2 = 0$. For $\lambda_1 = -1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t}.$$

If

$$\mathbf{X}(0) = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

then $c_1 = 2$, $c_2 = 3$, and $c_3 = 2$.

31. In this case $\det(\mathbf{A} - \lambda\mathbf{I}) = (2 - \lambda)^5$, and $\lambda_1 = 2$ is an eigenvalue of multiplicity 5. Linearly independent eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{K}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{K}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

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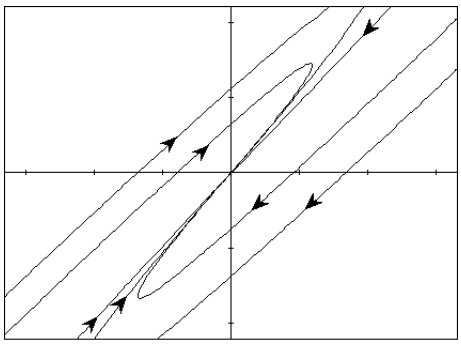
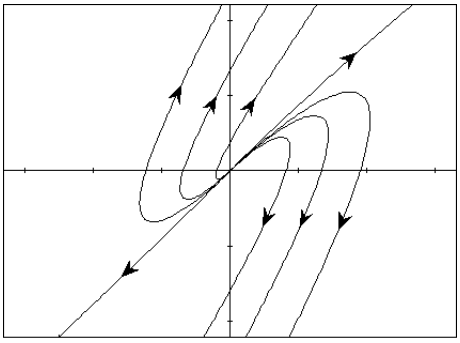
32. Letting $c_1 = 1$ and $c_2 = 0$ we get $x = e^t$, $y = e^t$. Eliminating the parameter we find $y = x$, $x > 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = x$, $x < 0$.



Phase portrait for the system in Problem 20

Letting $c_1 = 1$ and $c_2 = 0$ we get $x = e^{2t}$, $y = e^{2t}$. Eliminating the parameter we find $y = x$, $x > 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = x$, $x < 0$.



Phase portrait for the system in Problem 21

In Problems 33-46 the form of the answer will vary according to the choice of eigenvector. For example, in Problem 33, if $\mathbf{K}_1$ is chosen to be $\begin{pmatrix} 1 \\ 2-i \end{pmatrix}$ the solution has the form

$$\mathbf{X} = c_1 \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} e^{4t}.$$

33. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 8\lambda + 17 = 0$. For $\lambda_1 = 4 + i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 2+i \\ 5 \end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix} 2+i \\ 5 \end{pmatrix} e^{(4+i)t} = \begin{pmatrix} 2\cos t - \sin t \\ 5\cos t \end{pmatrix} e^{4t} + i \begin{pmatrix} \cos t + 2\sin t \\ 5\sin t \end{pmatrix} e^{4t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 2\cos t - \sin t \\ 5\cos t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ 5\sin t \end{pmatrix} e^{4t}.$$

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34. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 1 = 0$. For $\lambda_1 = i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} -1 - i \\ 2 \end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix} -1 - i \\ 2 \end{pmatrix} e^{it} = \begin{pmatrix} \sin t - \cos t \\ 2\cos t \end{pmatrix} + i \begin{pmatrix} -\cos t - \sin t \\ 2\sin t \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} \sin t - \cos t \\ 2\cos t \end{pmatrix} + c_2 \begin{pmatrix} -\cos t - \sin t \\ 2\sin t \end{pmatrix}.$$

35. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 8\lambda + 17 = 0$. For $\lambda_1 = 4 + i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} -1 - i \\ 2 \end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix} -1 - i \\ 2 \end{pmatrix} e^{(4+i)t} = \begin{pmatrix} \sin t - \cos t \\ 2\cos t \end{pmatrix} e^{4t} + i \begin{pmatrix} -\sin t - \cos t \\ 2\sin t \end{pmatrix} e^{4t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} \sin t - \cos t \\ 2\cos t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -\sin t - \cos t \\ 2\sin t \end{pmatrix} e^{4t}.$$

36. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 10\lambda + 34 = 0$. For $\lambda_1 = 5 + 3i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 - 3i \\ 2 \end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix} 1 - 3i \\ 2 \end{pmatrix} e^{(5+3i)t} = \begin{pmatrix} \cos 3t + 3\sin 3t \\ 2\cos 3t \end{pmatrix} e^{5t} + i \begin{pmatrix} \sin 3t - 3\cos 3t \\ 2\cos 3t \end{pmatrix} e^{5t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} \cos 3t + 3\sin 3t \\ 2\cos 3t \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} \sin 3t - 3\cos 3t \\ 2\cos 3t \end{pmatrix} e^{5t}.$$

37. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 9 = 0$. For $\lambda_1 = 3i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 4 + 3i \\ 5 \end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix} 4 + 3i \\ 5 \end{pmatrix} e^{3it} = \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ 5\cos 3t \end{pmatrix} + i \begin{pmatrix} 4\sin 3t + 3\cos 3t \\ 5\sin 3t \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ 5\cos 3t \end{pmatrix} + c_2 \begin{pmatrix} 4\sin 3t + 3\cos 3t \\ 5\sin 3t \end{pmatrix}.$$

38. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 2\lambda + 5 = 0$. For $\lambda_1 = -1 + 2i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 2 + 2i \\ 1 \end{pmatrix}$$

so that

$$\begin{aligned} \mathbf{X}_1 &= \begin{pmatrix} 2 + 2i \\ 1 \end{pmatrix} e^{(-1+2i)t} \\ &= (\, 2\cos 2t - 2\sin 2t\cos 2t \,)\, e^{-t} + i \begin{pmatrix} 2\cos 2t + 2\sin 2t \\ \sin 2t \end{pmatrix} e^{-t}. \end{aligned}$$

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Then

$$\mathbf{X} = c_1 \begin{pmatrix} 2\cos 2t - 2\sin 2t \\ \cos 2t \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2\cos 2t + 2\sin 2t \\ \sin 2t \end{pmatrix} e^{-t}.$$

39. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda\left(\lambda^2 + 1\right) = 0$. For $\lambda_1 = 0$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} -i \\ i \\ 1 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} -i \\ i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} \sin t \\ -\sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} -\cos t \\ \cos t \\ \sin t \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\sin t \\ \cos t \end{pmatrix} + c_3 \begin{pmatrix} -\cos t \\ \cos t \\ \sin t \end{pmatrix}.$$

40. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda + 3)(\lambda^2 - 2\lambda + 5) = 0$. For $\lambda_1 = -3$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1 + 2i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} -2 - i \\ -3i \\ 2 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} -2\cos 2t + \sin 2t \\ 3\sin 2t \\ 2\cos 2t \end{pmatrix} e^{t} + i \begin{pmatrix} -\cos 2t - 2\sin 2t \\ -3\cos 2t \\ 2\sin 2t \end{pmatrix} e^{t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} -2\cos 2t + \sin 2t \\ 3\sin 2t \\ 2\cos 2t \end{pmatrix} e^{t} + c_3 \begin{pmatrix} -\cos 2t - 2\sin 2t \\ -3\cos 2t \\ 2\sin 2t \end{pmatrix} e^{t}.$$

41. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (1 - \lambda)(\lambda^2 - 2\lambda + 2) = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1 + i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix}$$

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so that

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix} e^{(1+i)t} = \begin{pmatrix} \cos t \\ -\sin t \\ -\sin t \end{pmatrix} e^t + i \begin{pmatrix} \sin t \\ \cos t \\ \cos t \end{pmatrix} e^t.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos t \\ -\sin t \\ -\sin t \end{pmatrix} e^t + c_3 \begin{pmatrix} \sin t \\ \cos t \\ \cos t \end{pmatrix} e^t.$$

42. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda - 6)(\lambda^2 - 8\lambda + 20) = 0$. For $\lambda_1 = 6$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = 4 + 2i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} -i \\ 0 \\ 2 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} -i \\ 0 \\ 2 \end{pmatrix} e^{(4+2i)t} = \begin{pmatrix} \sin 2t \\ 0 \\ 2\cos 2t \end{pmatrix} e^{4t} + i \begin{pmatrix} -\cos 2t \\ 0 \\ 2\sin 2t \end{pmatrix} e^{4t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} \sin 2t \\ 0 \\ 2\cos 2t \end{pmatrix} e^{4t} + c_3 \begin{pmatrix} -\cos 2t \\ 0 \\ 2\sin 2t \end{pmatrix} e^{4t}.$$

43. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (2 - \lambda)(\lambda^2 + 4\lambda + 13) = 0$. For $\lambda_1 = 2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix}.$$

For $\lambda_2 = -2 + 3i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 4 + 3i \\ -5 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} 4 + 3i \\ -5 \\ 0 \end{pmatrix} e^{(-2+3i)t} = \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ -5\cos 3t \\ 0 \end{pmatrix} e^{-2t} + i \begin{pmatrix} 4\sin 3t + 3\cos 3t \\ -5\sin 3t \\ 0 \end{pmatrix} e^{-2t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ -5\cos 3t \\ 0 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 4\sin 3t + 3\cos 3t \\ -5\sin 3t \\ 0 \end{pmatrix} e^{-2t}.$$

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44. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda + 2)(\lambda^2 + 4) = 0$. For $\lambda_1 = -2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 2i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} -2 - 2i \\ 1 \\ 1 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} -2 - 2i \\ 1 \\ 1 \end{pmatrix} e^{2it} = \begin{pmatrix} -2\cos 2t + 2\sin 2t \\ \cos 2t \\ \cos 2t \end{pmatrix} + i \begin{pmatrix} -2\cos 2t - 2\sin 2t \\ \sin 2t \\ \sin 2t \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -2\cos 2t + 2\sin 2t \\ \cos 2t \\ \cos 2t \end{pmatrix} + c_3 \begin{pmatrix} -2\cos 2t - 2\sin 2t \\ \sin 2t \\ \sin 2t \end{pmatrix}.$$

45. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (1 - \lambda)(\lambda^2 + 25) = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix}.$$

For $\lambda_2 = 5i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 1 + 5i \\ 1 \\ 1 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} 1 + 5i \\ 1 \\ 1 \end{pmatrix} e^{5it} = \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + i \begin{pmatrix} \sin 5t + 5\cos 5t \\ \sin 5t \\ \sin 5t \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + c_3 \begin{pmatrix} \sin 5t + 5\cos 5t \\ \sin 5t \\ \sin 5t \end{pmatrix}.$$

If

$$\mathbf{X}(0) = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$$

then $c_1 = c_2 = -1$ and $c_3 = 6$.

46. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 10\lambda + 29 = 0$. For $\lambda_1 = 5 + 2i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 - 2i \end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ 1 - 2i \end{pmatrix} e^{(5+2i)t} = \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} e^{5t} + i \begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix} e^{5t}.$$

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and

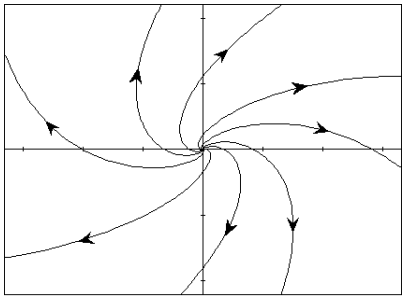
$$\mathbf{X} = c_1 \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} e^{5t} + c_3 \begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix} e^{5t}.$$

If

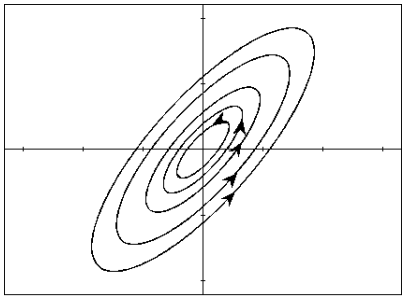
$$\mathbf{X}(0) = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$$

then $c_1 = -2$ and $c_2 = 5$.

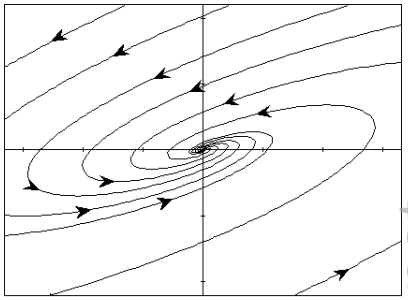
47.



Phase portrait for the system in Problem 36



Phase portrait for the system in Problem 37



Phase portrait for the system in Problem 38

48. (a) From $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda(\lambda - 2) = 0$ we get $\lambda_1 = 0$ and $\lambda_2 = 2$. For $\lambda_1 = 0$ we obtain

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

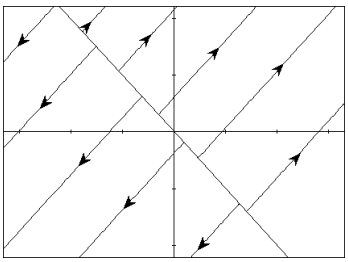
For $\lambda_2 = 2$ we obtain

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

The line $y = -x$ is not a trajectory of the system. Trajectories are $x = -c_1 + c_2 e^{2t}$, $y = c_1 + c_2 e^{2t}$ or $y = x + 2c_1$. This is a family of lines perpendicular to the line $y = -x$. All of the constant solutions of the system do, however, lie on the line $y = -x$.



(b) From $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 = 0$ we get $\lambda_1 = 0$ and

$$\mathbf{K} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right].$$

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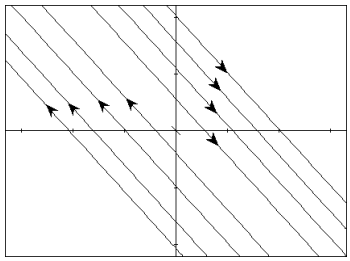
All trajectories are parallel to $y = -x$, but $y = -x$ is not a trajectory. There are constant solutions of the system, however, that do lie on the line $y = -x$.



49. The system of differential equations is

$$\begin{aligned} x_1' &= 2x_1 + x_2 \\ x_2' &= 2x_2 \\ x_3' &= 2x_3 \\ x_4' &= 2x_4 + x_5 \\ x_5' &= 2x_5. \end{aligned}$$

We see immediately that $x_2 = c_2e^{2t}$, $x_3 = c_3e^{2t}$, and $x_5 = c_5e^{2t}$. Then

$$x_1' = 2x_1 + c_2e^{2t} \implies x_1 = c_2te^{2t} + c_1e^{2t}$$

and

$$x_4' = 2x_4 + c_5e^{2t} \implies x_4 = c_5te^{2t} + c_4e^{2t}.$$

The general solution of the system is

$$\begin{aligned} \mathbf{X} &= \begin{pmatrix} c_2te^{2t} + c_1e^{2t} \\ c_2e^{2t} \\ c_3e^{2t} \\ c_5te^{2t} + c_4e^{2t} \\ c_5e^{2t} \end{pmatrix} \\ &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{2t} \right] \\ &\quad + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_5 \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} \right] \end{aligned}$$

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$$= c_1\mathbf{K}_1e^{2t} + c_2\left[\mathbf{K}_1te^{2t} + \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}e^{2t}\right]$$

$$+ c_3\mathbf{K}_2e^{2t} + c_4\mathbf{K}_3e^{2t} + c_5\left[\mathbf{K}_3te^{2t} + \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}e^{2t}\right].$$

There are three solutions of the form $\mathbf{X} = \mathbf{K}e^{2t}$, where $\mathbf{K}$ is an eigenvector, and two solutions of the form $\mathbf{X} = \mathbf{K}te^{2t} + \mathbf{P}e^{2t}$. See (12) in the text. From (13) and (14) in the text

$$(\mathbf{A} - 2\mathbf{I})\mathbf{K}_1 = \mathbf{0}$$

and

$$(\mathbf{A} - 2\mathbf{I})\mathbf{K}_2 = \mathbf{K}_1.$$

This implies

$$\begin{pmatrix}0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}p_1\\p_2\\p_3\\p_4\\p_5\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},$$

so $p_2 = 1$ and $p_5 = 0$, while p_1, p_3, and p_4 are arbitrary. Choosing $p_1 = p_3 = p_4 = 0$ we have

$$\mathbf{P} = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}.$$

Therefore a solution is

$$\mathbf{X} = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}te^{2t} + \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}e^{2t}.$$

Repeating for $\mathbf{K}_3$ we find

$$\mathbf{P} = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix},$$

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so another solution is

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

50. From $x = 2\cos 2t - 2\sin 2t$, $y = -\cos 2t$ we find $x + 2y = -2\sin 2t$. Then

$$(x + 2y)^2 = 4\sin^2 2t = 4(1 - \cos^2 2t) = 4 - 4\cos^2 2t = 4 - 4y^2$$

and

$$x^2 + 4xy + 4y^2 = 4 - 4y^2 \qquad \text{or} \qquad x^2 + 4xy + 8y^2 = 4.$$

This is a rotated conic section and, from the discriminant $b^2 - 4ac = 16 - 32 < 0$, we see that the curve is an ellipse.

51. Suppose the eigenvalues are $\alpha \pm i\beta$, $\beta > 0$. In Problem 36 the eigenvalues are $5 \pm 3i$, in Problem 37 they are $\pm 3i$, and in Problem 38 they are $-1 \pm 2i$. From Problem 47 we deduce that the phase portrait will consist of a family of closed curves when $\alpha = 0$ and spirals when $\alpha \neq 0$. The origin will be a repellor when $\alpha > 0$, and an attractor when $\alpha < 0$.

52. **(a)** The given system can be written as

$$x_1'' = -\frac{k_1 + k_2}{m_1}\, x_1 + \frac{k_2}{m_1}\, x_2, \qquad x_2'' = \frac{k_2}{m_1}\, x_1 - \frac{k_2}{m_1}\, x_2.$$

In terms of matrices this is $\mathbf{X}'' = \mathbf{A}\mathbf{X}$ where

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} -\dfrac{k_1 + k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_1} & -\dfrac{k_2}{m_1} \end{bmatrix}.$$

(b) If $\mathbf{X} = \mathbf{K}e^{\omega t}$ then $\mathbf{X}'' = \omega^2\mathbf{K}e^{\omega t}$ and $\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{K}e^{\omega t}$ so that $\mathbf{X}'' = \mathbf{A}\mathbf{X}$ becomes $\omega^2\mathbf{K}e^{\omega t} = \mathbf{A}\mathbf{K}e^{\omega t}$ or $(\mathbf{A} - \omega^2\mathbf{I})\mathbf{K} = 0$. Now let $\omega^2 = \lambda$.

(c) When $m_1 = 1$, $m_2 = 1$, $k_1 = 3$, and $k_2 = 2$ we obtain $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$. The eigenvalues and corresponding eigenvectors of $\mathbf{A}$ are $\lambda_1 = -1$, $\lambda_2 = -6$, $\mathbf{K}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{K}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Since $\omega_1 = i$, $\omega_2 = -i$, $\omega_3 = \sqrt{6}\, i$, and $\omega_4 = -\sqrt{6}\, i$ a solution is

$$\mathbf{X} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{it} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-it} + c_3 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{\sqrt{6}\, it} + c_4 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-\sqrt{6}\, it}.$$

Using $e^{it} = \cos t + i\sin t$ and $e^{\sqrt{6}\, it} = \cos\sqrt{6}\, t + i\sin\sqrt{6}\, t$ the preceding solution can be rewritten as

$$\mathbf{X} = b_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos t + b_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin t + b_3 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \cos\sqrt{6}\, t + b_4 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \sin\sqrt{6}\, t$$

where b_1, b_2, b_3, and b_4 are constants.

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Exercises 10.3

1. $\lambda_1 = 7,\ \lambda_2 = -4,\ \mathbf{K}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix};$

$$\mathbf{X} = \mathbf{PY} = \begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 e^{7t} \\ c_2 e^{-4t} \end{bmatrix} = \begin{bmatrix} 3c_1 e^{7t} - 2c_2 e^{-4t} \\ c_1 e^{7t} + 3c_2 e^{-4t} \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-4t}$$

2. $\lambda_1 = 0,\ \lambda_2 = 1,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix};$

$$\mathbf{X} = \mathbf{PY} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 e^{t} \end{bmatrix} = \begin{bmatrix} c_1 + c_2 e^{t} \\ -c_1 + c_2 e^{t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$$

3. $\lambda_1 = \frac{1}{2},\ \lambda_2 = \frac{3}{2},\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix};$

$$\mathbf{X} = \mathbf{PY} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} c_1 e^{t/2} \\ c_2 e^{3t/2} \end{bmatrix} = \begin{bmatrix} c_1 e^{t/2} + c_2 e^{3t/2} \\ -2c_1 e^{t/2} + 2c_2 e^{3t/2} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{t/2} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t/2}$$

4. $\lambda_1 = -\sqrt{2},\ \lambda_2 = \sqrt{2},\ \mathbf{K}_1 = \begin{bmatrix} -1 \\ 1+\sqrt{2} \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} -1 \\ 1-\sqrt{2} \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} -1 & -1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{bmatrix};$

$$\mathbf{X} = \mathbf{PY} = \begin{bmatrix} -1 & -1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{bmatrix} \begin{bmatrix} c_1 e^{-\sqrt{2}\,t} \\ c_2 e^{\sqrt{2}\,t} \end{bmatrix} = \begin{bmatrix} -c_1 e^{-\sqrt{2}\,t} - c_2 e^{\sqrt{2}\,t} \\ (1+\sqrt{2})c_1 e^{-\sqrt{2}\,t} + (1-\sqrt{2})c_2 e^{\sqrt{2}\,t} \end{bmatrix}$$

$$= c_1 \begin{bmatrix} -1 \\ 1+\sqrt{2} \end{bmatrix} e^{-\sqrt{2}\,t} + c_2 \begin{bmatrix} -1 \\ 1-\sqrt{2} \end{bmatrix} e^{\sqrt{2}\,t}$$

5. $\lambda_1 = -4,\ \lambda_2 = 2,\ \lambda_3 = 6,\ \mathbf{K}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix};$

$$\mathbf{X} = \mathbf{PY} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-4t} \\ c_2 e^{2t} \\ c_3 e^{6t} \end{bmatrix} = \begin{bmatrix} -c_1 e^{-4t} + c_2 e^{2t} \\ c_1 e^{-4t} + c_2 e^{2t} \\ c_2 e^{2t} + c_3 e^{6t} \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{6t}$$

6. $\lambda_1 = -1,\ \lambda_2 = 1,\ \lambda_3 = 4,\ \mathbf{K}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix};$

$$\mathbf{X} = \mathbf{PY} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{t} \\ c_3 e^{4t} \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 e^{t} + c_3 e^{4t} \\ -2c_2 e^{t} + c_3 e^{4t} \\ c_1 e^{-t} + c_2 e^{t} + c_3 e^{4t} \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t}$$

7. $\lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 2,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix};$

$$\mathbf{X} = \mathbf{PY} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{2t} \\ c_3 e^{2t} \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} - c_2 e^{2t} - c_3 e^{2t} \\ c_1 e^{-t} + c_2 e^{2t} \\ c_1 e^{-t} + c_3 e^{2t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

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8. $\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = 4,\ \mathbf{K}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{K}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$

$$\mathbf{P} = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix};$$

$$\mathbf{X} = \mathbf{PY} = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 e^{4t} \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 - c_3 + c_4 e^{4t} \\ c_1 + c_4 e^{4t} \\ c_2 + c_4 e^{4t} \\ c_3 + c_4 e^{4t} \end{bmatrix}$$

$$= c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} e^{4t}$$

9. $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix};$

$$\mathbf{X} = \mathbf{PY} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 e^t \\ c_2 e^{2t} \\ c_3 e^{3t} \end{bmatrix} = \begin{bmatrix} c_1 e^t + 2c_2 e^{2t} + 3c_3 e^{3t} \\ c_1 e^t + 2c_2 e^{2t} + 4c_3 e^{3t} \\ c_1 e^t + 3c_2 e^{2t} + 5c_3 e^{3t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} e^{3t}$$

10. $\lambda_1 = 0,\ \lambda_2 = -2\sqrt{2},\ \lambda_3 = 2\sqrt{2},\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix},\ \mathbf{K}_3 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -\sqrt{2} & \sqrt{2} \\ -1 & 1 & 1 \end{bmatrix};$

$$\mathbf{X} = \mathbf{PY} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -\sqrt{2} & \sqrt{2} \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 e^{-2\sqrt{2}\,t} \\ c_3 e^{2\sqrt{2}\,t} \end{bmatrix} = \begin{bmatrix} c_1 + c_2 e^{-2\sqrt{2}\,t} + c_3 e^{2\sqrt{2}\,t} \\ -\sqrt{2}\,c_2 e^{-2\sqrt{2}\,t} + \sqrt{2}\,c_3 e^{2\sqrt{2}\,t} \\ -c_1 + c_2 e^{-2\sqrt{2}\,t} + c_3 e^{2\sqrt{2}\,t} \end{bmatrix}$$

$$= c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} e^{-2\sqrt{2}\,t} + c_3 \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} e^{2\sqrt{2}\,t}$$

Exercises 10.4

1. From

$$\mathbf{X}' = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{pmatrix}$$

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so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} -11 \\ 5e^{-t} \end{pmatrix} dt = \begin{pmatrix} -11t \\ -5e^{-t} \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -11 \\ -11 \end{pmatrix} t + \begin{pmatrix} -15 \\ -10 \end{pmatrix}.$$

2. From

$$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} t$$

we obtain $$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{3}{2}e^{-t} & -\frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} -2te^{-t} \\ 2te^{t} \end{pmatrix} dt = \begin{pmatrix} 2te^{-t} + 2e^{-t} \\ 2te^{t} - 2e^{t} \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} t + \begin{pmatrix} 0 \\ -4 \end{pmatrix}.$$

3. From

$$\mathbf{X}' = \begin{pmatrix} 3 & -5 \\ 3/4 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2}$$

we obtain $$\mathbf{X}_c = c_1 \begin{pmatrix} 10 \\ 3 \end{pmatrix} e^{3t/2} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/2}.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} 10e^{3t/2} & 2e^{t/2} \\ 3e^{3t/2} & e^{t/2} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{4}e^{-3t/2} & -\frac{1}{2}e^{-3t/2} \\ -\frac{3}{4}e^{-t/2} & \frac{5}{2}e^{-t/2} \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \frac{3}{4}e^{-t} \\ -\frac{13}{4} \end{pmatrix} dt = \begin{pmatrix} -\frac{3}{4}e^{-t} \\ -\frac{13}{4}t \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -13/2 \\ -13/4 \end{pmatrix} te^{t/2} + \begin{pmatrix} -15/2 \\ -9/4 \end{pmatrix} e^{t/2}.$$

4. From

$$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}$$

we obtain $$\mathbf{X}_c = c_1 \begin{pmatrix} -\sin 2t \\ 2\cos 2t \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} e^{2t}.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} -e^{2t}\sin 2t & e^{2t}\cos 2t \\ 2e^{2t}\cos 2t & 2e^{2t}\sin 2t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\frac{1}{2}e^{-2t}\sin 2t & \frac{1}{4}e^{-2t}\cos 2t \\ \frac{1}{2}e^{-2t}\cos 2t & \frac{1}{4}e^{-2t}\sin 2t \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \frac{1}{2}\cos 4t \\ \frac{1}{2}\sin 4t \end{pmatrix} dt = \begin{pmatrix} \frac{1}{8}\sin 4t \\ -\frac{1}{8}\cos 4t \end{pmatrix}$$

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and $$\mathbf{X}_p = \mathbf{\Phi U} = \begin{pmatrix} -\frac{1}{8}\sin 2t\cos 4t - \frac{1}{8}\cos 2t\cos 4t \\ \frac{1}{4}\cos 2t\sin 4t - \frac{1}{4}\sin 2t\cos 4t \end{pmatrix} e^{2t}.$$

5. From

$$\mathbf{X}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

we obtain $$\mathbf{X}_c = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 2 \\ -3e^{-t} \end{pmatrix} dt = \begin{pmatrix} 2t \\ 3e^{-t} \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi U} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^t.$$

6. From

$$\mathbf{X}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 2 \\ e^{-3t} \end{pmatrix}$$

we obtain $$\mathbf{X}_c = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 2e^{-t} - e^{-4t} \\ -2e^{-2t} + 2e^{-5t} \end{pmatrix} dt = \begin{pmatrix} -2e^{-t} + \frac{1}{4}e^{-4t} \\ e^{-2t} - \frac{2}{5}e^{-5t} \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi U} = \begin{pmatrix} \frac{1}{10}e^{-3t} - 3 \\ -\frac{3}{20}e^{-3t} - 1 \end{pmatrix}.$$

7. From

$$\mathbf{X}' = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 12 \\ 12 \end{pmatrix} t$$

we obtain $$\mathbf{X}_c = c_1\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t}.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t} \\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t} \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 6te^{-3t} \\ 6te^{3t} \end{pmatrix} dt = \begin{pmatrix} -2te^{-3t} - \frac{2}{3}e^{-3t} \\ 2te^{3t} - \frac{2}{3}e^{3t} \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi U} = \begin{pmatrix} -12 \\ 0 \end{pmatrix} t + \begin{pmatrix} -4/3 \\ -4/3 \end{pmatrix}.$$

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8. From

$$\mathbf{X}' = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} e^{-t} \\ te^{t} \end{pmatrix}$$

we obtain $$\mathbf{X}_c = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t}.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} 4e^{3t} & -2e^{3t} \\ e^{3t} & e^{-3t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t} \\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t} \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} \frac{1}{6}e^{-4t} + \frac{1}{3}te^{-2t} \\ -\frac{1}{6}e^{2t} + \frac{2}{3}te^{4t} \end{pmatrix} dt = \begin{pmatrix} -\frac{1}{24}e^{-4t} - \frac{1}{6}te^{-2t} - \frac{1}{12}e^{-2t} \\ -\frac{1}{12}e^{2t} + \frac{1}{6}te^{4t} - \frac{1}{24}e^{4t} \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -te^{t} - \frac{1}{4}e^{t} \\ -\frac{1}{8}e^{-t} - \frac{1}{8}e^{t} \end{pmatrix}.$$

9. From

$$\mathbf{X}' = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$$

we obtain $$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{t} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} e^{t} \right].$$

Then $$\mathbf{\Phi} = \begin{pmatrix} e^{t} & te^{t} \\ -e^{t} & \frac{1}{2}e^{t} - te^{t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} e^{-t} - 2te^{-t} & -2te^{-t} \\ 2e^{-t} & 2e^{-t} \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} 2e^{-2t} - 6te^{-2t} \\ 6e^{-2t} \end{pmatrix} dt = \begin{pmatrix} \frac{1}{2}e^{-2t} + 3te^{-2t} \\ -3e^{-2t} \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 1/2 \\ -2 \end{pmatrix} e^{-t}.$$

10. From

$$\mathbf{X}' = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

we obtain $$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{t} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} e^{t} \right].$$

Then $$\mathbf{\Phi} = \begin{pmatrix} e^{t} & te^{t} \\ -e^{t} & \frac{1}{2}e^{t} - te^{t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} e^{-t} - 2te^{-t} & -2te^{-t} \\ 2e^{-t} & 2e^{-t} \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} e^{-t} - 4te^{-t} \\ 2e^{-t} \end{pmatrix} dt = \begin{pmatrix} 3e^{-t} + 4te^{-t} \\ -2e^{-t} \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}.$$

11. From

$$\mathbf{X}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \sec t \\ 0 \end{pmatrix}$$

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we obtain $$\mathbf{X}_c = c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} 1 \\ \tan t \end{pmatrix} dt = \begin{pmatrix} t \\ \ln|\sec t| \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} t\cos t + \sin t \ln|\sec t| \\ t\sin t - \cos t \ln|\sec t| \end{pmatrix}.$$

12. From

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^t$$

we obtain $$\mathbf{X}_c = c_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} e^t \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} e^{-t}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} -3\sin t + 3\cos t \\ 3\cos t + 3\sin t \end{pmatrix} dt = \begin{pmatrix} 3\cos t + 3\sin t \\ 3\sin t - 3\cos t \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} e^t.$$

13. From

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t$$

we obtain $$\mathbf{X}_c = c_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} e^t \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} e^{-t}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = \begin{pmatrix} 0 \\ t \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} te^t.$$

14. From

$$\mathbf{X}' = \begin{pmatrix} 2 & -2 \\ 8 & -6 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} \frac{1}{t} e^{-2t}$$

we obtain $$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{-2t} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} e^{-2t} \right].$$

Then $$\mathbf{\Phi} = \begin{pmatrix} 1 & 2t+1 \\ 2 & 4t+1 \end{pmatrix} e^{-2t} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -(4t+1) & 2t+1 \\ 2 & -1 \end{pmatrix} e^{2t}$$

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so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 2t + 2\ln t \\ -\ln t \end{pmatrix} dt$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 2t + \ln t - 2t\ln t \\ 4t + 3\ln t - 4t\ln t \end{pmatrix} e^{-2t}.$$

15. From

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 0 \\ \sec t\tan t \end{pmatrix}$$

we obtain $$\mathbf{X}_c = c_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} t \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} -\tan^2 t \\ \tan t \end{pmatrix} dt = \begin{pmatrix} t - \tan t \\ \ln|\sec t| \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} t + \begin{pmatrix} -\sin t \\ \sin t\tan t \end{pmatrix} + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \ln|\sec t|.$$

16. From

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1 \\ \cot t \end{pmatrix}$$

we obtain $$\mathbf{X}_c = c_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 0 \\ \csc t \end{pmatrix} dt = \begin{pmatrix} 0 \\ \ln|\csc t - \cot t| \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} \sin t\ln|\csc t - \cot t| \\ \cos t\ln|\csc t - \cot t| \end{pmatrix}.$$

17. From

$$\mathbf{X}' = \begin{pmatrix} 1 & 2 \\ -1/2 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \csc t \\ \sec t \end{pmatrix} e^t$$

we obtain $$\mathbf{X}_c = c_1\begin{pmatrix} 2\sin t \\ \cos t \end{pmatrix} e^t + c_2\begin{pmatrix} 2\cos t \\ -\sin t \end{pmatrix} e^t.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} 2\sin t & 2\cos t \\ \cos t & -\sin t \end{pmatrix} e^t \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{2}\sin t & \cos t \\ \frac{1}{2}\cos t & -\sin t \end{pmatrix} e^{-t}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2}\cos t - \tan t \end{pmatrix} dt = \begin{pmatrix} \frac{3}{2}t \\ \frac{1}{2}\ln|\sin t| - \ln|\sec t| \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 3\sin t \\ \frac{3}{2}\cos t \end{pmatrix} te^t + \begin{pmatrix} \cos t \\ -\frac{1}{2}\sin t \end{pmatrix} \ln|\sin t| + \begin{pmatrix} -2\cos t \\ \sin t \end{pmatrix} \ln|\sec t|.$$

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18. From

$$\mathbf{X}' = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \tan t \\ 1 \end{pmatrix}$$

we obtain $$\mathbf{X}_c = c_1 \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix}.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} \cos t - \sin t & \cos t + \sin t \\ \cos t & \sin t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\sin t & \cos t + \sin t \\ \cos t & \sin t - \cos t \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} 2\cos t + \sin t - \sec t \\ 2\sin t - \cos t \end{pmatrix} dt = \begin{pmatrix} 2\sin t - \cos t - \ln|\sec t + \tan t| \\ -2\cos t - \sin t \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 3\sin t\cos t - \cos^2 t - 2\sin^2 t + (\sin t - \cos t)\ln|\sec t + \tan t| \\ \sin^2 t - \cos^2 t - \cos t(\ln|\sec t + \tan t|) \end{pmatrix}.$$

19. From

$$\mathbf{X}' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{X} + \begin{pmatrix} e^t \\ e^{2t} \\ te^{3t} \end{pmatrix}$$

we obtain $$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t}.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} 1 & e^{2t} & 0 \\ -1 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix}$$

so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} \frac{1}{2}e^t - \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{-t} + \frac{1}{2} \\ t \end{pmatrix} dt = \begin{pmatrix} \frac{1}{2}e^t - \frac{1}{4}e^{2t} \\ -\frac{1}{2}e^{-t} + \frac{1}{2}t \\ \frac{1}{2}t^2 \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -\frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} \\ -e^t + \frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} \\ \frac{1}{2}t^2e^{3t} \end{pmatrix}.$$

20. From

$$\mathbf{X}' = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ t \\ 2e^t \end{pmatrix}$$

we obtain $$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

Then $$\mathbf{\Phi} = \begin{pmatrix} e^t & e^{2t} & e^{2t} \\ e^t & e^{2t} & 0 \\ e^t & 0 & e^{2t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -e^{-t} & e^{-t} & e^{-t} \\ e^{-2t} & 0 & -e^{-2t} \\ e^{-2t} & -e^{-2t} & 0 \end{pmatrix}$$

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so that $$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} te^{-t} + 2 \\ -2e^{-t} \\ -te^{-2t} \end{pmatrix} dt = \begin{pmatrix} -te^{-t} - e^{-t} + 2t \\ 2e^{-t} \\ \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} \end{pmatrix}$$

and $$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -1/2 \\ -1 \\ -1/2 \end{pmatrix} t + \begin{pmatrix} -3/4 \\ -1 \\ -3/4 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} te^t.$$

21. From

$$\mathbf{X}' = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 4e^{2t} \\ 4e^{4t} \end{pmatrix}$$

we obtain $$\mathbf{\Phi} = \begin{pmatrix} -e^{4t} & e^{2t} \\ e^{4t} & e^{2t} \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\frac{1}{2}e^{-4t} & \frac{1}{2}e^{-4t} \\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} \end{pmatrix},$$

and $$\mathbf{X} = \mathbf{\Phi}\mathbf{\Phi}^{-1}(0)\mathbf{X}(0) + \mathbf{\Phi}\int_0^t \mathbf{\Phi}^{-1}\mathbf{F}\,ds = \mathbf{\Phi}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathbf{\Phi}\cdot\begin{pmatrix} e^{-2t} + 2t - 1 \\ e^{2t} + 2t - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{4t}.$$

22. From

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1/t \\ 1/t \end{pmatrix}$$

we obtain $$\mathbf{\Phi} = \begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -t & 1+t \\ 1 & -1 \end{pmatrix},$$

and

$$\mathbf{X} = \mathbf{\Phi}\mathbf{\Phi}^{-1}(1)\mathbf{X}(1) + \mathbf{\Phi}\int_1^t \mathbf{\Phi}^{-1}\mathbf{F}\,ds = \mathbf{\Phi}\cdot\begin{pmatrix} -4 \\ 3 \end{pmatrix} + \mathbf{\Phi}\cdot\begin{pmatrix} \ln t \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} t - \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ln t.$$

23. $\lambda_1 = -1$, $\lambda_2 = -2$, $\mathbf{K}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{K}_2 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$, $\mathbf{P} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$, $\mathbf{P}^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$, $\mathbf{P}^{-1}\mathbf{F} = \begin{bmatrix} 34 \\ -14 \end{bmatrix}$;

$$\mathbf{Y}' = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{Y} + \begin{bmatrix} 34 \\ -14 \end{bmatrix}$$

$$y_1 = 34 + c_1e^{-t}, \qquad y_2 = -7 + c_2e^{-2t}$$

$$\mathbf{X} = \mathbf{P}\mathbf{Y} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 34 + c_1e^{-t} \\ -7 + c_2e^{-2t} \end{bmatrix} = \begin{bmatrix} 20 + c_1e^{-t} + 2c_2e^{-2t} \\ 53 + 3c_1e^{-t} + 7c_2e^{-2t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 7 \end{bmatrix} e^{-2t} + \begin{bmatrix} 20 \\ 53 \end{bmatrix}$$

24. $\lambda_1 = -1$, $\lambda_2 = 4$, $\mathbf{K}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{P} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$, $\mathbf{P}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{P}^{-1}\mathbf{F} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$;

$$\mathbf{Y}' = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{Y} + \begin{bmatrix} 0 \\ e^t \end{bmatrix}$$

$$y_1 = c_1e^{-t}, \qquad y_2 = -\frac{1}{3}e^t + c_2e^{4t}$$

$$\mathbf{X} = \mathbf{P}\mathbf{Y} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} c_1e^{-t} \\ -\frac{1}{3}e^t + c_2e^{4t} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}e^t + 3c_1e^{-t} + c_2e^{4t} \\ -\frac{1}{3}e^t - 2c_1e^{-t} + c_2e^{4t} \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$$

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25. $\lambda_1 = 0,\ \lambda_2 = 10,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},\ \mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\ \mathbf{P}^{-1}\mathbf{F} = \begin{bmatrix} t-4 \\ t+4 \end{bmatrix};$

$$\mathbf{Y}' = \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}\mathbf{Y} + \begin{bmatrix} t-4 \\ t+4 \end{bmatrix}$$

$$y_1 = \frac{1}{2}t^2 - 4t + c_1, \qquad y_2 = -\frac{1}{10}t - \frac{41}{100} + c_2e^{10t}$$

$$\mathbf{X} = \mathbf{PY} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2}t^2 - 4t + c_1 \\ -\frac{1}{10}t - \frac{41}{100} + c_2e^{10t} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t^2 - \frac{41}{10}t - \frac{41}{100} + c_1 + c_2e^{10t} \\ -\frac{1}{2}t^2 + \frac{39}{10}t - \frac{41}{100} - c_1 + c_2e^{10t} \end{bmatrix}$$

$$= c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{10t} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}t^2 + \frac{1}{10}\begin{bmatrix} -41 \\ 39 \end{bmatrix}t - \frac{41}{100}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

26. $\lambda_1 = -1,\ \lambda_2 = 1,\ \mathbf{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},\ \mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\ \mathbf{P}^{-1}\mathbf{F} = \begin{bmatrix} 2 - 4e^{-2t} \\ 2 + 4e^{-2t} \end{bmatrix};$

$$\mathbf{Y}' = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{Y} + \begin{bmatrix} 2 - 4e^{-2t} \\ 2 + 4e^{-2t} \end{bmatrix}$$

$$y_1 = 2 + 4e^{-2t} + c_1e^{-t}, \qquad y_2 = -2 - \frac{4}{3}e^{-2t} + c_2e^{t}$$

$$\mathbf{X} = \mathbf{PY} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 + 4e^{-2t} + c_1e^{-t} \\ -2 - \frac{4}{3}e^{-2t} + c_2e^{t} \end{bmatrix} = \begin{bmatrix} \frac{8}{3}e^{-2t} + c_1e^{-t} + c_2e^{t} \\ -4 - \frac{16}{3}e^{-2t} - c_1e^{-t} + c_2e^{t} \end{bmatrix}$$

$$= c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t} + \frac{8}{3}\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-2t} + \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

27. Let $\mathbf{I} = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$ so that

$$\mathbf{I}' = \begin{pmatrix} -11 & 3 \\ 3 & -3 \end{pmatrix}\mathbf{I} + \begin{pmatrix} 100\sin t \\ 0 \end{pmatrix}$$

and

$$\mathbf{X}_c = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} 3 \\ -1 \end{pmatrix}e^{-12t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} e^{-2t} & 3e^{-12t} \\ 3e^{-2t} & -e^{-12t} \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{10}e^{2t} & \frac{3}{10}e^{2t} \\ \frac{3}{10}e^{12t} & -\frac{1}{10}e^{12t} \end{pmatrix},$$

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 10e^{2t}\sin t \\ 30e^{12t}\sin t \end{pmatrix} dt = \begin{pmatrix} 2e^{2t}(2\sin t - \cos t) \\ \frac{6}{29}e^{12t}(12\sin t - \cos t) \end{pmatrix},$$

and

$$\mathbf{I}_p = \mathbf{\Phi U} = \begin{pmatrix} \frac{332}{29}\sin t - \frac{76}{29}\cos t \\ \frac{276}{29}\sin t - \frac{168}{29}\cos t \end{pmatrix}$$

so that

$$\mathbf{I} = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} 3 \\ -1 \end{pmatrix}e^{-12t} + \mathbf{I}_p.$$

If $\mathbf{I}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then $c_1 = 2$ and $c_2 = \frac{6}{29}$.

28. For the system

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ q \end{bmatrix} = \begin{bmatrix} -\frac{20}{3} & -\frac{50}{3} \\ \frac{1}{3} & -\frac{5}{3} \end{bmatrix}\begin{bmatrix} i_1 \\ q \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

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we find $\lambda_1 = -5$, $\lambda_2 = -\frac{10}{3}$, $\mathbf{K}_1 = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$, $\mathbf{K}_2 = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, $\mathbf{X}_p = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{25} \end{bmatrix}$ and so

$$\mathbf{X} = \begin{bmatrix} i_1 \\ q \end{bmatrix} = c_1 \begin{bmatrix} 10 \\ -1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 5 \\ -1 \end{bmatrix} e^{-10t/3} + \frac{1}{25} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

29. Let $\mathbf{I} = \begin{bmatrix} i_2 \\ i_3 \end{bmatrix}$ so that

$$\mathbf{I}' = \begin{bmatrix} -2 & -2 \\ -2 & -5 \end{bmatrix} \mathbf{I} + \begin{bmatrix} 60 \\ 60 \end{bmatrix} \quad \text{and} \quad \mathbf{X}_c = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-6t}.$$

If $\mathbf{X}_p = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ then $\mathbf{X}_p = \begin{bmatrix} 30 \\ 0 \end{bmatrix}$ so that $\mathbf{X} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-6t} + \begin{bmatrix} 30 \\ 0 \end{bmatrix}$. For $\mathbf{I}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

we find $c_1 = -12$ and $c_2 = -6$. Thus

$$i_1(t) = i_2(t) + i_3(t) = -12e^{-t} - 18e^{-6t} + 30.$$

30. **(a)** The eigenvalues are 0, 1, 3, and 4, with corresponding eigenvectors

$$\begin{pmatrix} -6 \\ -4 \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

(b) $\mathbf{\Phi} = \begin{pmatrix} -6 & 2e^t & 3e^{3t} & -e^{4t} \\ -4 & e^t & e^{3t} & e^{4t} \\ 1 & 0 & 2e^{3t} & 0 \\ 2 & 0 & e^{3t} & 0 \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} 0 & 0 & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3}e^{-t} & \frac{1}{3}e^{-t} & -2e^{-t} & \frac{8}{3}e^{-t} \\ 0 & 0 & \frac{2}{3}e^{-3t} & -\frac{1}{3}e^{-3t} \\ -\frac{1}{3}e^{-4t} & \frac{2}{3}e^{-4t} & 0 & \frac{1}{3}e^{-4t} \end{pmatrix}$

(c) $\mathbf{\Phi}^{-1}(t)\mathbf{F}(t) = \begin{pmatrix} \frac{2}{3} - \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{-2t} + \frac{8}{3}e^{-t} - 2e^t + \frac{1}{3}t \\ -\frac{1}{3}e^{-3t} + \frac{2}{3}e^{-t} \\ \frac{2}{3}e^{-5t} + \frac{1}{3}e^{-4t} - \frac{1}{3}te^{-3t} \end{pmatrix},$

$$\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt = \begin{pmatrix} -\frac{1}{6}e^{2t} + \frac{2}{3}t \\ -\frac{1}{6}e^{-2t} - \frac{8}{3}e^{-t} - 2e^t + \frac{1}{6}t^2 \\ \frac{1}{9}e^{-3t} - \frac{2}{3}e^{-t} \\ -\frac{2}{15}e^{-5t} - \frac{1}{12}e^{-4t} + \frac{1}{27}e^{-3t} + \frac{1}{9}te^{-3t} \end{pmatrix},$$

$$\mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt = \begin{pmatrix} -5e^{2t} - \frac{1}{5}e^{-t} - \frac{1}{27}e^t - \frac{1}{9}te^t + \frac{1}{3}t^2e^t - 4t - \frac{59}{12} \\ -2e^{2t} - \frac{3}{10}e^{-t} + \frac{1}{27}e^t + \frac{1}{9}te^t + \frac{1}{6}t^2e^t - \frac{8}{3}t - \frac{95}{36} \\ -\frac{3}{2}e^{2t} + \frac{2}{3}t + \frac{2}{9} \\ -e^{2t} + \frac{4}{3}t - \frac{1}{9} \end{pmatrix},$$

$$\mathbf{\Phi}(t)\mathbf{C} = \begin{pmatrix} -6c_1 + 2c_2e^t + 3c_3e^{3t} - c_4e^{4t} \\ -4c_1 + c_2e^t + c_3e^{3t} + c_4e^{4t} \\ c_1 + 2c_3e^{3t} \\ 2c_1 + c_3e^{3t} \end{pmatrix},$$

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$$\mathbf{\Phi}(t)\mathbf{C} + \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt$$

$$= \begin{pmatrix} -6c_1 + 2c_2e^t + 3c_3e^{3t} - c_4e^{4t} \\ -4c_1 + c_2e^t + c_3e^{3t} + c_4e^{4t} \\ c_1 + 2c_3e^{3t} \\ 2c_1 + c_3e^{3t} \end{pmatrix} + \begin{pmatrix} -5e^{2t} - \frac{1}{5}\,e^{-t} - \frac{1}{27}\,e^t - \frac{1}{9}\,te^t + \frac{1}{3}\,t^2e^t - 4t - \frac{59}{12} \\ -2e^{2t} - \frac{3}{10}\,e^{-t} + \frac{1}{27}e^t + \frac{1}{9}\,te^t + \frac{1}{6}\,t^2e^t - \frac{8}{3}t - \frac{95}{36} \\ -\frac{3}{2}\,e^{2t} + \frac{2}{3}\,t + \frac{2}{9} \\ -e^{2t} + \frac{4}{3}\,t - \frac{1}{9} \end{pmatrix}$$

(d) $\mathbf{X}(t) = c_1\begin{pmatrix} -6 \\ -4 \\ 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix} + c_4\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$+ \begin{pmatrix} -5e^{2t} - \frac{1}{5}\,e^{-t} - \frac{1}{27}\,e^t - \frac{1}{9}\,te^t + \frac{1}{3}\,t^2e^t - 4t - \frac{59}{12} \\ -2e^{2t} - \frac{3}{10}\,e^{-t} + \frac{1}{27}e^t + \frac{1}{9}\,te^t + \frac{1}{6}\,t^2e^t - \frac{8}{3}t - \frac{95}{36} \\ -\frac{3}{2}\,e^{2t} + \frac{2}{3}\,t + \frac{2}{9} \\ -e^{2t} + \frac{4}{3}\,t - \frac{1}{9} \end{pmatrix}$$

Exercises 10.5

1. For $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ we have

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix},$$

$$\mathbf{A}^4 = \mathbf{A}\mathbf{A}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix},$$

and so on. In general

$$\mathbf{A}^k = \begin{pmatrix} 1 & 0 \\ 0 & 2^k \end{pmatrix} \quad \text{for} \quad k = 1,\, 2,\, 3,\, \ldots\,.$$

Thus

$$e^{\mathbf{A}t} = \mathbf{I} + \frac{\mathbf{A}}{1!}t + \frac{\mathbf{A}^2}{2!}t^2 + \frac{\mathbf{A}^3}{3!}t^3 + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{1!}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}t + \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}t^2 + \frac{1}{3!}\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}t^3 + \cdots$$

$$= \begin{pmatrix} 1 + t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + \cdots & 0 \\ 0 & 1 + t + \dfrac{(2t)^2}{2!} + \dfrac{(2t)^3}{3!} + \cdots \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$

and

$$e^{-\mathbf{A}t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}.$$

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2. For $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we have

$$\mathbf{A}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbf{A}$$

$$\mathbf{A}^4 = (\mathbf{A}^2)^2 = \mathbf{I}$$

$$\mathbf{A}^5 = \mathbf{A}\mathbf{A}^4 = \mathbf{A}\mathbf{I} = \mathbf{A}$$

and so on. In general

$$\mathbf{A}^k = \begin{cases} \mathbf{A}, & k = 1, 3, 5, \ldots \\ \mathbf{I}, & k = 2, 4, 6, \ldots \end{cases}.$$

Thus

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \frac{\mathbf{A}}{1!}t + \frac{\mathbf{A}^2}{2!}t^2 + \frac{\mathbf{A}^3}{3!}t^3 + \cdots \\ &= \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{I}t^2 + \frac{1}{3!}\mathbf{A}t^3 + \cdots \\ &= \mathbf{I}\left(1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \cdots\right) + \mathbf{A}\left(t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots\right) \\ &= \mathbf{I}\cosh t + \mathbf{A}\sinh t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \end{aligned}$$

and

$$e^{-\mathbf{A}t} = \begin{pmatrix} \cosh(-t) & \sinh(-t) \\ \sinh(-t) & \cosh(-t) \end{pmatrix} = \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}.$$

3. For

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix}$$

we have

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathbf{A}^3 = \mathbf{A}^4 = \mathbf{A}^5 = \cdots = \mathbf{0}$ and

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} t & t & t \\ t & t & t \\ -2t & -2t & -2t \end{pmatrix} = \begin{pmatrix} t+1 & t & t \\ t & t+1 & t \\ -2t & -2t & -2t+1 \end{pmatrix}.$$

4. For

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}$$

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$$\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathbf{A}^4 = \mathbf{A}^5 = \mathbf{A}^6 = \cdots = \mathbf{0}$ and

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 3t & 0 & 0 \\ 5t & t & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3}{2}t^2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ \frac{3}{2}t^2 + 5t & t & 1 \end{pmatrix}. \end{aligned}$$

5. Using the result of Problem 1

$$\mathbf{X} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1\begin{pmatrix} e^t \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

6. Using the result of Problem 2

$$\mathbf{X} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}.$$

7. Using the result of Problem 3

$$\mathbf{X} = \begin{pmatrix} t+1 & t & t \\ t & t+1 & t \\ -2t & -2t & -2t+1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1\begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} + c_2\begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + c_3\begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}.$$

8. Using the result of Problem 4

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ \frac{3}{2}t^2 + 5t & t & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 3t \\ \frac{3}{2}t^2 + 5t \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

9. To solve

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

we identify $t_0 = 0$, $\mathbf{F}(s) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, and use the results of Problem 1 and equation (6) in the text.

$$\begin{aligned} \mathbf{X}(t) &= e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds \\ &= \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\int_0^t \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} ds \\ &= \begin{pmatrix} c_1e^t \\ c_2e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\int_0^t \begin{pmatrix} 3e^{-s} \\ -e^{-2s} \end{pmatrix} ds \end{aligned}$$

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$$= \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -3e^{-s} \\ \frac{1}{2}e^{-2s} \end{pmatrix} \Big|_0^t$$

$$= \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -3e^{-t} - 3 \\ \frac{1}{2}e^{-2t} - \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} -3 - 3e^t \\ \frac{1}{2} - \frac{1}{2}e^{2t} \end{pmatrix} = c_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix}.$$

10. To solve

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} t \\ e^{4t} \end{pmatrix}$$

we identify $t_0 = 0$, $\mathbf{F}(s) = \begin{pmatrix} t \\ e^{4t} \end{pmatrix}$, and use the results of Problem 1 and equation (6) in the text.

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t} \int_{t_0}^t e^{-\mathbf{A}s}\mathbf{F}(s)\, ds$$

$$= \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{pmatrix} \begin{pmatrix} s \\ e^{4s} \end{pmatrix} ds$$

$$= \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \int_0^t \begin{pmatrix} se^{-s} \\ e^{2s} \end{pmatrix} ds$$

$$= \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -se^{-s} - e^{-s} \\ \frac{1}{2}e^{2s} \end{pmatrix} \Big|_0^t$$

$$= \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -te^{-t} - e^{-t} + 1 \\ \frac{1}{2}e^{2t} - \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} -t - 1 + e^t \\ \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} \end{pmatrix} = c_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -t - 1 \\ \frac{1}{2}e^{4t} \end{pmatrix}.$$

11. To solve

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

we identify $t_0 = 0$, $\mathbf{F}(s) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and use the results of Problem 2 and equation (6) in the text.

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t} \int_{t_0}^t e^{-\mathbf{A}s}\mathbf{F}(s)\, ds$$

$$= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \int_0^t \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} ds$$

$$= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \int_0^t \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix} ds$$

$$= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} \sinh s - \cosh s \\ -\cosh s + \sinh s \end{pmatrix} \Big|_0^t$$

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$$= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} \sinh t - \cosh t \\ -\cosh t + \sinh t \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \sinh^2 t - \cosh^2 t \\ \sinh^2 t - \cosh^2 t \end{pmatrix} = c_1 \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2 \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

12. To solve

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}$$

we identify $t_0 = 0$, $\mathbf{F}(s) = \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}$, and use the results of Problem 2 and equation (6) in the text.

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t} \int_{t_0}^{t} e^{-\mathbf{A}s} \mathbf{F}(s)\, ds$$

$$= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \int_0^t \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix} \begin{pmatrix} \cosh s \\ \sinh s \end{pmatrix} ds$$

$$= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \int_0^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} ds$$

$$= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} s \\ 0 \end{pmatrix} \Bigg|_0^t$$

$$= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} t \cosh t \\ t \sinh t \end{pmatrix} = c_1 \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2 \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} + t \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}.$$

13. We have

$$\mathbf{X}(0) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix}.$$

Thus, the solution of the initial-value problem is

$$\mathbf{X} = \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} - 4 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + 6 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}.$$

14. We have

$$\mathbf{X}(0) = c_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} c_3 - 3 \\ c_4 + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Thus, $c_3 = 7$ and $c_4 = \frac{5}{2}$, so

$$\mathbf{X} = 7 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \frac{5}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix}.$$

15. From $s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s-4 & -3 \\ 4 & s+4 \end{pmatrix}$ we find

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} \dfrac{3/2}{s-2} - \dfrac{1/2}{s+2} & \dfrac{3/4}{s-2} - \dfrac{3/4}{s+2} \\ \dfrac{-1}{s-2} + \dfrac{1}{s+2} & \dfrac{-1/2}{s-2} + \dfrac{3/2}{s+2} \end{pmatrix}$$

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and

$$e^{\mathbf{A}t} = \begin{pmatrix} \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t} & \frac{3}{4}e^{2t} - \frac{3}{4}e^{-2t} \\ -e^{2t} + e^{-2t} & -\frac{1}{2}e^{2t} + \frac{3}{2}e^{-2t} \end{pmatrix}.$$

The general solution of the system is then

$$\begin{aligned} \mathbf{X} = e^{\mathbf{A}t}\mathbf{C} &= \begin{pmatrix} \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t} & \frac{3}{4}e^{2t} - \frac{3}{4}e^{-2t} \\ -e^{2t} + e^{-2t} & -\frac{1}{2}e^{2t} + \frac{3}{2}e^{-2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 3/2 \\ -1 \end{pmatrix} e^{2t} + c_1 \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3/4 \\ -1/2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -3/4 \\ 3/2 \end{pmatrix} e^{-2t} \\ &= \left(\frac{1}{2}c_1 + \frac{1}{4}c_2\right) \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{2t} + \left(-\frac{1}{2}c_1 - \frac{3}{4}c_2\right) \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} \\ &= c_3 \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{2t} + c_4 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}. \end{aligned}$$

16. From $s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s-4 & 2 \\ -1 & s-1 \end{pmatrix}$ we find

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} \frac{2}{s-3} - \frac{1}{s-2} & -\frac{2}{s-3} + \frac{2}{s-2} \\ \frac{1}{s-3} - \frac{1}{s-2} & \frac{-1}{s-3} + \frac{2}{s-2} \end{pmatrix}$$

and

$$e^{\mathbf{A}t} = \begin{pmatrix} 2e^{3t} - e^{2t} & -2e^{3t} + 2e^{2t} \\ e^{3t} - e^{2t} & -e^{3t} + 2e^{2t} \end{pmatrix}.$$

The general solution of the system is then

$$\begin{aligned} \mathbf{X} = e^{\mathbf{A}t}\mathbf{C} &= \begin{pmatrix} 2e^{3t} - e^{2t} & -2e^{3t} + 2e^{2t} \\ e^{3t} - e^{2t} & -e^{3t} + 2e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + c_1 \begin{pmatrix} -1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -2 \\ -1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{2t} \\ &= (c_1 - c_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + (-c_1 + 2c_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \\ &= c_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + c_4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}. \end{aligned}$$

17. From $s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s-5 & 9 \\ -1 & s+1 \end{pmatrix}$ we find

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} \frac{1}{s-2} + \frac{3}{(s-2)^2} & -\frac{9}{(s-2)^2} \\ \frac{1}{(s-2)^2} & \frac{1}{s-2} - \frac{3}{(s-2)^2} \end{pmatrix}$$

and

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{2t} + 3te^{2t} & -9te^{2t} \\ te^{2t} & e^{2t} - 3te^{2t} \end{pmatrix}.$$

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$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} = \begin{pmatrix} e^{2t} + 3te^{2t} & -9te^{2t} \\ te^{2t} & e^{2t} - 3te^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -9 \\ -3 \end{pmatrix} te^{2t}$$
$$= c_1 \begin{pmatrix} 1 + 3t \\ t \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -9t \\ 1 - 3t \end{pmatrix} e^{2t}.$$

18. From $s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s & -1 \\ 2 & s+2 \end{pmatrix}$ we find

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} \dfrac{s+1+1}{(s+1)^2+1} & \dfrac{1}{(s+1)^2+1} \\ \dfrac{-2}{(s+1)^2+1} & \dfrac{s+1+1}{(s+1)^2+1} \end{pmatrix}$$

and

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{-t}\cos t + e^{-t}\sin t & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}\cos t - e^{-t}\sin t \end{pmatrix}.$$

The general solution of the system is then

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} = \begin{pmatrix} e^{-t}\cos t + e^{-t}\sin t & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}\cos t - e^{-t}\sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}\cos t + c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}\sin t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}\cos t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}\sin t$$
$$= c_1 \begin{pmatrix} \cos t + \sin t \\ -2\sin t \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix} e^{-t}.$$

19. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 6$. This leads to the system

$$e^t = b_0 + b_1$$
$$e^{6t} = b_0 + 6b_1,$$

which has the solution $b_0 = \frac{6}{5}e^t - \frac{1}{5}e^{6t}$ and $b_1 = -\frac{1}{5}e^t + \frac{1}{5}e^{6t}$. Then

$$e^{\mathbf{A}t} = b_0\mathbf{I} + b_1\mathbf{A} = \begin{pmatrix} \frac{4}{5}e^t + \frac{1}{5}e^{6t} & \frac{2}{5}e^t - \frac{2}{5}e^{6t} \\ \frac{2}{5}e^t - \frac{2}{5}e^{6t} & \frac{1}{5}e^t + \frac{4}{5}e^{6t} \end{pmatrix}.$$

The general solution of the system is then

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} = \begin{pmatrix} \frac{4}{5}e^t + \frac{1}{5}e^{6t} & \frac{2}{5}e^t - \frac{2}{5}e^{6t} \\ \frac{2}{5}e^t - \frac{2}{5}e^{6t} & \frac{1}{5}e^t + \frac{4}{5}e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= c_1 \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} e^t + c_1 \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix} e^t + c_2 \begin{pmatrix} -2/5 \\ 4/5 \end{pmatrix} e^{6t}$$
$$= \left(\frac{2}{5}c_1 + \frac{1}{5}c_2\right) \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + \left(\frac{1}{5}c_1 - \frac{2}{5}c_2\right) \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{6t}$$
$$= c_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_4 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{6t}.$$

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20. The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. This leads to the system

$$e^{2t} = b_0 + 2b_1$$
$$e^{3t} = b_0 + 3b_1,$$

which has the solution $b_0 = 3e^{2t} - 2e^{3t}$ and $b_1 = -e^{2t} + e^{3t}$. Then

$$e^{\mathbf{A}t} = b_0\mathbf{I} + b_1\mathbf{A} = \begin{pmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{pmatrix}.$$

The general solution of the system is then

$$\begin{aligned} \mathbf{X} = e^{\mathbf{A}t}\mathbf{C} &= \begin{pmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_1 \begin{pmatrix} -1 \\ -1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{3t} \\ &= (c_1 - c_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + (-c_1 + 2c_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} \\ &= c_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}. \end{aligned}$$

21. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$. This leads to the system

$$e^{-t} = b_0 - b_1$$
$$e^{3t} = b_0 + 3b_1,$$

which has the solution $b_0 = \frac{3}{4}e^{-t} + \frac{1}{4}e^{3t}$ and $b_1 = -\frac{1}{4}e^{-t} + \frac{1}{4}e^{3t}$. Then

$$e^{\mathbf{A}t} = b_0\mathbf{I} + b_1\mathbf{A} = \begin{pmatrix} e^{3t} & -2e^{-t} + 2e^{3t} \\ 0 & e^{-t} \end{pmatrix}.$$

The general solution of the system is then

$$\begin{aligned} \mathbf{X} = e^{\mathbf{A}t}\mathbf{C} &= \begin{pmatrix} e^{3t} & -2e^{-t} + 2e^{3t} \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{3t} \\ &= c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} + (c_1 + 2c_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} \\ &= c_3 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} + c_4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t}. \end{aligned}$$

22. The eigenvalues are $\lambda_1 = \frac{1}{4}$ and $\lambda_2 = \frac{1}{2}$. This leads to the system

$$e^{t/4} = b_0 + \frac{1}{4}b_1$$
$$e^{t/2} = b_0 + \frac{1}{2}b_1,$$

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which has the solution $b_0 = 2e^{t/4} + e^{t/2}$ and $b_1 = -4e^{t/4} + 4e^{t/2}$. Then

$$e^{\mathbf{A}t} = b_0\mathbf{I} + b_1\mathbf{A} = \begin{pmatrix} -2e^{t/4} + 3e^{t/2} & 6e^{t/4} - 6e^{t/2} \\ -e^{t/4} + e^{t/2} & 3e^{t/4} - 2e^{t/2} \end{pmatrix}.$$

The general solution of the system is then

$$\begin{aligned} \mathbf{X} = e^{\mathbf{A}t}\mathbf{C} &= \begin{pmatrix} -2e^{t/4} + 3e^{t/2} & 6e^{t/4} - 6e^{t/2} \\ -e^{t/4} + e^{t/2} & 3e^{t/4} - 2e^{t/2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} -2 \\ -1 \end{pmatrix} e^{t/4} + c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{t/2} + c_2 \begin{pmatrix} 6 \\ 3 \end{pmatrix} e^{t/4} + c_2 \begin{pmatrix} -6 \\ -2 \end{pmatrix} e^{t/2} \\ &= (-c_1 + 3c_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/4} + (c_1 - 2c_2) \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{t/2} \\ &= c_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/4} + c_4 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{t/2}. \end{aligned}$$

23. From equation (2) in the text we have $e^{\mathbf{D}t} = \mathbf{I} + t\mathbf{D} + \dfrac{t^2}{2!}\mathbf{D}^2 + \dfrac{t^3}{3!}\mathbf{D}^3 + \cdots$ so that

$$\mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \mathbf{P}\mathbf{P}^{-1} + t(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}) + \frac{t^2}{2!}(\mathbf{P}\mathbf{D}^2\mathbf{P}^{-1}) + \frac{t^3}{3!}(\mathbf{P}\mathbf{D}^3\mathbf{P}^{-1}) + \cdots.$$

But $\mathbf{P}\mathbf{P}^{-1} + \mathbf{I}$, $\mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{A}$ and $\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} = \mathbf{A}^n$ (see Problem 37, Exercises 8.12). Thus,

$$\mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \cdots = e^{\mathbf{A}t}.$$

24. From equation (2) in the text

$$\begin{aligned} e^{\mathbf{D}t} &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} + \frac{1}{2!}t^2 \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix} \\ &\qquad + \frac{1}{3!}t^3 \begin{bmatrix} \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \lambda_2^3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^3 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 + \lambda_1 t + \frac{1}{2!}(\lambda_1 t)^2 + \cdots & 0 & \cdots & 0 \\ 0 & 1 + \lambda_2 t + \frac{1}{2!}(\lambda_2 t)^2 + \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 + \lambda_n t + \frac{1}{2!}(\lambda_n t)^2 + \cdots \end{bmatrix} \\ &= \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \end{aligned}$$

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25. From Problems 23 and 24 and equation (1) in the text

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}\mathbf{C} = \begin{bmatrix} e^{3t} & e^{5t} \\ e^{3t} & 3e^{5t} \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} \frac{3}{2}e^{-3t} & -\frac{1}{2}e^{-3t} \\ -\frac{1}{2}e^{-5t} & \frac{1}{2}e^{-5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}e^{3t} - \frac{1}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ \frac{3}{2}e^{3t} - \frac{3}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{3}{2}e^{5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

26. From Problems 23 and 24 and equation (1) in the text

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}\mathbf{C} = \begin{bmatrix} -e^{t} & e^{3t} \\ e^{t} & e^{3t} \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2}e^{-t} & \frac{1}{2}e^{-t} \\ \frac{1}{2}e^{3t} & \frac{1}{2}e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}e^{t} + \frac{1}{2}e^{9t} & -\frac{1}{2}e^{t} + \frac{1}{2}e^{3t} \\ -\frac{1}{2}e^{t} + \frac{1}{2}e^{9t} & \frac{1}{2}e^{t} + \frac{1}{2}e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

27. (a) The following commands can be used in *Mathematica*:

A={{4, 2},{3, 3}};
c={c1, c2};
m=MatrixExp[A t];
sol=Expand[m.c]
Collect[sol, {c1, c2}] // MatrixForm

The output gives

$$x(t) = c_1\left(\frac{2}{5}e^{t} + \frac{3}{5}e^{6t}\right) + c_2\left(-\frac{2}{5}e^{t} + \frac{2}{5}e^{6t}\right)$$
$$y(t) = c_1\left(-\frac{3}{5}e^{t} + \frac{3}{5}e^{6t}\right) + c_2\left(\frac{3}{5}e^{t} + \frac{2}{5}e^{6t}\right).$$

The eigenvalues are 1 and 6 with corresponding eigenvectors

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so the solution of the system is

$$\mathbf{X}(t) = b_1\begin{pmatrix} -2 \\ 3 \end{pmatrix}e^{t} + b_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{6t}$$

or

$$x(t) = -2b_1e^{t} + b_2e^{6t}$$
$$y(t) = 3b_1e^{t} + b_2e^{6t}.$$

If we replace b_1 with $-\frac{1}{5}c_1 + \frac{1}{5}c_2$ and b_2 with $\frac{3}{5}c_1 + \frac{2}{5}c_2$, we obtain the solution found using the matrix exponential.

(b) $x(t) = c_1e^{-2t}\cos t - (c_1 + c_2)e^{-2t}\sin t$
$y(t) = c_2e^{-2t}\cos t + (2c_1 + c_2)e^{-2t}\sin t$

28. $x(t) = c_1(3e^{-2t} - 2e^{-t}) + c_3(-6e^{-2t} + 6e^{-t})$
$y(t) = c_2(4e^{-2t} - 3e^{-t}) + c_4(4e^{-2t} - 4e^{-t})$
$z(t) = c_1(e^{-2t} - e^{-t}) + c_3(-2e^{-2t} + 3e^{-t})$
$w(t) = c_2(-3e^{-2t} + 3e^{-t}) + c_4(-3e^{-2t} + 4e^{-t})$

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Chapter 10 Review Exercises

1. If $\mathbf{X} = k\begin{pmatrix} 4 \\ 5 \end{pmatrix}$, then $\mathbf{X}' = \mathbf{0}$ and

$$k\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 8 \\ 1 \end{pmatrix} = k\begin{pmatrix} 24 \\ 3 \end{pmatrix} - \begin{pmatrix} 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We see that $k = \frac{1}{3}$.

2. Solving for c_1 and c_2 we find $c_1 = -\frac{3}{4}$ and $c_2 = \frac{1}{4}$.

3. Since

$$\begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ -4 \end{pmatrix} = 4\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix},$$

we see that $\lambda = 4$ is an eigenvalue with eigenvector $\mathbf{K}_3$. The corresponding solution is $\mathbf{X} = \mathbf{K}_3 e^{4t}$.

4. The other eigenvalue is $\lambda_2 = 1 - 2i$ with corresponding eigenvector $\mathbf{K}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. The general solution is

$$\mathbf{X}(t) = c_1\begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix}e^t + c_2\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}e^t.$$

5. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 1)^2 = 0$ and $\mathbf{K} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. A solution to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{P} = \mathbf{K}$ is $\mathbf{P} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so that

$$\mathbf{X} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^t + c_2\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}te^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}e^t\right].$$

6. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda + 6)(\lambda + 2) = 0$ so that

$$\mathbf{X} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-6t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t}.$$

7. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 2\lambda + 5 = 0$. For $\lambda = 1 + 2i$ we obtain $\mathbf{K}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}e^{(1+2i)t} = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix}e^t + i\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}e^t.$$

Then

$$\mathbf{X} = c_1\begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix}e^t + c_2\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}e^t.$$

8. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 2\lambda + 2 = 0$. For $\lambda = 1 + i$ we obtain $\mathbf{K}_1 = \begin{pmatrix} 3 - i \\ 2 \end{pmatrix}$ and

$$\mathbf{X}_1 = \begin{pmatrix} 3 - i \\ 2 \end{pmatrix}e^{(1+i)t} = \begin{pmatrix} 3\cos t + \sin t \\ 2\cos t \end{pmatrix}e^t + i\begin{pmatrix} -\cos t + 3\sin t \\ 2\sin t \end{pmatrix}e^t.$$

Then

$$\mathbf{X} = c_1\begin{pmatrix} 3\cos t + \sin t \\ 2\cos t \end{pmatrix}e^t + c_2\begin{pmatrix} -\cos t + 3\sin t \\ 2\sin t \end{pmatrix}e^t.$$

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9. We have $\det(\mathbf{A}-\lambda\mathbf{I}) = -(\lambda-2)(\lambda-4)(\lambda+3) = 0$ so that

$$\mathbf{X} = c_1\begin{pmatrix}-2\\3\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}0\\1\\1\end{pmatrix}e^{4t} + c_3\begin{pmatrix}7\\12\\-16\end{pmatrix}e^{-3t}.$$

10. We have $\det(\mathbf{A}-\lambda\mathbf{I}) = -(s+2)(s^2-2s+3) = 0$. The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 1+\sqrt{2}i$, and $\lambda_2 = 1-\sqrt{2}i$, with eigenvectors

$$\mathbf{K}_1 = \begin{pmatrix}-7\\5\\4\end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix}1\\\sqrt{2}i/2\\1\end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix}1\\-\sqrt{2}i/2\\1\end{pmatrix}.$$

Thus

$$\begin{aligned}\mathbf{X} &= c_1\begin{pmatrix}-7\\5\\4\end{pmatrix}e^{-2t} + c_2\left[\begin{pmatrix}1\\0\\1\end{pmatrix}\cos\sqrt{2}t - \begin{pmatrix}0\\\sqrt{2}/2\\0\end{pmatrix}\sin\sqrt{2}t\right]e^{t}\\ &\quad + c_3\left[\begin{pmatrix}0\\\sqrt{2}/2\\0\end{pmatrix}\cos\sqrt{2}t + \begin{pmatrix}1\\0\\1\end{pmatrix}\sin\sqrt{2}t\right]e^{t}\\ &= c_1\begin{pmatrix}-7\\5\\4\end{pmatrix}e^{-2t} + c_2\begin{pmatrix}\cos\sqrt{2}t\\-\sqrt{2}\sin\sqrt{2}t/2\\\cos\sqrt{2}t\end{pmatrix}e^{t} + c_3\begin{pmatrix}\sin\sqrt{2}t\\\sqrt{2}\cos\sqrt{2}t/2\\\sin\sqrt{2}t\end{pmatrix}e^{t}.\end{aligned}$$

11. We have

$$\mathbf{X}_c = c_1\begin{pmatrix}1\\0\end{pmatrix}e^{2t} + c_2\begin{pmatrix}4\\1\end{pmatrix}e^{4t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix}e^{2t} & 4e^{4t}\\0 & e^{4t}\end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix}e^{-2t} & -4e^{-2t}\\0 & e^{-4t}\end{pmatrix},$$

and

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int\begin{pmatrix}2e^{-2t} - 64te^{-2t}\\16te^{-4t}\end{pmatrix}dt = \begin{pmatrix}15e^{-2t} + 32te^{-2t}\\-e^{-4t} - 4te^{-4t}\end{pmatrix},$$

so that

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix}11+16t\\-1-4t\end{pmatrix}.$$

12. We have

$$\mathbf{X}_c = c_1\begin{pmatrix}2\cos t\\-\sin t\end{pmatrix}e^{t} + c_2\begin{pmatrix}2\sin t\\\cos t\end{pmatrix}e^{t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix}2\cos t & 2\sin t\\-\sin t & \cos t\end{pmatrix}e^{t}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix}\frac{1}{2}\cos t & -\sin t\\\frac{1}{2}\sin t & \cos t\end{pmatrix}e^{-t},$$

and

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int\begin{pmatrix}\cos t - \sec t\\\sin t\end{pmatrix}dt = \begin{pmatrix}\sin t - \ln|\sec t + \tan t|\\-\cos t\end{pmatrix},$$

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so that

$$\mathbf{X}_p = \mathbf{\Phi U} = \begin{pmatrix} -2\cos t \ln|\sec t + \tan t| \\ -1 + \sin t \ln|\sec t + \tan t| \end{pmatrix}$$

13. We have

$$\mathbf{X}_c = c_1 \begin{pmatrix} \cos t + \sin t \\ 2\cos t \end{pmatrix} + c_2 \begin{pmatrix} \sin t - \cos t \\ 2\sin t \end{pmatrix}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} \cos t + \sin t & \sin t - \cos t \\ 2\cos t & 2\sin t \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \sin t & \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ -\cos t & \frac{1}{2}\cos t + \frac{1}{2}\sin t \end{pmatrix},$$

and

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} \frac{1}{2}\sin t - \frac{1}{2}\cos t + \frac{1}{2}\csc t \\ -\frac{1}{2}\sin t - \frac{1}{2}\cos t + \frac{1}{2}\csc t \end{pmatrix} dt$$

$$= \begin{pmatrix} -\frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}\ln|\csc t - \cot t| \\ \frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}\ln|\csc t - \cot t| \end{pmatrix},$$

so that

$$\mathbf{X}_p = \mathbf{\Phi U} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix} \ln|\csc t - \cot t|.$$

14. We have

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} \right].$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} e^{2t} & te^{2t} + e^{2t} \\ -e^{2t} & -te^{2t} \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -te^{-2t} & -te^{-2t} - e^{-2t} \\ e^{-2t} & e^{-2t} \end{pmatrix},$$

and

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} t - 1 \\ -1 \end{pmatrix} dt = \begin{pmatrix} \frac{1}{2}t^2 - t \\ -t \end{pmatrix},$$

so that

$$\mathbf{X}_p = \mathbf{\Phi U} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} t^2 e^{2t} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} te^{2t}.$$

15. (a) Letting

$$\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

we note that $(\mathbf{A} - 2\mathbf{I})\mathbf{K} = \mathbf{0}$ implies that $3k_1 + 3k_2 + 3k_3 = 0$, so $k_1 = -(k_2 + k_3)$. Choosing $k_2 = 0$, $k_3 = 1$ and then $k_2 = 1$, $k_3 = 0$ we get

$$\mathbf{K}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

respectively. Thus,

$$\mathbf{X}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \quad \text{and} \quad \mathbf{X}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$$

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are two solutions.

(b) From $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2(3 - \lambda) = 0$ we see that $\lambda_1 = 3$, and 0 is an eigenvalue of multiplicity two. Letting

$$\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix},$$

as in part (**a**), we note that $(\mathbf{A} - 0\mathbf{I})\mathbf{K} = \mathbf{AK} = \mathbf{0}$ implies that $k_1 + k_2 + k_3 = 0$, so $k_1 = -(k_2 + k_3)$. Choosing $k_2 = 0$, $k_3 = 1$, and then $k_2 = 1$, $k_3 = 0$ we get

$$\mathbf{K}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

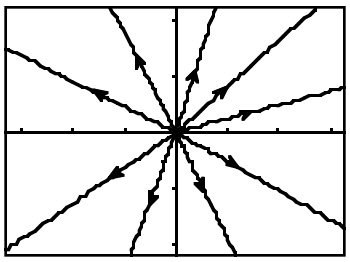
respectively. Since the eigenvector corresponding to $\lambda_1 = 3$ is

$$\mathbf{K}_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

the general solution of the system is

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

16. For $\mathbf{X} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^t$ we have $\mathbf{X}' = \mathbf{X} = \mathbf{IX}$.



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11 Systems of Nonlinear Differential Equations

Exercises 11.1

1. The corresponding plane autonomous system is

$$x' = y, \quad y' = -9 \sin x.$$

If (x, y) is a critical point, $y = 0$ and $-9 \sin x = 0$. Therefore $x = \pm n\pi$ and so the critical points are $(\pm n\pi, 0)$ for $n = 0, 1, 2, \ldots$.

2. The corresponding plane autonomous system is

$$x' = y, \quad y' = -2x - y^2.$$

If (x, y) is a critical point, then $y = 0$ and so $-2x - y^2 = -2x = 0$. Therefore $(0, 0)$ is the sole critical point.

3. The corresponding plane autonomous system is

$$x' = y, \quad y' = x^2 - y(1 - x^3).$$

If (x, y) is a critical point, $y = 0$ and so $x^2 - y(1 - x^3) = x^2 = 0$. Therefore $(0, 0)$ is the sole critical point.

4. The corresponding plane autonomous system is

$$x' = y, \quad y' = -4 \frac{x}{1 + x^2} - 2y.$$

If (x, y) is a critical point, $y = 0$ and so $-4 \dfrac{x}{1 + x^2} - 2(0) = 0$. Therefore $x = 0$ and so $(0, 0)$ is the sole critical point.

5. The corresponding plane autonomous system is

$$x' = y, \quad y' = -x + \epsilon x^3.$$

If (x, y) is a critical point, $y = 0$ and $-x + \epsilon x^3 = 0$. Hence $x(-1 + \epsilon x^2) = 0$ and so $x = 0$, $\sqrt{1/\epsilon}$, $-\sqrt{1/\epsilon}$. The critical points are $(0, 0)$, $(\sqrt{1/\epsilon}, 0)$ and $(-\sqrt{1/\epsilon}, 0)$.

6. The corresponding plane autonomous system is

$$x' = y, \quad y' = -x + \epsilon x|x|.$$

If (x, y) is a critical point, $y = 0$ and $-x + \epsilon x|x| = x(-1 + \epsilon|x|) = 0$. Hence $x = 0$, $1/\epsilon$, $-1/\epsilon$. The critical points are $(0, 0)$, $(1/\epsilon, 0)$ and $(-1/\epsilon, 0)$.

7. From $x + xy = 0$ we have $x(1 + y) = 0$. Therefore $x = 0$ or $y = -1$. If $x = 0$, then, substituting into $-y - xy = 0$, we obtain $y = 0$, Likewise, if $y = -1$, $1 + x = 0$ or $x = -1$. We may conclude that $(0, 0)$ and $(-1, -1)$ are critical points of the system.

8. From $y^2 - x = 0$ we have $x = y^2$. Substituting into $x^2 - y = 0$, we obtain $y^4 - y = 0$ or $y(y^3 - 1) = 0$. It follows that $y = 0$, 1 and so $(0, 0)$ and $(1, 1)$ are the critical points of the system.

9. From $x - y = 0$ we have $y = x$. Substituting into $3x^2 - 4y = 0$ we obtain $3x^2 - 4x = x(3x - 4) = 0$. It follows that $(0, 0)$ and $(4/3, 4/3)$ are the critical points of the system.

10. From $x^3 - y = 0$ we have $y = x^3$. Substituting into $x - y^3 = 0$ we obtain $x - x^9 = 0$ or $x(1 - x^8)$. Therefore $x = 0$, 1, -1 and so the critical points of the system are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

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11. From $x(10 - x - \frac{1}{2}y) = 0$ we obtain $x = 0$ or $x + \frac{1}{2}y = 10$. Likewise $y(16 - y - x) = 0$ implies that $y = 0$ or $x + y = 16$. We therefore have four cases. If $x = 0$, $y = 0$ or $y = 16$. If $x + \frac{1}{2}y = 10$, we may conclude that $y(-\frac{1}{2}y + 6) = 0$ and so $y = 0$, 12. Therefore the critical points of the system are $(0, 0)$, $(0, 16)$, $(10, 0)$, and $(4, 12)$.

12. Adding the two equations we obtain $10 - 15\,\dfrac{y}{y+5} = 0$. It follows that $y = 10$, and from $-2x + y + 10 = 0$ we may conclude that $x = 10$. Therefore $(10, 10)$ is the sole critical point of the system.

13. From $x^2e^y = 0$ we have $x = 0$. Since $e^x - 1 = e^0 - 1 = 0$, the second equation is satisfied for an arbitrary value of y. Therefore any point of the form $(0, y)$ is a critical point.

14. From $\sin y = 0$ we have $y = \pm n\pi$. From $e^{x-y} = 1$, we may conclude that $x - y = 0$ or $x = y$. The critical points of the system are therefore $(\pm n\pi, \pm n\pi)$ for $n = 0$, 1, 2, ...

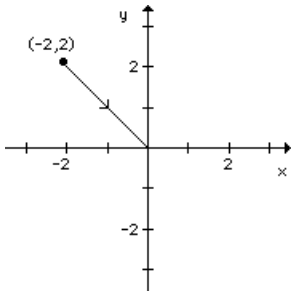
15. From $x(1 - x^2 - 3y^2) = 0$ we have $x = 0$ or $x^2 + 3y^2 = 1$. If $x = 0$, then substituting into $y(3 - x^2 - 3y^2)$ gives $y(3-3y^2) = 0$. Therefore $y = 0$, 1, -1. Likewise $x^2 = 1-3y^2$ yields $2y = 0$ so that $y = 0$ and $x^2 = 1-3(0)^2 = 1$. The critical points of the system are therefore $(0, 0)$, $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$.

16. From $-x(4 - y^2) = 0$ we obtain $x = 0$, $y = 2$, or $y = -2$. If $x = 0$, then substituting into $4y(1 - x^2)$ yields $y = 0$. Likewise $y = 2$ gives $8(1 - x^2) = 0$ or $x = 1$, -1. Finally $y = -2$ yields $-8(1 - x^2) = 0$ or $x = 1$, -1. The critical points of the system are therefore $(0, 0)$, $(1, 2)$, $(-1, 2)$, $(1, -2)$, and $(-1, -2)$.

17. **(a)** From Exercises 10.2, Problem 1, $x = c_1e^{5t} - c_2e^{-t}$ and $y = 2c_1e^{5t} + c_2e^{-t}$.

(b) From $\mathbf{X}(0) = (2, -1)$ it follows that $c_1 = 0$ and $c_2 = 2$. Therefore $x = -2e^{-t}$ and $y = 2e^{-t}$.

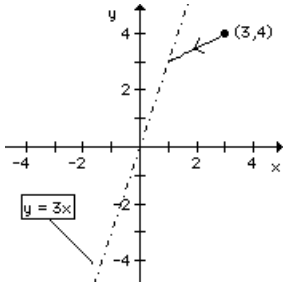
(c)



18. **(a)** From Exercises 10.2, Problem 6, $x = c_1 + 2c_2e^{-5t}$ and $y = 3c_1 + c_2e^{-5t}$.

(b) From $\mathbf{X}(0) = (3, 4)$ it follows that $c_1 = c_2 = 1$. Therefore $x = 1 + 2e^{-5t}$ and $y = 3 + e^{-5t}$.

(c)



19. **(a)** From Exercises 10.2, Problem 37, $x = c_1(4\cos 3t - 3\sin 3t) + c_2(4\sin 3t + 3\cos 3t)$ and $y = c_1(5\cos 3t) + c_2(5\sin 3t)$. All solutions are one periodic with $p = 2\pi/3$.

(b) From $\mathbf{X}(0) = (4, 5)$ it follows that $c_1 = 1$ and $c_2 = 0$. Therefore $x = 4\cos 3t - 3\sin 3t$ and $y = 5\cos 3t$.

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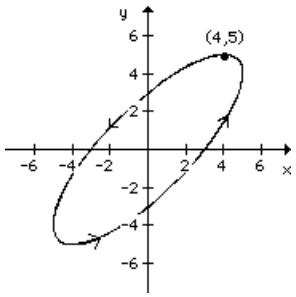
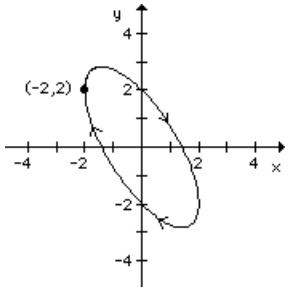
(c)



20. **(a)** From Exercises 10.2, Problem 34, $x = c_1(\sin t - \cos t) + c_2(-\cos t - \sin t)$ and $y = 2c_1 \cos t + 2c_2 \sin t$. All solutions are periodic with $p = 2\pi$.

(b) From $\mathbf{X}(0) = (-2, 2)$ it follows that $c_1 = c_2 = 1$. Therefore $x = -2\cos t$ and $y = 2\cos t + 2\sin t$.

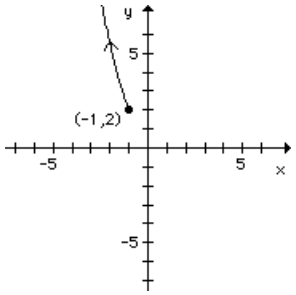
(c)



21. **(a)** From Exercises 10.2, Problem 35, $x = c_1(\sin t - \cos t)e^{4t} + c_2(-\sin t - \cos t)e^{4t}$ and $y = 2c_1(\cos t)\, e^{4t} + 2c_2(\sin t)\, e^{4t}$. Because of the presence of e^{4t}, there are no periodic solutions.

(b) From $\mathbf{X}(0) = (-1, 2)$ it follows that $c_1 = 1$ and $c_2 = 0$. Therefore $x = (\sin t - \cos t)e^{4t}$ and $y = 2(\cos t)\, e^{4t}$.

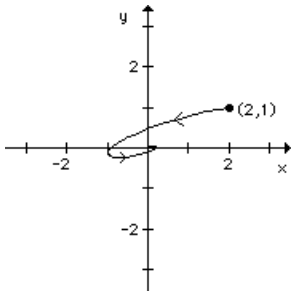
(c)



22. **(a)** From Exercises 10.2, Problem 38, $x = c_1e^{-t}(2\cos 2t - 2\sin 2t) + c_2e^{-t}(2\cos 2t + 2\sin 2t)$ and $y = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t$. Because of the presence of e^{-t}, there are no periodic solutions.

(b) From $\mathbf{X}(0) = (2, 1)$ it follows that $c_1 = 1$ and $c_2 = 0$. Therefore $x = e^{-t}(2\cos 2t - 2\sin 2t)$ and $y = e^{-t}\cos 2t$.

(c)



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23. Switching to polar coordinates,

$$\frac{dr}{dt} = \frac{1}{r}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) = \frac{1}{r}(-xy - x^2r^4 + xy - y^2r^4) = -r^5$$
$$\frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\,\frac{dx}{dt} + x\,\frac{dy}{dt}\right) = \frac{1}{r^2}(y^2 + xyr^4 + x^2 - xyr^4) = 1\,.$$

If we use separation of variables on $\dfrac{dr}{dt} = -r^5$ we obtain

$$r = \left(\frac{1}{4t + c_1}\right)^{1/4} \quad \text{and} \quad \theta = t + c_2.$$

Since $\mathbf{X}(0) = (4, 0)$, $r = 4$ and $\theta = 0$ when $t = 0$. It follows that $c_2 = 0$ and $c_1 = \dfrac{1}{256}$. The final solution may be written as

$$r = \frac{4}{\sqrt[4]{1024t + 1}}\,, \qquad \theta = t$$

and so the solution spirals toward the origin as t increases.

24. Switching to polar coordinates,

$$\frac{dr}{dt} = \frac{1}{r}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) = \frac{1}{r}(xy - x^2r^2 - xy + y^2r^2) = r^3$$
$$\frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\,\frac{dx}{dt} + x\,\frac{dy}{dt}\right) = \frac{1}{r^2}(-y^2 - xyr^2 - x^2 + xyr^2) = -1\,.$$

If we use separation of variables, it follows that

$$r = \frac{1}{\sqrt{-2t + c_1}} \quad \text{and} \quad \theta = -t + c_2.$$

Since $\mathbf{X}(0) = (4, 0)$, $r = 4$ and $\theta = 0$ when $t = 0$. It follows that $c_2 = 0$ and $c_1 = \frac{1}{16}$. The final solution may be written as

$$r = \frac{4}{\sqrt{1 - 32t}}\,, \qquad \theta = -t.$$

Note that $r \to \infty$ as $t \to \left(\frac{1}{32}\right)^-$. Because $0 \le t \le \frac{1}{32}$, the curve is not a spiral.

25. Switching to polar coordinates,

$$\frac{dr}{dt} = \frac{1}{r}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) = \frac{1}{r}[-xy + x^2(1 - r^2) + xy + y^2(1 - r^2)] = r(1 - r^2)$$
$$\frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\,\frac{dx}{dt} + x\,\frac{dy}{dt}\right) = \frac{1}{r^2}[y^2 - xy(1 - r^2) + x^2 + xy(1 - r^2)] = 1.$$

Now $\dfrac{dr}{dt} = r - r^3$ or $\dfrac{dr}{dt} - r = -r^3$ is a Bernoulli differential equation. Following the procedure in Section 2.5 of the text, we let $w = r^{-2}$ so that $w' = -2r^{-3}\,\dfrac{dr}{dt}$. Therefore $w' + 2w = 2$, a linear first order differential equation. It follows that $w = 1 + c_1e^{-2t}$ and so $r^2 = \dfrac{1}{1 + c_1e^{-2t}}$. The general solution may be written as

$$r = \frac{1}{\sqrt{1 + c_1e^{-2t}}}\,, \qquad \theta = t + c_2.$$

If $\mathbf{X}(0) = (1, 0)$, $r = 1$ and $\theta = 0$ when $t = 0$. Therefore $c_1 = 0 = c_2$ and so $x = r\cos t = \cos t$ and $y = r\sin t = \sin t$. This solution generates the circle $r = 1$. If $\mathbf{X}(0) = (2, 0)$, $r = 2$ and $\theta = 0$ when $t = 0$.

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Therefore $c_1 = -3/4$, $c_2 = 0$ and so

$$r = \frac{1}{\sqrt{1 - \frac{3}{4}\,e^{-2t}}}\,, \qquad \theta = t.$$

This solution spirals toward the circle $r = 1$ as t increases.

26. Switching to polar coordinates,

$$\frac{dr}{dt} = \frac{1}{r}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) = \frac{1}{r}\left[xy - \frac{x^2}{r}(4 - r^2) - xy - \frac{y^2}{r}(4 - r^2)\right] = r^2 - 4$$

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\,\frac{dx}{dt} + x\,\frac{dy}{dt}\right) = \frac{1}{r^2}\left[-y^2 + \frac{xy}{r}(4 - r^2) - x^2 - \frac{xy}{r}(4 - r^2)\right] = -1.$$

From Example 3, Section 2.2,

$$r = 2\,\frac{1 + c_1 e^{4t}}{1 - c_1 e^{4t}} \quad \text{and} \quad \theta = -t + c_2.$$

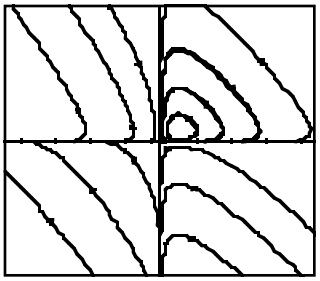
If $\mathbf{X}(0) = (1, 0)$, $r = 1$ and $\theta = 0$ when $t = 0$. It follows that $c_2 = 0$ and $c_1 = -\frac{1}{3}$. Therefore

$$r = 2\,\frac{1 - \frac{1}{3}e^{4t}}{1 + \frac{1}{3}e^{4t}} \quad \text{and} \quad \theta = -t.$$

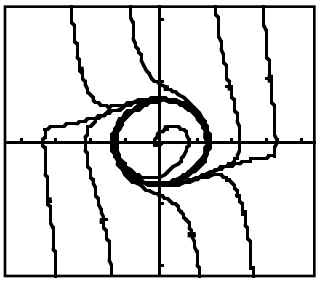
Note that $r = 0$ when $e^{4t} = 3$ or $t = \dfrac{\ln 3}{4}$ and $r \to -2$ as $t \to \infty$. The solution therefore approaches the circle $r = 2$. If $\mathbf{X}(0) = (2, 0)$, it follows that $c_1 = c_2 = 0$. Therefore $r = 2$ and $\theta = -t$ so that the solution generates the circle $r = 2$ traversed in the clockwise direction. Note also that the original system is not defined at $(0, 0)$ but the corresponding polar system is defined for $r = 0$. If the Runge-Kutta method is applied to the original system, the solution corresponding to $\mathbf{X}(0) = (1, 0)$ will stall at the origin.

27. The system has no critical points, so there are no periodic solutions.

28. From $x(6y - 1) = 0$ and $y(2 - 8x) = 0$ we see that $(0, 0)$ and $(1/4, 1/6)$ are critical points. From the graph we see that there are periodic solutions around $(1/4, 1/6)$.



29. The only critical point is $(0, 0)$. There appears to be a single periodic solution around $(0, 0)$.



30. The system has no critical points, so there are no periodic solutions.

31. If $\mathbf{X}(t) = (x(t), y(t))$ is a solution,

$$\frac{d}{dt}\,f(x(t), y(t)) = \frac{\partial f}{\partial x}\,\frac{dx}{dt} + \frac{\partial f}{\partial y}\,\frac{dy}{dt} = QP - PQ = 0,$$

using the chain rule. Therefore $f(x(t), y(t)) = c$ for some constant c, and the solution lies on a level curve of f.

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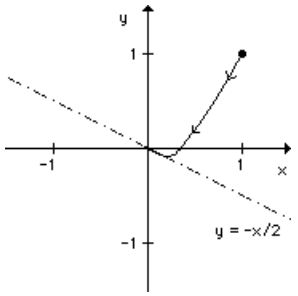
Exercises 11.2

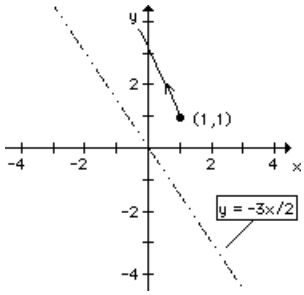
1. **(a)** If $\mathbf{X}(0) = \mathbf{X}_0$ lies on the line $y = 2x$, then $\mathbf{X}(t)$ approaches $(0, 0)$ along this line. For all other initial conditions, $\mathbf{X}(t)$ approaches $(0, 0)$ from the direction determined by the line $y = -x/2$.

(b)



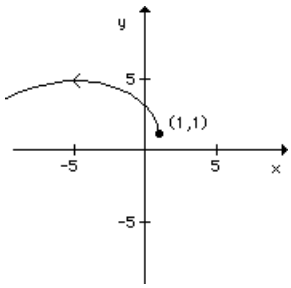
2. **(a)** If $\mathbf{X}(0) = \mathbf{X}_0$ lies on the line $y = -x$, then $\mathbf{X}(t)$ becomes unbounded along this line. For all other initial conditions, $\mathbf{X}(t)$ becomes unbounded and $y = -3x/2$ serves as an asymptote.

(b)



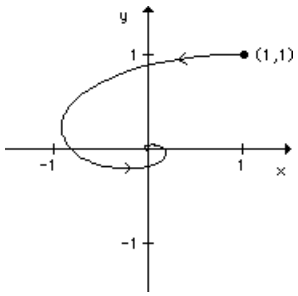
3. **(a)** All solutions are unstable spirals which become unbounded as t increases.

(b)



4. **(a)** All solutions are spirals which approach the origin.

(b)



5. **(a)** All solutions approach $(0, 0)$ from the direction specified by the line $y = x$.

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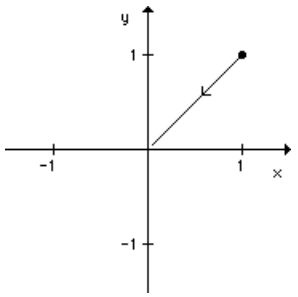
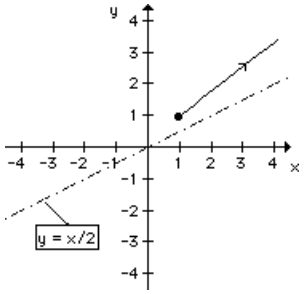
(b)



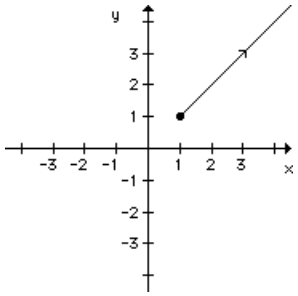
6. (a) All solutions become unbounded and $y = x/2$ serves as the asymptote.

(b)



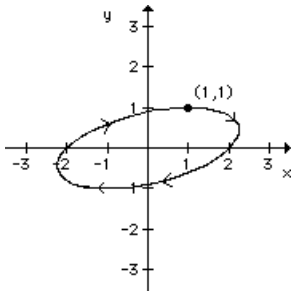
7. (a) If $\mathbf{X}(0) = \mathbf{X}_0$ lies on the line $y = 3x$, then $\mathbf{X}(t)$ approaches $(0, 0)$ along this line. For all other initial conditions, $\mathbf{X}(t)$ becomes unbounded and $y = x$ serves as the asymptote.

(b)



8. (a) The solutions are ellipses which encircle the origin.

(b)



9. Since $\Delta = -41 < 0$, we may conclude from Figure 11.18 that $(0, 0)$ is a saddle point.

10. Since $\Delta = 29$ and $\tau = -12$, $\tau^2 - 4\Delta > 0$ and so from Figure 11.18, $(0, 0)$ is a stable node.

11. Since $\Delta = -19 < 0$, we may conclude from Figure 11.18 that $(0, 0)$ is a saddle point.

12. Since $\Delta = 1$ and $\tau = -1$, $\tau^2 - 4\Delta = -3$ and so from Figure 11.18, $(0, 0)$ is a stable spiral point.

13. Since $\Delta = 1$ and $\tau = -2$, $\tau^2 - 4\Delta = 0$ and so from Figure 11.18, $(0, 0)$ is a degenerate stable node.

14. Since $\Delta = 1$ and $\tau = 2$, $\tau^2 - 4\Delta = 0$ and so from Figure 11.18, $(0, 0)$ is a degenerate unstable node.

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15. Since $\Delta = 0.01$ and $\tau = -0.03$, $\tau^2 - 4\Delta < 0$ and so from Figure 11.18, $(0, 0)$ is a stable spiral point.

16. Since $\Delta = 0.0016$ and $\tau = 0.08$, $\tau^2 - 4\Delta = 0$ and so from Figure 11.18, $(0, 0)$ is a degenerate unstable node.

17. $\Delta = 1 - \mu^2$, $\tau = 0$, and so we need $\Delta = 1 - \mu^2 > 0$ for $(0, 0)$ to be a center. Therefore $|\mu| < 1$.

18. Note that $\Delta = 1$ and $\tau = \mu$. Therefore we need both $\tau = \mu < 0$ and $\tau^2 - 4\Delta = \mu^2 - 4 < 0$ for $(0, 0)$ to be a stable spiral point. These two conditions may be written as $-2 < \mu < 0$.

19. Note that $\Delta = \mu + 1$ and $\tau = \mu + 1$ and so $\tau^2 - 4\Delta = (\mu + 1)^2 - 4(\mu + 1) = (\mu + 1)(\mu - 3)$. It follows that $\tau^2 - 4\Delta < 0$ if and only if $-1 < \mu < 3$. We may conclude that $(0, 0)$ will be a saddle point when $\mu < -1$. Likewise $(0, 0)$ will be an unstable spiral point when $\tau = \mu + 1 > 0$ and $\tau^2 - 4\Delta < 0$. This condition reduces to $-1 < \mu < 3$.

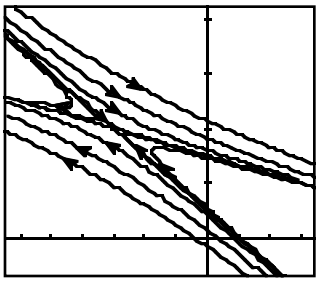
20. $\tau = 2\alpha$, $\Delta = \alpha^2 + \beta^2 > 0$, and $\tau^2 - 4\Delta = -4\beta < 0$. If $\alpha < 0$, $(0, 0)$ is a stable spiral point. If $\alpha > 0$, $(0, 0)$ is an unstable spiral point. Therefore $(0, 0)$ cannot be a node or saddle point.

21. $\mathbf{AX}_1 + \mathbf{F} = \mathbf{0}$ implies that $\mathbf{AX}_1 = -\mathbf{F}$ or $\mathbf{X}_1 = -\mathbf{A}^{-1}\mathbf{F}$. Since $\mathbf{X}_p(t) = -\mathbf{A}^{-1}\mathbf{F}$ is a particular solution, it follows from Theorem 10.6 that $\mathbf{X}(t) = \mathbf{X}_c(t) + \mathbf{X}_1$ is the general solution to $\mathbf{X}' = \mathbf{AX} + \mathbf{F}$. If $\tau < 0$ and $\Delta > 0$ then $\mathbf{X}_c(t)$ approaches $(0, 0)$ by Theorem 11.1(a). It follows that $\mathbf{X}(t)$ approaches $\mathbf{X}_1$ as $t \to \infty$.

22. If $bc < 1$, $\Delta = ad\hat{x}\hat{y}(1 - bc) > 0$ and $\tau^2 - 4\Delta = (a\hat{x} - d\hat{y})^2 + 4abcd\hat{x}\hat{y} > 0$. Therefore $(0, 0)$ is a stable node.

23. **(a)** The critical point is $\mathbf{X}_1 = (-3, 4)$.

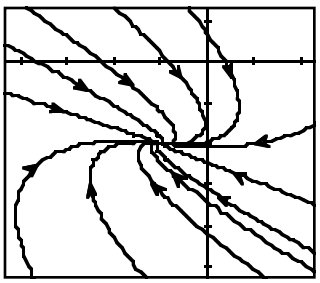
(b) From the graph, $\mathbf{X}_1$ appears to be an unstable node or a saddle point.



(c) Since $\Delta = -1$, $(0, 0)$ is a saddle point.

24. **(a)** The critical point is $\mathbf{X}_1 = (-1, -2)$.

(b) From the graph, $\mathbf{X}_1$ appears to be a stable node or a degenerate stable node.



(c) Since $\tau = -16$, $\Delta = 64$, and $\tau^2 - 4\Delta = 0$, $(0, 0)$ is a degenerate stable node.

25. **(a)** The critical point is $\mathbf{X}_1 = (0.5, 2)$.

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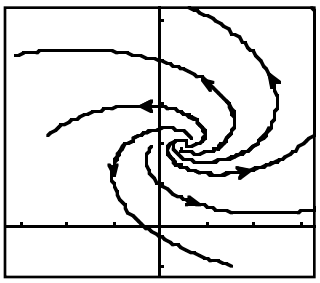
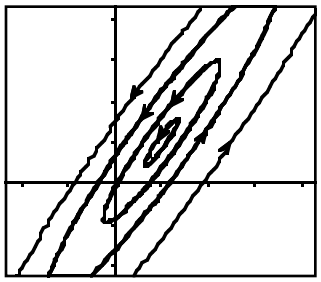
(b) From the graph, $\mathbf{X}_1$ appears to be an unstable spiral point.



(c) Since $\tau = 0.2$, $\Delta = 0.03$, and $\tau 2 - 4\Delta = -0.08$, $(0, 0)$ is an unstable spiral point.

26. (a) The critical point is $\mathbf{X}_1 = (1, 1)$.

(b) From the graph, $\mathbf{X}_1$ appears to be a center.



(c) Since $\tau = 0$ and $\Delta = 1$, $(0, 0)$ is a center.

Exercises 11.3

1. Switching to polar coordinates,

$$\frac{dr}{dt} = \frac{1}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{1}{r}(\alpha x^2 - \beta xy + xy^2 + \beta xy + \alpha y^2 - xy^2) = \frac{1}{r}\,\alpha r^2 = \alpha r.$$

Therefore $r = ce^{\alpha t}$ and so $r \to 0$ if and only if $\alpha < 0$.

2. The differential equation $\dfrac{dr}{dt} = \alpha r(5 - r)$ is a logistic differential equation. [See Section 2.8, (4) and (5).] It follows that

$$r = \frac{5}{1 + c_1 e^{-5\alpha t}} \quad \text{and} \quad \theta = -t + c_2.$$

If $\alpha > 0$, $r \to 5$ as $t \to +\infty$ and so the critical point $(0, 0)$ is unstable. If $\alpha < 0$, $r \to 0$ as $t \to +\infty$ and so $(0, 0)$ is asymptotically stable.

3. The critical points are $x = 0$ and $x = n + 1$. Since $g'(x) = k(n + 1) - 2kx$, $g'(0) = k(n + 1) > 0$ and $g'(n + 1) = -k(n + 1) < 0$. Therefore $x = 0$ is unstable while $x = n + 1$ is asymptotically stable. See Theorem 11.2.

4. Note that $x = k$ is the only critical point since $\ln(x/k)$ is not defined at $x = 0$. Since $g'(x) = -k - k\ln(x/k)$, $g'(k) = -k < 0$. Therefore $x = k$ is an asymptotically stable critical point by Theorem 11.2.

5. The only critical point is $T = T_0$. Since $g'(T) = k$, $g'(T_0) = k > 0$. Therefore $T = T_0$ is unstable by Theorem 11.2.

6. The only critical point is $v = mg/k$. Now $g(v) = g - (k/m)v$ and so $g'(v) = -k/m < 0$. Therefore $v = mg/k$ is an asymptotically stable critical point by Theorem 11.2.

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7. Critical points occur at $x = \alpha$, β. Since $g'(x) = k(-\alpha - \beta + 2x)$, $g'(\alpha) = k(\alpha - \beta)$ and $g'(\beta) = k(\beta - \alpha)$. Since $\alpha > \beta$, $g'(\alpha) > 0$ and so $x = \alpha$ is unstable. Likewise $x = \beta$ is asymptotically stable.

8. Critical points occur at $x = \alpha$, β, γ. Since

$$g'(x) = k(\alpha - x)(-\beta - \gamma - 2x) + k(\beta - x)(\gamma - x)(-1),$$

$g'(\alpha) = -k(\beta - \alpha)(\gamma - \alpha) < 0$ since $\alpha > \beta > \gamma$. Therefore $x = \alpha$ is asymptotically stable. Similarly $g'(\beta) > 0$ and $g'(\gamma) < 0$. Therefore $x = \beta$ is unstable while $x = \gamma$ is asymptotically stable.

9. Critical points occur at $P = a/b$, c but not at $P = 0$. Since $g'(P) = (a - bP) + (P - c)(-b)$,

$$g'(a/b) = (a/b - c)(-b) = -a + bc \quad \text{and} \quad g'(c) = a - bc.$$

Since $a < bc$, $-a + bc > 0$ and $a - bc < 0$. Therefore $P = a/b$ is unstable while $P = c$ is asymptotically stable.

10. Since $A > 0$, the only critical point is $A = K^2$. Since $g'(A) = \frac{1}{2}kKA^{-1/2} - k$, $g'(K^2) = -k/2 < 0$. Therefore $A = K^2$ is asymptotically stable.

11. The sole critical point is $(1/2, 1)$ and

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -2y & -2x \\ 2y & 2x - 1 \end{pmatrix}.$$

Computing $\mathbf{g}'((1/2, 1))$ we find that $\tau = -2$ and $\Delta = 2$ so that $\tau^2 - 4\Delta = -4 < 0$. Therefore $(1/2, 1)$ is a stable spiral point.

12. Critical points are $(1, 0)$ and $(-1, 0)$, and

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 2x & -2y \\ 0 & 2 \end{pmatrix}.$$

At $\mathbf{X} = (1, 0)$, $\tau = 4$, $\Delta = 4$, and so $\tau^2 - 4\Delta = 0$. We may conclude that $(1, 0)$ is unstable but we are unable to classify this critical point any further. At $\mathbf{X} = (-1, 0)$, $\Delta = -4 < 0$ and so $(-1, 0)$ is a saddle point.

13. $y' = 2xy - y = y(2x - 1)$. Therefore if (x, y) is a critical point, either $x = 1/2$ or $y = 0$. The case $x = 1/2$ and $y - x^2 + 2 = 0$ implies that $(x, y) = (1/2, -7/4)$. The case $y = 0$ leads to the critical points $(\sqrt{2}\,, 0)$ and $(-\sqrt{2}\,, 0)$. We next use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -2x & 1 \\ 2y & 2x - 1 \end{pmatrix}$$

to classify these three critical points. For $\mathbf{X} = (\sqrt{2}\,, 0)$ or $(-\sqrt{2}\,, 0)$, $\tau = -1$ and $\Delta < 0$. Therefore both critical points are saddle points. For $\mathbf{X} = (1/2, -7/4)$, $\tau = -1$, $\Delta = 7/2$ and so $\tau^2 - 4\Delta = -13 < 0$. Therefore $(1/2, -7/4)$ is a stable spiral point.

14. $y' = -y + xy = y(-1 + x)$. Therefore if (x, y) is a critical point, either $y = 0$ or $x = 1$. The case $y = 0$ and $2x - y^2 = 0$ implies that $(x, y) = (0, 0)$. The case $x = 1$ leads to the critical points $(1, \sqrt{2}\,)$ and $(1, -\sqrt{2}\,)$. We next use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 2 & -2y \\ y & x - 1 \end{pmatrix}$$

to classify these critical points. For $\mathbf{X} = (0, 0)$, $\Delta = -2 < 0$ and so $(0, 0)$ is a saddle point. For either $(1, \sqrt{2}\,)$ or $(1, -\sqrt{2}\,)$, $\tau = 2$, $\Delta = 4$, and so $\tau^2 - 4\Delta = -12$. Therefore $(1, \sqrt{2}\,)$ and $(1, -\sqrt{2}\,)$ are unstable spiral points.

15. Since $x^2 - y^2 = 0$, $y^2 = x^2$ and so $x^2 - 3x + 2 = (x - 1)(x - 2) = 0$. It follows that the critical points are $(1, 1)$, $(1, -1)$, $(2, 2)$, and $(2, -2)$. We next use the Jacobian

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -3 & 2y \\ 2x & -2y \end{pmatrix}$$

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to classify these four critical points. For $\mathbf{X} = (1, 1)$, $\tau = -5$, $\Delta = 2$, and so $\tau^2 - 4\Delta = 17 > 0$. Therefore $(1, 1)$ is a stable node. For $\mathbf{X} = (1, -1)$, $\Delta = -2 < 0$ and so $(1, -1)$ is a saddle point. For $\mathbf{X} = (2, 2)$, $\Delta = -4 < 0$ and so we have another saddle point. Finally, if $\mathbf{X} = (2, -2)$, $\tau = 1$, $\Delta = 4$, and so $\tau^2 - 4\Delta = -15 < 0$. Therefore $(2, -2)$ is an unstable spiral point.

16. From $y^2 - x^2 = 0$, $y = x$ or $y = -x$. The case $y = x$ leads to $(4, 4)$ and $(-1, 1)$ but the case $y = -x$ leads to $x^2 - 3x + 4 = 0$ which has no real solutions. Therefore $(4, 4)$ and $(-1, 1)$ are the only critical points. We next use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} y & x - 3 \\ -2x & 2y \end{pmatrix}$$

to classify these two critical points. For $\mathbf{X} = (4, 4)$, $\tau = 12$, $\Delta = 40$, and so $\tau^2 - 4\Delta < 0$. Therefore $(4, 4)$ is an unstable spiral point. For $\mathbf{X} = (-1, 1)$, $\tau = -3$, $\Delta = 10$, and so $x^2 - 4\Delta < 0$. It follows that $(-1, -1)$ is a stable spiral piont.

17. Since $x' = -2xy = 0$, either $x = 0$ or $y = 0$. If $x = 0$, $y(1 - y^2) = 0$ and so $(0, 0)$, $(0, 1)$, and $(0, -1)$ are critical points. The case $y = 0$ leads to $x = 0$. We next use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -2y & -2x \\ -1 + y & 1 + x - 3y^2 \end{pmatrix}$$

to classify these three critical points. For $\mathbf{X} = (0, 0)$, $\tau = 1$ and $\Delta = 0$ and so the test is inconclusive. For $\mathbf{X} = (0, 1)$, $\tau = -4$, $\Delta = 4$ and so $\tau^2 - 4\Delta = 0$. We can conclude that $(0, 1)$ is a stable critical point but we are unable to classify this critical point further in this borderline case. For $\mathbf{X} = (0, -1)$, $\Delta = -4 < 0$ and so $(0, -1)$ is a saddle point.

18. We found that $(0, 0)$, $(0, 1)$, $(0, -1)$, $(1, 0)$ and $(-1, 0)$ were the critical points in Exercise 15, Section 11.1. The Jacobian is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 1 - 3x^2 - 3y^2 & -6xy \\ -2xy & 3 - x^2 - 9y^2 \end{pmatrix}.$$

For $\mathbf{X} = (0, 0)$, $\tau = 4$, $\Delta = 3$ and so $\tau^2 - 4\Delta = 4 > 0$. Therefore $(0, 0)$ is an unstable node. Both $(0, 1)$ and $(0, -1)$ give $\tau = -8$, $\Delta = 12$, and $\tau^2 - 4\Delta = 16 > 0$. These two critical points are therefore stable nodes. For $\mathbf{X} = (1, 0)$ or $(-1, 0)$, $\Delta = -4 < 0$ and so saddle points occur.

19. We found the critical points $(0, 0)$, $(10, 0)$, $(0, 16)$ and $(4, 12)$ in Exercise 11, Section 11.1. Since the Jacobian is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 10 - 2x - \frac{1}{2}y & -\frac{1}{2}x \\ -y & 16 - 2y - x \end{pmatrix}$$

we may classify the critical points as follows:

$\mathbf{X}$	τ	Δ	$\tau^2 - 4\Delta$	Conclusion
$(0, 0)$	26	160	36	unstable node
$(10, 0)$	-4	-60	–	saddle point
$(0, 16)$	-14	-32	–	saddle point
$(4, 12)$	-16	24	160	stable node

20. We found the sole critical point $(10, 10)$ in Exercise 12, Section 11.1. The Jacobian is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -2 & 1 \\ 2 & -1 - \dfrac{15}{(y + 5)^2} \end{pmatrix},$$

$\mathbf{g}'((10, 10))$ has trace $\tau = -46/15$, $\Delta = 2/15$, and $\tau^2 - 4\Delta > 0$. Therefore $(0, 0)$ is a stable node.

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21. The corresponding plane autonomous system is

$$\theta' = y, \quad y' = (\cos\theta - \frac{1}{2})\sin\theta.$$

Since $|\theta| < \pi$, it follows that critical points are $(0,0)$, $(\pi/3,0)$ and $(-\pi/3,0)$. The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ \cos 2\theta - \frac{1}{2}\cos\theta & 0 \end{pmatrix}$$

and so at $(0,0)$, $\tau = 0$ and $\Delta = -1/2$. Therefore $(0,0)$ is a saddle point. For $\mathbf{X} = (\pm\pi/3, 0)$, $\tau = 0$ and $\Delta = 3/4$. It is not possible to classify either critical point in this borderline case.

22. The corresponding plane autonomous system is

$$x' = y, \quad y' = -x + \left(\frac{1}{2} - 3y^2\right)y - x^2.$$

If (x, y) is a critical point, $y = 0$ and so $-x - x^2 = -x(1 + x) = 0$. Therefore $(0,0)$ and $(-1,0)$ are the only two critical points. We next use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ -1 - 2x & \frac{1}{2} - 9y^2 \end{pmatrix}$$

to classify these critical points. For $\mathbf{X} = (0,0)$, $\tau = 1/2$, $\Delta = 1$, and $\tau^2 - 4\Delta < 0$. Therefore $(0,0)$ is an unstable spiral point. For $\mathbf{X} = (-1,0)$, $\tau = 1/2$, $\Delta = -1$ and so $(-1,0)$ is a saddle point.

23. The corresponding plane autonomous system is

$$x' = y, \quad y' = x^2 - y(1 - x^3)$$

and the only critical point is $(0,0)$. Since the Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ 2x + 3x^2y & x^3 - 1 \end{pmatrix},$$

$\tau = -1$ and $\Delta = 0$, and we are unable to classify the critical point in this borderline case.

24. The corresponding plane autonomous system is

$$x' = y, \quad y' = -\frac{4x}{1 + x^2} - 2y$$

and the only critical point is $(0,0)$. Since the Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ -4\dfrac{1 - x^2}{(1 + x^2)^2} & -2 \end{pmatrix},$$

$\tau = -2$, $\Delta = 4$, $\tau^2 - 4\Delta = -12$, and so $(0,0)$ is a stable spiral point.

25. In Exercise 5, Section 11.1, we showed that $(0,0)$, $(\sqrt{1/\epsilon}\,, 0)$ and $(-\sqrt{1/\epsilon}\,, 0)$ are the critical points. We will use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ -1 + 3\epsilon x^2 & 0 \end{pmatrix}$$

to classify these three critical points. For $\mathbf{X} = (0,0)$, $\tau = 0$ and $\Delta = 1$ and we are unable to classify this critical point. For $(\pm\sqrt{1/\epsilon}\,, 0)$, $\tau = 0$ and $\Delta = -2$ and so both of these critical points are saddle points.

26. In Exercise 6, Section 11.1, we showed that $(0,0)$, $(1/\epsilon, 0)$, and $(-1/\epsilon, 0)$ are the critical points. Since $D_x x|x| = 2|x|$, the Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ 2\epsilon|x| - 1 & 0 \end{pmatrix}.$$

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For $\mathbf{X} = (0,0)$, $\tau = 0$, $\Delta = 1$ and we are unable to classify this critical point. For $(\pm 1/\epsilon, 0)$, $\tau = 0$, $\Delta = -1$, and so both of these critical points are saddle points.

27. The corresponding plane autonomous system is

$$x' = y, \quad y' = -\frac{(\beta + \alpha^2 y^2)x}{1 + \alpha^2 x^2}$$

and the Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ \dfrac{(\beta + \alpha y^2)(\alpha^2 x^2 - 1)}{(1 + \alpha^2 x^2)^2} & \dfrac{-2\alpha^2 yx}{1 + \alpha^2 x^2} \end{pmatrix}.$$

For $\mathbf{X} = (0,0)$, $\tau = 0$ and $\Delta = \beta$. Since $\beta < 0$, we may conclude that $(0,0)$ is a saddle point.

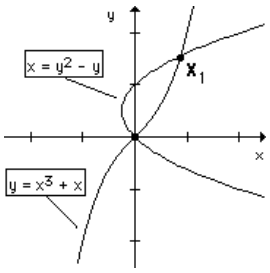
28. From $x' = -\alpha x + xy = x(-\alpha + y) = 0$, either $x = 0$ or $y = \alpha$. If $x = 0$, then $1 - \beta y = 0$ and so $y = 1/\beta$. The case $y = \alpha$ implies that $1 - \beta\alpha - x^2 = 0$ or $x^2 = 1 - \alpha\beta$. Since $\alpha\beta > 1$, this equation has no real solutions. It follows that $(0, 1/\beta)$ is the unique critical point. Since the Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -\alpha + y & x \\ -2x & -\beta \end{pmatrix},$$

$\tau = -\alpha - \beta + \dfrac{1}{\beta} = -\beta + \dfrac{1 - \alpha\beta}{\beta} < 0$ and $\Delta = \alpha\beta - 1 > 0$. Therefore $(0, 1/\beta)$ is a stable critical point.

29. **(a)** The graphs of $-x + y - x^3 = 0$ and $-x - y + y^2 = 0$ are shown in the figure. The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -1 - 3x^2 & 1 \\ -1 & -1 + 2y \end{pmatrix}.$$



For $\mathbf{X} = (0,0)$, $\tau = -2$, $\Delta = 2$, $\tau^2 - 4\Delta = -4$, and so $(0,0)$ is a stable spiral point.

(b) For $\mathbf{X}_1$, $\Delta = -6.07 < 0$ and so a saddle point occurs at $\mathbf{X}_1$.

30. **(a)** The corresponding plane autonomous system is

$$x' = y, \quad y' = \epsilon(y - \frac{1}{3}y^3) - x$$

and so the only critical point is $(0,0)$. Since the Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon(1 - y^2) \end{pmatrix},$$

$\tau = \epsilon$, $\Delta = 1$, and so $\tau^2 - 4\Delta = \epsilon^2 - 4$ at the critical point $(0,0)$.

(b) When $\tau = \epsilon > 0$, $(0,0)$ is an unstable critical point.

(c) When $\epsilon < 0$ and $\tau^2 - 4\Delta = \epsilon^2 - 4 < 0$, $(0,0)$ is a stable spiral point. These two requirements can be written as $-2 < \epsilon < 0$.

(d) When $\epsilon = 0$, $x'' + x = 0$ and so $x = c_1 \cos t + c_2 \sin t$. Therefore all solutions are periodic (with period 2π) and so $(0,0)$ is a center.

31. $\dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{-2x^3}{y}$ may be solved by separating variables. It follows that $y^2 + x^4 = c$. If $\mathbf{X}(0) = (x_0, 0)$ where $x_0 > 0$, then $c = x_0^4$ so that $y^2 = x_0^4 - x^4$. Therefore if $-x_0 < x < x_0$, $y^2 > 0$ and so there are two values of y corresponding to each value of x. Therefore the solution $\mathbf{X}(t)$ with $\mathbf{X}(0) = (x_0, 0)$ is periodic and so $(0,0)$ is a center.

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32. $\dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{x^2-2x}{y}$ may be solved by separating variables. It follows that $\dfrac{y^2}{2} = \dfrac{x^3}{3} - x^2 + c$ and since $\mathbf{X}(0) = (x(0), x'(0)) = (1,0)$, $c = \dfrac{2}{3}$. Therefore

$$\frac{y^2}{2} = \frac{x^3 - 3x^2 + 2}{3} = \frac{(x-1)(x^2-2x-2)}{3}.$$

But $(x-1)(x^2-2x-2) > 0$ for $1-\sqrt{3} < x < 1$ and so each x in this interval has 2 corresponding values of y. therefore $\mathbf{X}(t)$ is a periodic solution.

33. **(a)** $x' = 2xy = 0$ implies that either $x = 0$ or $y = 0$. If $x = 0$, then from $1 - x^2 + y^2 = 0$, $y^2 = -1$ and there are no real solutions. If $y = 0$, $1 - x^2 = 0$ and so $(1,0)$ and $(-1,0)$ are critical points. The Jacobian matrix is

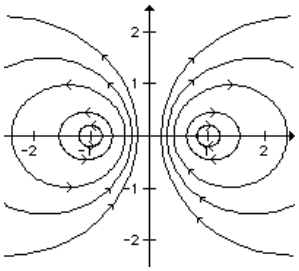
$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 2y & 2x \\ -2x & 2y \end{pmatrix}$$

and so $\tau = 0$ and $\Delta = 4$ at either $\mathbf{X} = (1,0)$ or $(-1,0)$. We obtain no information about these critical points in this borderline case.

(b) $\dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{1-x^2+y^2}{2xy}$ or $2xy\dfrac{dy}{dx} = 1 - x^2 + y^2$. Letting $\mu = \dfrac{y^2}{x}$, it follows that $\dfrac{d\mu}{dx} = \dfrac{1}{x^2} - 1$ and so $\mu = -\dfrac{1}{x} - x + 2c$. Therefore $\dfrac{y^2}{x} = -\dfrac{1}{x} - x + 2c$ which can be put in the form

$$(x-c)^2 + y^2 = c^2 - 1.$$

The solution curves are shown and so both $(1,0)$ and $(-1,0)$ are centers.



34. **(a)** $\dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{-x-y^2}{y} = -\dfrac{x}{y} - y$ and so $\dfrac{dy}{dx} + y = -xy^{-1}$.

(b) Let $w = y^{1-n} = y^2$. It follows that $\dfrac{dw}{dx} + 2w = -2x$, a linear first order differential equation whose solution is

$$y^2 = w = ce^{-2x} + \left(\frac{1}{2} - x\right).$$

Since $x(0) = \frac{1}{2}$ and $y(0) = x'(0) = 0$, $0 = c$ and so

$$y^2 = \frac{1}{2} - x,$$

a parabola with vertex at $(1/2, 0)$. Therefore the solution $\mathbf{X}(t)$ with $\mathbf{X}(0) = (1/2, 0)$ is not periodic.

35. $\dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{x^3-x}{y}$ and so $\dfrac{y^2}{2} = \dfrac{x^4}{4} - \dfrac{x^2}{2} + c$ or $y^2 = \dfrac{x^4}{2} - x^2 + c_1$. Since $x(0) = 0$ and $y(0) = x'(0) = v_0$, it follows that $c_1 = v_0^2$ and so

$$y^2 = \frac{1}{2}x^4 - x^2 + v_0^2 = \frac{(x^2-1)^2 + 2v_0^2 - 1}{2}.$$

The x-intercepts on this graph satisfy

$$x^2 = 1 \pm \sqrt{1 - 2v_0^2}$$

and so we must require that $1 - 2v_0^2 \geq 0$ (or $|v_0| \leq \frac{1}{2}\sqrt{2}$) for real solutions to exist. If $x_0^2 = 1 - \sqrt{1-2v_0^2}$ and $-x_0 < x < x_0$, then $(x^2-1)^2 + 2v_0^2 - 1 > 0$ and so there are two corresponding values of y. Therefore $\mathbf{X}(t)$ with $\mathbf{X}(0) = (0, v_0)$ is periodic provided that $|v_0| \leq \frac{1}{2}\sqrt{2}$.

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36. The corresponding plane autonomous system is

$$x' = y, \quad y' = \epsilon x^2 - x + 1$$

and so the critical points must satisfy $y = 0$ and

$$x = \frac{1 \pm \sqrt{1 - 4\epsilon}}{2\epsilon}.$$

Therefore we must require that $\epsilon \le \frac{1}{4}$ for real solutions to exist. We will use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ 2\epsilon x - 1 & 0 \end{pmatrix}$$

to attempt to classify $((1 \pm \sqrt{1 - 4\epsilon})/2\epsilon, 0)$ when $\epsilon \le 1/4$. Note that $\tau = 0$ and $\Delta = \mp\sqrt{1 - 4\epsilon}$. For $\mathbf{X} = ((1 + \sqrt{1 - 4\epsilon})/2\epsilon, 0)$ and $\epsilon < 1/4$, $\Delta < 0$ and so a saddle point occurs. For $\mathbf{X} = ((1 - \sqrt{1 - 4\epsilon})/2\epsilon, 0)$ $\Delta \ge 0$ and we are not able to classify this critical point using linearization.

37. The corresponding plane autonomous system is

$$x' = y, \quad y' = -\frac{\alpha}{L}x - \frac{\beta}{L}x^3 - \frac{R}{L}y$$

where $x = q$ and $y = q'$. If $\mathbf{X} = (x, y)$ is a critical point, $y = 0$ and $-\alpha x - \beta x^3 = -x(\alpha + \beta x^2) = 0$. If $\beta > 0$, $\alpha + \beta x^2 = 0$ has no real solutions and so $(0, 0)$ is the only critical point. Since

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ \dfrac{-\alpha - 3\beta x^2}{L} & -\dfrac{R}{L} \end{pmatrix},$$

$\tau = -R/L < 0$ and $\Delta = \alpha/L > 0$. Therefore $(0, 0)$ is a stable critical point. If $\beta < 0$, $(0, 0)$ and $(\pm\hat{x}, 0)$, where $\hat{x}^2 = -\alpha/\beta$ are critical points. At $\mathbf{X}(\pm\hat{x}, 0)$, $\tau = -R/L < 0$ and $\Delta = -2\alpha/L < 0$. Therefore both critical points are saddles.

38. If we let $dx/dt = y$, then $dy/dt = -x^3 - x$. From this we obtain the first-order differential equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{x^3 + x}{y}.$$

Separating variables and integrating we obtain

$$\int y\, dy = -\int (x^3 + x)\, dx$$

and

$$\frac{1}{2}y^2 = -\frac{1}{4}x^4 - \frac{1}{2}x^2 + c_1.$$

Completing the square we can write the solution as $y^2 = -\frac{1}{2}(x^2+1)^2 + c_2$. If $\mathbf{X}(0) = (x_0, 0)$, then $c_2 = \frac{1}{2}(x_0^2+1)^2$ and so

$$y^2 = -\frac{1}{2}(x^2 + 1)^2 + \frac{1}{2}(x_0^2 + 1)^2 = \frac{x_0^4 + 2x_0^2 + 1 - x^4 - 2x^2 - 1}{2}$$

$$= \frac{(x_0^2 + x^2)(x_0^2 - x^2) + 2(x_0^2 - x^2)}{2} = \frac{(x_0^2 + x^2 + 2)(x_0^2 - x^2)}{2}.$$

Note that $y = 0$ when $x = -x_0$. In addition, the right-hand side is positive for $-x_0 < x < x_0$, and so there are two corresponding values of y for each x between $-x_0$ and x_0. The solution $\mathbf{X} = \mathbf{X}(t)$ that satisfies $\mathbf{X}(0) = (x_0, 0)$ is therefore periodic, and so $(0, 0)$ is a center.

39. **(a)** Letting $x = \theta$ and $y = x'$ we obtain the system $x' = y$ and $y' = 1/2 - \sin x$. Since $\sin \pi/6 = \sin 5\pi/6 = 1/2$ we see that $(\pi/6, 0)$ and $(5\pi/6, 0)$ are critical points of the system.

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(b) The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix}$$

and so

$$\mathbf{A}_1 = \mathbf{g}' = ((\pi/6, 0)) = \begin{pmatrix} 0 & 1 \\ -\sqrt{3}/2 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_2 = \mathbf{g}' = ((5\pi/6, 0)) = \begin{pmatrix} 0 & 1 \\ \sqrt{3}/2 & 0 \end{pmatrix}.$$

Since $\det \mathbf{A}_1 > 0$ and the trace of $\mathbf{A}_1$ is 0, no conclusion can be drawn regarding the critical point $(\pi/6, 0)$. Since $\det \mathbf{A}_2 < 0$, we see that $(5\pi/6, 0)$ is a saddle point.

(c) From the system in part (a) we obtain the first-order differential equation

$$\frac{dy}{dx} = \frac{1/2 - \sin x}{y}.$$

Separating variables and integrating we obtain

$$\int y\, dy = \int \left(\frac{1}{2} - \sin x\right) dx$$

and

$$\frac{1}{2}y^2 = \frac{1}{2}x + \cos x + c_1$$

or

$$y^2 = x + 2\cos x + c_2.$$

For x_0 near $\pi/6$, if $\mathbf{X}(0) = (x_0, 0)$ then $c_2 = -x_0 - 2\cos x_0$ and $y^2 = x + 2\cos x - x_0 - 2\cos x_0$. Thus, there are two values of y for each x in a sufficiently small interval around $\pi/6$. Therefore $(\pi/6, 0)$ is a center.

40. (a) Writing the system as $x' = x(x^3 - 2y^3)$ and $y' = y(2x^3 - y^3)$ we see that $(0, 0)$ is a critical point. Setting $x^3 - 2y^3 = 0$ we have $x^3 = 2y^3$ and $2x^3 - y^3 = 4y^3 - y^3 = 3y^3$. Thus, $(0, 0)$ is the only critical point of the system.

(b) From the system we obtain the first-order differential equation

$$\frac{dy}{dx} = \frac{2x^3y - y^4}{x^4 - 2xy^3}$$

or

$$(2x^3y - y^4)\, dx + (2xy^3 - x^4)\, dy = 0$$

which is homogeneous. If we let $y = ux$ it follows that

$$\begin{aligned} (2x^4u - x^4u^4)\, dx + (2x^4u^3 - x^4)(u\, dx + x\, du) &= 0 \\ x^4u(1 + u^3)\, dx + x^5(2u^3 - 1)\, du &= 0 \\ \frac{1}{x}\, dx + \frac{2u^3 - 1}{u(u^3 + 1)}\, du &= 0 \\ \frac{1}{x}\, dx + \left(\frac{1}{u+1} - \frac{1}{u} + \frac{2u - 1}{u^2 - u + 1}\right) du &= 0. \end{aligned}$$

Integrating gives

$$\ln|x| + \ln|u + 1| - \ln|u| + \ln|u^2 - u + 1| = c_1$$

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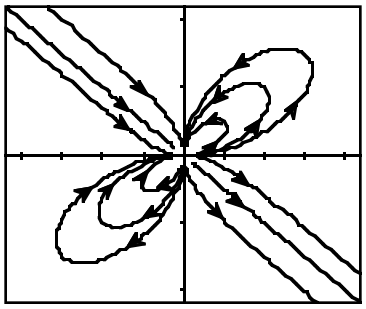
or

$$x\left(\frac{u+1}{u}\right)(u^2-u+1)=c_2$$
$$x\left(\frac{y+x}{y}\right)\left(\frac{y^2}{x^2}-\frac{y}{x}+1\right)=c_2$$
$$(xy+x^2)(y^2-xy+x^2)=c_2x^2y$$
$$xy^3+x^4=c_2x^2y$$
$$x^3+y^2=3c_3xy.$$

(c) We see from the graph that $(0,0)$ is unstable. It is not possible to classify the critical point as a node, saddle, center, or spiral point.



Exercises 11.4

1. We are given that $x(0)=\theta(0)=\dfrac{\pi}{3}$ and $y(0)=\theta'(0)=w_0$. Since $y^2=\dfrac{2g}{l}\cos x+c$, $w_0^2=\dfrac{2g}{l}\cos\dfrac{\pi}{3}+c=\dfrac{g}{l}+c$ and so $c=w_0^2-\dfrac{g}{l}$. Therefore

$$y^2=\frac{2g}{l}\left(\cos x-\frac{1}{2}+\frac{l}{2g}w_0^2\right)$$

and the x-intercepts occur where $\cos x=\dfrac{1}{2}-\dfrac{l}{2g}w_0^2$ and so $\dfrac{1}{2}-\dfrac{l}{2g}w_0^2$ must be greater than -1 for solutions to exist. This condition is equivalent to $|w_0|<\sqrt{\dfrac{3g}{l}}$.

2. **(a)** Since $y^2=\dfrac{2g}{l}\cos x+c$, $x(0)=\theta(0)=\theta_0$ and $y(0)=\theta'(0)=0$, $c=-\dfrac{2g}{l}\cos\theta_0$ and so $y^2=\dfrac{2g}{l}(\cos\theta-\cos\theta_0)$. When $\theta=-\theta_0$, $y^2=\dfrac{2g}{l}(\cos(-\theta_0)-\cos(\theta_0))=0$. Therefore $y=\dfrac{d\theta}{dt}=0$ when $\theta=\theta_0$.

(b) Since $y=\dfrac{d\theta}{dt}$ and θ is decreasing between the time when $\theta=\theta_0$, $t=0$, and $\theta=-\theta_0$, that is, $t=T$,

$$\frac{d\theta}{dt}=-\sqrt{\frac{2g}{l}}\,\sqrt{\cos\theta-\cos\theta_0}\,.$$

Therefore $\dfrac{dt}{d\theta}=-\sqrt{\dfrac{l}{2g}}\,\dfrac{1}{\sqrt{\cos\theta-\cos\theta_0}}$ and so

$$T=-\sqrt{\frac{l}{2g}}\int_{\theta=\theta_0}^{\theta=-\theta_0}\frac{1}{\sqrt{\cos\theta-\cos\theta_0}}\,d\theta=\sqrt{\frac{l}{2g}}\int_{-\theta_0}^{\theta_0}\frac{1}{\sqrt{\cos\theta-\cos\theta_0}}\,d\theta.$$

3. The corresponding plane autonomous system is

$$x'=y,\quad y'=-g\,\frac{f'(x)}{1+[f'(x)]^2}-\frac{\beta}{m}\,y$$

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and

$$\frac{\partial}{\partial x}\left(-g\,\frac{f'(x)}{1+[f'(x)]^2}-\frac{\beta}{m}\,y\right)=-g\,\frac{(1+[f'(x)]^2)f''(x)-f'(x)2f'(x)f''(x)}{(1+[f'(x)]^2)^2}\,.$$

If $\mathbf{X}_1=(x_1,y_1)$ is a critical point, $y_1=0$ and $f'(x_1)=0$. The Jacobian at this critical point is therefore

$$\mathbf{g}'(\mathbf{X}_1)=\begin{pmatrix}0 & 1\\ -gf''(x_1) & -\frac{\beta}{m}\end{pmatrix}.$$

4. When $\beta=0$ the Jacobian matrix is

$$\begin{pmatrix}0 & 1\\ -gf''(x_1) & 0\end{pmatrix}$$

which has complex eigenvalues $\lambda=\pm\sqrt{gf''(x_1)}\,i$. The approximating linear system with $x'(0)=0$ has solution

$$x(t)=x(0)\cos\sqrt{gf''(x_1)}\,t$$

and period $2\pi/\sqrt{gf''(x_1)}$. Therefore $p\approx 2\pi/\sqrt{gf''(x_1)}$ for the actual solution.

5. **(a)** If $f(x)=\dfrac{x^2}{2}$, $f'(x)=x$ and so

$$\frac{dy}{dx}=\frac{y'}{x'}=-g\,\frac{x}{1+x^2}\,\frac{1}{y}\,.$$

We may separate variables to show that $y^2=-g\ln(1+x^2)+c$. But $x(0)=x_0$ and $y(0)=x'(0)=v_0$. Therefore $c=v_0^2+g\ln(1+x_0^2)$ and so

$$y^2=v_0^2-g\ln\left(\frac{1+x^2}{1+x_0^2}\right).$$

Now

$$v_0^2-g\ln\left(\frac{1+x^2}{1+x_0^2}\right)\ge 0\quad\text{if and only if}\quad x^2\le e^{v_0^2/g}(1+x_0^2)-1.$$

Therefore, if $|x|\le[e^{v_0^2/g}(1+x_0^2)-1]^{1/2}$, there are two values of y for a given value of x and so the solution is periodic.

(b) Since $z=\dfrac{x^2}{2}$, the maximum height occurs at the largest value of x on the cycle. From (a), $x_{\max}=[e^{v_0^2/g}(1+x_0^2)-1]^{1/2}$ and so

$$z_{\max}=\frac{x_{\max}^2}{2}=\frac{1}{2}[e^{v_0^2/g}(1+x_0^2)-1].$$

6. **(a)** If $f(x)=\cosh x$, $f'(x)=\sinh x$ and $[f'(x)]^2+1=\sinh^2 x+1=\cosh^2 x$. Therefore

$$\frac{dy}{dx}=\frac{y'}{x'}=-g\,\frac{\sinh x}{\cosh^2 x}\,\frac{1}{y}\,.$$

We may separate variables to show that $y^2=\dfrac{2g}{\cosh x}+c$. But $x(0)=x_0$ and $y(0)=x'(0)=v_0$. Therefore $c=v_0^2-\dfrac{2g}{\cosh x_0}$ and so

$$y^2=\frac{2g}{\cosh x}-\frac{2g}{\cosh x_0}+v_0^2.$$

Now

$$\frac{2g}{\cosh x}-\frac{2g}{\cosh x_0}+v_0^2\ge 0\quad\text{if and only if}\quad\cosh x\le\frac{2g\cosh x_0}{2g-v_0^2\cosh x_0}$$

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and the solution to this inequality is an interval $[-a, a]$. Therefore each x in $(-a, a)$ has two corresponding values of y and so the solution is periodic.

(b) Since $z = \cosh x$, the maximum height occurs at the largest value of x on the cycle. From (a), $x_{\max} = a$ where $\cosh a = \dfrac{2g\cosh x_0}{2g - v_0^2\cosh x_0}$. Therefore

$$z_{\max} = \frac{2g\cosh x_0}{2g - v_0^2\cosh x_0}.$$

7. If $x_m < x_1 < x_n$, then $F(x_1) > F(x_m) = F(x_n)$. Letting $x = x_1$,

$$G(y) = \frac{c_0}{F(x_1)} = \frac{F(x_m)G(a/b)}{F(x_1)} < G(a/b).$$

Therefore from Figure 11.35(b), $G(y) = \dfrac{c_0}{F(x_1)}$ has two solutions y_1 and y_2 that satisfy $y_1 < a/b < y_2$.

8. From (i) when $y = a/b$, x_n is taken on at some time t. From (iii) if $x > x_n$ there is no corresponding value of y. Therefore the maximum number of predators is x_n and x_n occurs when $y = a/b$.

9. (a) In the Lotka-Volterra Model the average number of predators is d/c and the average number of prey is a/b. But

$$x' = -ax + bxy - \epsilon_1 x = -(a + \epsilon_1)x + bxy$$
$$y' = -cxy + dy - \epsilon_2 y = -cxy + (d - \epsilon_2)y$$

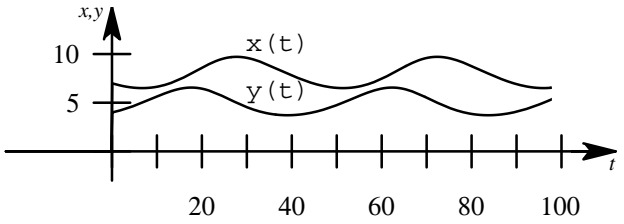
and so the new critical point in the first quadrant is $(d/c - \epsilon_2/c, a/b + \epsilon_1/b)$.

(b) The average number of predators $d/c - \epsilon_2/c$ has decreased while the average number of prey $a/b + \epsilon_1/b$ has increased. The fishery science model is consistent with Volterra's principle.

10. (a) Solving

$$x(-0.1 + 0.02y) = 0$$
$$y(0.2 - 0.025x) = 0$$

in the first quadrant we obtain the critical point $(8, 5)$. The graphs are plotted using $x(0) = 7$ and $y(0) = 4$.



(b) The graph in part (a) was obtained using **NDSolve** in *Mathematica*. We see that the period is around 40. Since $x(0) = 7$, we use the **FindRoot** equation solver in *Mathematica* to approximate the solution of $x(t) = 7$ for t near 40. From this we see that the period is more closely approximated by $t = 44.65$.

11. Solving

$$x(20 - 0.4x - 0.3y) = 0$$
$$y(10 - 0.1y - 0.3x) = 0$$

we see that critical points are $(0, 0)$, $(0, 100)$, $(50, 0)$, and $(20, 40)$. The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0.08(20 - 0.8x - 0.3y) & -0.024x \\ -0.018y & 0.06(10 - 0.2y - 0.3x) \end{pmatrix}$$

and so

$$\mathbf{A}_1 = \mathbf{g}'((0,0)) = \begin{pmatrix} 1.6 & 0 \\ 0 & 0.6 \end{pmatrix} \qquad \mathbf{A}_2 = \mathbf{g}'((0,100)) = \begin{pmatrix} -0.8 & 0 \\ -1.8 & -0.6 \end{pmatrix}$$

$$\mathbf{A}_3 = \mathbf{g}'((50,0)) = \begin{pmatrix} -1.6 & -1.2 \\ 0 & -0.3 \end{pmatrix} \qquad \mathbf{A}_4 = \mathbf{g}'((20,40)) = \begin{pmatrix} -0.64 & -0.48 \\ -0.72 & -0.24 \end{pmatrix}.$$

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Since $\det(\mathbf{A}_1) = \Delta_1 = 0.96 > 0$, $\tau = 2.2 > 0$, and $\tau_1^2 - 4\Delta_1 = 1 > 0$, we see that $(0,0)$ is an unstable node. Since $\det(\mathbf{A}_2) = \Delta_2 = 0.48 > 0$, $\tau = -1.4 < 0$, and $\tau_2^2 - 4\Delta_2 = 0.04 > 0$, we see that $(0,100)$ is a stable node. Since $\det(\mathbf{A}_3) = \Delta_3 = 0.48 > 0$, $\tau = -1.9 < 0$, and $\tau_3^2 - 4\Delta_3 = 1.69 > 0$, we see that $(50,0)$ is a stable node. Since $\det(\mathbf{A}_4) = -0.192 < 0$ we see that $(20,40)$ is a saddle point.

12. $\Delta = r_1 r_2$, $\tau = r_1 + r_2$ and $\tau^2 - 4\Delta = (r_1 + r_2)^2 - 4r_1 r_2 = (r_1 - r_2)^2$. Therefore when $r_1 \neq r_2$, $(0,0)$ is an unstable node.

13. For $\mathbf{X} = (K_1, 0)$, $\tau = -r_1 + r_2\left(1 - \dfrac{K_1}{K_2}\,\alpha_{21}\right)$ and $\Delta = -r_1 r_2\left(1 - \dfrac{K_1}{K_2}\,\alpha_{21}\right)$. If we let $c = 1 - \dfrac{K_1}{K_2}\,\alpha_{21}$, $\tau^2 - 4\Delta = (cr_2 + r_1)^2 > 0$. Now if $k_1 > \dfrac{K_2}{\alpha_{21}}$, $c < 0$ and so $\tau < 0$, $\Delta > 0$. Therefore $(K_1, 0)$ is a stable node. If $K_1 < \dfrac{K_2}{\alpha_{21}}$, $c > 0$ and so $\Delta < 0$. In this case $(K_1, 0)$ is a saddle point.

14. $(\hat{x}, \hat{y})$ is a stable node if and only if $\dfrac{K_1}{\alpha_{12}} > K_2$ and $\dfrac{K_2}{\alpha_{21}} > K_1$. [See Figure 11.38(a) in the text.] From Problem 12, (0.0) is an unstable node and from Problem 13, since $K_1 < \dfrac{K_2}{\alpha_{21}}$, $(K_1, 0)$ is a saddle point. Finally, when $K_2 < \dfrac{K_1}{\alpha_{12}}$, $(0, K_2)$ is a saddle point. This is Problem 12 with the roles of 1 and 2 interchanged. Therefore $(0,0)$, $(K_1, 0)$, and $(0, K_2)$ are unstable.

15. $\dfrac{K_1}{\alpha_{12}} < K_2 < K_1\alpha_{21}$ and so $\alpha_{12}\alpha_{21} > 1$. Therefore $\Delta = (1 - \alpha_{12}\alpha_{21})\hat{x}\hat{y}\,\dfrac{r_1 r_2}{K_1 K_2} < 0$ and so $(\hat{x}, \hat{y})$ is a saddle point.

16. **(a)** The corresponding plane autonomous system is

$$x' = y, \quad y' = \frac{-g}{l}\sin x - \frac{\beta}{ml}\,y$$

and so critical points must satisfy both $y = 0$ and $\sin x = 0$. Therefore $(\pm n\pi, 0)$ are critical points.

(b) The Jacobian matrix

$$\begin{pmatrix} 0 & 1 \\ -\dfrac{g}{l}\cos x & -\dfrac{\beta}{ml} \end{pmatrix}$$

has trace $\tau = -\dfrac{\beta}{ml}$ and determinant $\Delta = \dfrac{g}{l} > 0$ at $(0,0)$. Therefore

$$\tau^2 - 4\Delta = \frac{\beta^2}{m^2 l^2} - 4\,\frac{g}{l} = \frac{\beta^2 - 4glm^2}{m^2 l^2}\,.$$

We may conclude that $(0,0)$ is a stable spiral point provided $\beta^2 - 4glm^2 < 0$ or $\beta < 2m\sqrt{gl}\,$.

17. **(a)** The corresponding plane autonomous system is

$$x = y, \quad y' = -\frac{\beta}{m}\,y|y| - \frac{k}{m}\,x$$

and so a critical point must satisfy both $y = 0$ and $x = 0$. Therefore $(0,0)$ is the unique critical point.

(b) The Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{\beta}{m}\,2|y| \end{pmatrix}$$

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and so $\tau = 0$ and $\Delta = \dfrac{k}{m} > 0$. Therefore $(0, 0)$ is a center, stable spiral point, or an unstable spiral point. Physical considerations suggest that $(0, 0)$ must be asymptotically stable and so $(0, 0)$ must be a stable spiral point.

18. (a) The magnitude of the frictional force between the bead and the wire is $\mu(mg\cos\theta)$ for some $\mu > 0$. The component of this frictional force in the x-direction is

$$(\mu mg\cos\theta)\cos\theta = \mu mg\cos^2\theta.$$

But

$$\cos\theta = \frac{1}{\sqrt{1 + [f'(x)]^2}} \quad \text{and so} \quad \mu mg\cos^2\theta = \frac{\mu mg}{1 + [f'(x)]^2}.$$

It follows from Newton's Second Law that

$$mx'' = -mg\,\frac{f'(x)}{1 + [f'(x)]^2} - \beta x' + mg\,\frac{\mu}{1 + [f'(x)]^2}$$

and so

$$x'' = g\,\frac{\mu - f'(x)}{1 + [f'(x)]^2} - \frac{\beta}{m}\,x'.$$

(b) A critical point (x, y) must satisfy $y = 0$ and $f'(x) = \mu$. Therefore critical points occur at $(x_1, 0)$ where $f'(x_1) = \mu$. The Jacobian matrix of the plane autonomous system is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ g\,\dfrac{(1 + [f'(x)]^2)(-f''(x)) - (\mu - f'(x))2f'(x)f''(x)}{(1 + [f'(x)]^2)^2} & -\dfrac{\beta}{m} \end{pmatrix}$$

and so at a critical point $\mathbf{X}_1$,

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ \dfrac{-gf''(x_1)}{1 + \mu^2} & -\dfrac{\beta}{m} \end{pmatrix}.$$

Therefore $\tau = -\dfrac{\beta}{m} < 0$ and $\Delta = \dfrac{gf''(x_1)}{1 + \mu^2}$. When $f''(x_1) < 0$, $\Delta < 0$ and so a saddle point occurs. When $f''(x_1) > 0$ and

$$\tau^2 - 4\Delta = \frac{\beta^2}{m^2} - 4g\,\frac{f''(x_1)}{1 + \mu^2} < 0,$$

$(x_1, 0)$ is a stable spiral point. This condition may also be written as

$$\beta^2 < 4gm^2\,\frac{f''(x_1)}{1 + \mu^2}.$$

19. $\dfrac{dy}{dx} = \dfrac{y'}{x'} = -\dfrac{f(x)}{y}$ and so using separation of variables, $\dfrac{y^2}{2} = -\displaystyle\int_0^x f(\mu)\,d\mu + c$ or $y^2 + 2F(x) = c$. We may conclude that for a given value of x there are at most two corresponding values of y. If $(0, 0)$ were a stable spiral point there would exist an x with more than two corresponding values of y. Note that the condition $f(0) = 0$ is required for $(0, 0)$ to be a critical point of the corresponding plane autonomous system $x' = y$, $y' = -f(x)$.

20. (a) $x' = x(-a + by) = 0$ implies that $x = 0$ or $y = a/b$. If $x = 0$, then, from

$$-cxy + \frac{r}{K}\,y(K - y) = 0,$$

$y = 0$ or K. Therefore $(0, 0)$ and $(0, K)$ are critical points. If $\hat{y} = a/b$, then

$$\hat{y}\left[-cx + \frac{r}{K}(K - \hat{y})\right] = 0.$$

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The corresponding value of x, $x = \hat{x}$, therefore satisfies the equation $c\hat{x} = \dfrac{r}{K}(K - \hat{y})$.

(b) The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -a + by & bx \\ -cy & -cx + \dfrac{r}{K}(K - 2y) \end{pmatrix}$$

and so at $\mathbf{X}_1 = (0, 0)$, $\Delta = -ar < 0$. For $\mathbf{X}_1 = (0, K)$, $\Delta = n(Kb - a) = -rb\left(K - \dfrac{a}{b}\right)$. Since we are given that $K > \dfrac{a}{b}$, $\Delta < 0$ in this case. Therefore $(0, 0)$ and $(0, K)$ are each saddle points. For $\mathbf{X}_1 = (\hat{x}, \hat{y})$ where $\hat{y} = \dfrac{a}{b}$ and $c\hat{x} = \dfrac{r}{K}(K - \hat{y})$, we may write the Jacobian matrix as

$$\mathbf{g}'((\hat{x}, \hat{y})) = \begin{pmatrix} 0 & b\hat{x} \\ -c\hat{y} & -\dfrac{r}{K}\,\hat{y} \end{pmatrix}$$

and so $\tau = -\dfrac{r}{K}\,\hat{y} < 0$ and $\Delta = bc\hat{x}\hat{y} > 0$. Therefore $(\hat{x}, \hat{y})$ is a stable critical point and so it is either a stable node (perhaps degenerate) or a stable spiral point.

(c) Write

$$\tau^2 - 4\Delta = \frac{r^2}{K^2}\,\hat{y}^2 - 4bc\hat{x}\hat{y} = \hat{y}\left[\frac{r^2}{K^2}\,\hat{y} - 4bc\hat{x}\right] = \hat{y}\left[\frac{r^2}{K^2}\,\hat{y} - 4b\,\frac{r}{K}(K - \hat{y})\right]$$

using

$$c\hat{x} = \frac{r}{K}(K - \hat{y}) = \frac{r}{K}\,\hat{y}\left[\left(\frac{r}{K} + 4b\right)\hat{y} - 4bK\right].$$

Therefore $\tau^2 - 4\Delta < 0$ if and only if

$$\hat{y} < \frac{4bK}{\frac{r}{K} + 4b} = \frac{4bK^2}{r + 4bK}.$$

Note that

$$\frac{4bK^2}{r + 4bK} = \frac{4bK}{r + 4bK} \cdot K \approx K$$

where K is large, and $\hat{y} = \dfrac{a}{b} < K$. Therefore $\tau^2 - 4\Delta < 0$ when K is large and a stable spiral point will result.

21. The equation

$$x' = \alpha\,\frac{y}{1 + y}\,x - x = x\left(\frac{\alpha y}{1 + y} - 1\right) = 0$$

implies that $x = 0$ or $y = \dfrac{1}{\alpha - 1}$. When $\alpha > 0$, $\hat{y} = \dfrac{1}{\alpha - 1} > 0$. If $x = 0$, then from the differential equation for y', $y = \beta$. On the other hand, if $\hat{y} = \dfrac{1}{\alpha - 1}$, $\dfrac{\hat{y}}{1 + \hat{y}} = \dfrac{1}{\alpha}$ and so $\dfrac{1}{\alpha}\,\hat{x} - \dfrac{1}{\alpha - 1} + \beta = 0$. It follows that

$$\hat{x} = \alpha\left(\beta - \frac{1}{\alpha - 1}\right) = \frac{\alpha}{\alpha - 1}[(\alpha - 1)\beta - 1]$$

and if $\beta(\alpha - 1) > 1$, $\hat{x} > 0$. Therefore $(\hat{x}, \hat{y})$ is the unique critical point in the first quadrant. The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} \alpha\,\dfrac{y}{y + 1} - 1 & \dfrac{\alpha x}{(1 + y)^2} \\ -\dfrac{y}{1 + y} & \dfrac{-x}{(1 + y)^2} - 1 \end{pmatrix}$$

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and for $\mathbf{X} = (\hat{x}, \hat{y})$, the Jacobian can be written in the form

$$\mathbf{g}'((\hat{x}, \hat{y})) = \begin{pmatrix} 0 & \dfrac{(\alpha-1)^2}{\alpha}\,\hat{x} \\ -\dfrac{1}{\alpha} & -\dfrac{(\alpha-1)^2}{\alpha^2} - 1 \end{pmatrix}.$$

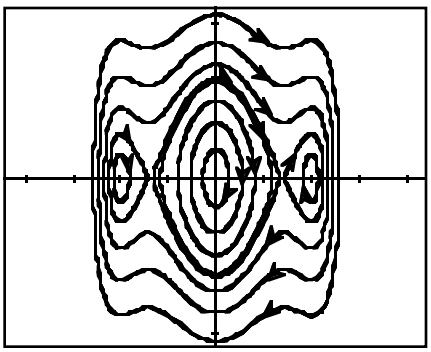
It follows that

$$\tau = -\left[\frac{(\alpha-1)^2}{\alpha^2}\,\hat{x} + 1\right] < 0, \quad \Delta = \frac{(\alpha-1)^2}{\alpha^2}\,\hat{x}$$

and so $\tau = -(\Delta + 1)$. Therefore $\tau^2 - 4\Delta = (\Delta+1)^2 - 4\Delta = (\Delta-1)^2 > 0$. Therefore $(\hat{x}, \hat{y})$ is a stable node.

22. Letting $y = x'$ we obtain the plane autonomous system

$$x' = y$$
$$y' = -8x + 6x^3 - x^5.$$



Solving $x^5 - 6x^3 + 8x = x(x^2-4)(x^2-2) = 0$ we see that critical points are $(0,0)$, $(0,-2)$, $(0,2)$, $(0,-\sqrt{2}\,)$, and $(0,\sqrt{2}\,)$. The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ -8 + 18x^2 - 5x^4 & 0 \end{pmatrix}$$

and we see that $\det(\mathbf{g}'(\mathbf{X})) = 5x^4 - 18x^2 + 8$ and the trace of $\mathbf{g}'(\mathbf{X})$ is 0. Since $\det(g'((\pm\sqrt{2}\,, 0))) = -8 < 0$, $(\pm\sqrt{2}\,, 0)$ are saddle points. For the other critical points the determinant is positive and linearization discloses no information. The graph of the phase plane suggests that $(0,0)$ and $(\pm 2, 0)$ are centers.

Exercises 11.5

1. $y' = x - y = 0$ implies that $y = x$ and so $2 + xy = 2 + x^2 > 0$. Therefore the system has no critical points, and so, by the corollary to Theorem 11.4, there are no periodic solutions.

2. $x' = 2x - xy = x(2-y) = 0$ implies that $x = 0$ or $y = 2$. If $x = 0$, then from $-1 - x^2 + 2x - y^2 = 0$, $y^2 = -1$ and there are no real solutions. If $y = 2$, $-x^2 + 2x - 5 = 0$ which has no real solutions. Therefore the system has no critical points and so, by the corollary to Theorem 11.4, there are no periodic solutions.

3. For $P = -x + y^2$ and $Q = x - y$, $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} = -2 < 0$. Therefore there are no periodic solutions by Theorem 11.5.

4. For $P = xy^2 - x^2y$ and $Q = x^2y - 1$, $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} = y^2 - 2xy + x^2 = (y-x)^2$. Therefore $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ does not change signs and so there are no periodic solutions by Theorem 11.5.

5. For $P = -\mu x - y$ and $Q = x + y^3$, $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} = -\mu + 9y^2 > 0$ since $\mu < 0$. Therefore there are no periodic solutions by Theorem 11.5.

6. From $y' = xy - y = y(x-1) = 0$ either $y = 0$ or $x = 1$. If $y = 0$, then from $2x + y^2 = 0$, $x = 0$. Likewise $x = 1$ implies that $2 + y^2 = 0$, which has no real solutions. Therefore $(0,0)$ is the only critical point. But $\mathbf{g}'((0,0))$ has determinant $\Delta = -2$. The single critical point is a saddle point and so, by the corollary to Theorem 11.4, there are no periodic solutions.

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7. The corresponding plane autonomous system is $x' = y$, $y' = 2x - y^4$. Therefore $(0, 0)$ is the only critical point. But $\mathbf{g}'((0, 0))$ has determinant $\Delta = -2 < 0$. The single critical point is a saddle point and so, by the corollary to Theorem 11.4, there are no periodic solutions.

8. The corresponding plane autonomous system is

$$x' = y, \quad y' = -x + \left(\frac{1}{2} + 3y^2\right) y - x^2$$

and so $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} = 0 + \dfrac{1}{2} + 9y^2 > 0$. Therefore there are no periodic solutions by Theorem 11.5.

9. For $\delta(x, y) = e^{ax+by}$, $\dfrac{\partial}{\partial x}(\delta P) + \dfrac{\partial}{\partial y}(\delta Q)$ can be simplified to

$$e^{ax+by}[-bx^2 - 2ax + axy + (2b + 1)y].$$

Setting $a = 0$ and $b = -1/2$,

$$\frac{\partial}{\partial}(\delta P) + \frac{\partial}{\partial y}(\delta Q) = \frac{1}{2}x^2 e^{-\frac{1}{2}y}$$

which does not change signs. Therefore by Theorem 11.6 there are no periodic solutions.

10. For $\delta(x, y) = ax^2 + by^2$, $\dfrac{\partial}{\partial x}(\delta P) + \dfrac{\partial}{\partial y}(\delta Q)$ can be simplified to

$$-5ax^4 - 3bx^2y^2 + 10(a - b)x^2y.$$

Setting $a = b = 1$,

$$\frac{\partial}{\partial x}(\delta P) + \frac{\partial}{\partial y}(\delta Q) = -5x^4 - 3x^2y^2$$

which does not change signs. Therefore by Theorem 11.6 there are no periodic solutions.

11. For $P = x(1 - x^2 - 3y^2)$ and $Q = y(3 - x^2 - 3y^2)$,

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 4(1 - x^2 - 3y^2) \quad \text{and so} \quad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} > 0$$

for $x^2 + 3y^2 < 1$. Therefore there are no periodic solutions in the elliptical region $x^2 + 3y^2 < 1$.

12. The corresponding plane autonomous system is $x' = y$, $y' = g(x, y)$ and so

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \frac{\partial g}{\partial y} = \frac{\partial g}{\partial x'} \neq 0$$

in the region R. Therefore $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ cannot change signs and so there are no periodic solutions by Theorem 11.5.

13. For $\delta(x, y) = \dfrac{1}{xy}$, $\delta P = -\dfrac{a}{y} + b$, $\delta Q = -c + \dfrac{r}{K}\dfrac{1}{x}(K - y)$ and so

$$\frac{\partial(\delta P)}{\partial x} + \frac{\partial(\delta Q)}{\partial y} = -\frac{r}{Kx} < 0$$

in the first quadrant. By Theorem 11.6, there are no periodic solutions in the first quadrant.

14. If $\mathbf{n} = (-2x, -2y)$,

$$\mathbf{V} \cdot \mathbf{n} = 2xy + 2x^2e^{x+y} - 2xy + 2y^2e^{x+y} = 2(x^2 + y^2)e^{x+y} \geq 0.$$

Therefore any circular region of the form $x^2 + y^2 \leq r^2$ is an invariant region by Theorem 11.7.

15. If $\mathbf{n} = (-2x, -2y)$,

$$\mathbf{V} \cdot \mathbf{n} = 2x^2 - 4xy + 2y^2 + 2y^4 = 2(x - y)^2 + 2y^4 \geq 0.$$

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Therefore $x^2 + y^2 \le r$ serves as an invariant region for any $r > 0$ by Theorem 11.7.

16. $\mathbf{n} = -\nabla t = (-6x^5, -6y)$. Since the corresponding plane autonomous system is $x' = y$, $y' = -y - y^3 - x^5$,

$$\mathbf{n} \cdot \mathbf{V} = -6x^5 y + 6y^2 + 6y^4 + 6x^5 y = 6y^2 + 6y^4.$$

Therefore $\mathbf{n} \cdot \mathbf{V} \ge 0$ and so by Theorem 11.7, the region $x^6 + 3y^2 \le 1$ serves as an invariant region.

17. We showed in Example 8 that $\frac{1}{16} \le x^2 + y^2 \le 1$ is an invariant region for the plane autonomous system. If the only critical point is $(0, 0)$, this critical point lies outside the invariant region and so Theorem 11.8(ii) is applicable. There is at least one periodic solution in R.

18. The corresponding plane autonomous system is

$$x' = y, \quad y' = y(1 - 3x^2 - 2y^2) - x$$

and it is easy to see that $(0, 0)$ is the only critical point. If $\mathbf{n} = (-2x, -2y)$ then

$$\mathbf{V} \cdot \mathbf{n} = -2xy - 2y^2(1 - 3x^2 - 2y^2) + 2xy = -2y^2(1 - 2r^2 - x^2).$$

If $r = \frac{1}{2}\sqrt{2}$, $2r^2 = 1$ and so $\mathbf{V} \cdot \mathbf{n} = 2x^2y^2 \ge 0$. Therefore $\frac{1}{4} \le x^2 + y^2 \le \frac{1}{2}$ serves as an invariant region. By Theorem 11.8(ii) there is at least one periodic solution.

19. If $r < 1$ and $\mathbf{n} = (-2x, -2y)$ then

$$\mathbf{V} \cdot \mathbf{n} = -2xy + 2xy + 2y^2(1 - x^2) = 2y^2(1 - x^2) \ge 0$$

since $x^2 < 1$. Therefore $x^2 + y^2 \le r^2$ serves as an invariant region. Now $(0, 0)$ is the only critical point and, since $\tau = -1$ and $\Delta = 1$, $\tau^2 - 4\Delta < 0$. Therefore $(0, 0)$ is a stable spiral point and so, by Theorem 11.9(ii), $\lim_{t\to\infty} \mathbf{X}(t) = (0, 0)$.

20. Since $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} = -1 - 3y^2 < 0$, there are no periodic solutions. If $\mathbf{n} = (-2x, -2y)$,

$$\mathbf{V} \cdot \mathbf{n} = -2xy + 2x^2 + 2xy + 2y^4 = 2(x^2 + y^4) \ge 0.$$

Therefore the circular region $x^2 + y^2 \le r^2$ serves as an invariant region for any $r > 0$. If (x, y) is a critical point. $y - x = 0$ or $y = x$. From $-x - y^3 = 0$ we have $-y(1 + y^2) = 0$. Therefore $y = 0$ and so $(0, 0)$ is the only critical point. It is easy to check that $\tau = -1$, $\Delta = 1$, $\tau^2 - 4\Delta = -3$ and so $(0, 0)$ is a stable spiral point. By Theorem 11.9(ii), $(0, 0)$ is globally stable. For any initial condition, $\lim_{t\to\infty} \mathbf{X}(t) = (0, 0)$.

21. **(a)** $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} = 2xy - 1 - x^2 \le 2x - 1 - x^2 = -(x - 1)^2 \le 0$. Therefore there are no periodic solutions.

(b) If (x, y) is a critical point, $x^2 y = \frac{1}{2}$ and so from $x' = x^2 y - x + 1$, $\frac{1}{2} - x + 1 = 0$. Therefore $x = 3/2$ and so $y = 2/9$. For this critical point, $\tau = -31/12 < 0$, $\Delta = 9/4 > 0$, and $\tau^2 - 4\Delta < 0$. Therefore $(3/2, 2/9)$ is a stable spiral point and so, from Theorem 11.9(ii), $\lim_{x\to\infty} \mathbf{X}(t) = (3/2, 2/9)$.

22. **(a)** From $x\left(\dfrac{2y}{y+2} - 1\right) = 0$, either $x = 0$ or $y = 2$. For the case $x = 0$, from $y\left(1 - \dfrac{2x}{y+2} - \dfrac{y}{8}\right) = 0$, $y\left(1 - \dfrac{y}{8}\right) = 0$. Therefore $(0, 0)$ and $(0, 8)$ are critical points. If $y = 2$, $1 - \dfrac{2x}{4} - \dfrac{1}{4} = 0$ and so $x = 3/2$. Therefore $(3/2, 2)$ is the additional critical point. We may classify these critical points as follows:

$\mathbf{X}$	τ	Δ	$\tau^2 - 4\Delta$	Conclusion
$(0, 0)$	–	-1	–	saddle point
$(0, 8)$	–	$-\frac{3}{5}$	–	saddle point
$(\frac{3}{2}, 2)$	$\frac{1}{8}$	$\frac{3}{8}$	$-\frac{95}{64}$	unstable spiral point

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(b) By Theorem 11.8(i), since there is a unique unstable critical point inside the invariant region, there is at least one periodic solution.

Chapter 11 Review Exercises

1. True

2. True

3. A center or a saddle point

4. Complex with negative real parts

5. False; there are initial conditions for which $\lim_{t\to\infty} \mathbf{X}(t) = (0,0)$.

6. True

7. False; this is a borderline case. See Figure 11.25 in the text.

8. False; see Figure 11.29 in the text.

9. True

10. False; we also need to have no critical points on the boundary of R.

11. Switching to polar coordinates,

$$\frac{dr}{dt} = \frac{1}{r}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) = \frac{1}{r}(-xy - x^2r^3 + xy - y^2r^3) = -r^4$$

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\,\frac{dx}{dt} + x\,\frac{dy}{dt}\right) = \frac{1}{r^2}(y^2 + xyr^3 + x^2 - xyr^3) = 1.$$

Using separation of variables it follows that $r = \dfrac{1}{\sqrt[3]{3t + c_1}}$ and $\theta = t + c_2$. Since $\mathbf{X}(0) = (1,0)$, $r = 1$ and $\theta = 0$. It follows that $c_1 = 1$, $c_2 = 0$, and so

$$r = \frac{1}{\sqrt[3]{3t+1}}, \quad \theta = t.$$

As $t \to \infty$, $r \to 0$ and the solution spirals toward the origin.

12. **(a)** If $\mathbf{X}(0) = \mathbf{X}_0$ lies on the line $y = -2x$, then $\mathbf{X}(t)$ approaches $(0,0)$ along this line. For all other initial conditions, $\mathbf{X}(t)$ approaches $(0,0)$ from the direction determined by the line $y = x$.

(b) If $\mathbf{X}(0) = \mathbf{X}_0$ lies on the line $y = -x$, then $\mathbf{X}(t)$ approaches $(0,0)$ along this line. For all other initial conditions, $\mathbf{X}(t)$ becomes unbounded and $y = 2x$ serves as an asymptote.

13. **(a)** $\tau = 0$, $\Delta = 11 > 0$ and so $(0,0)$ is a center.

(b) $\tau = -2$, $\Delta = 1$, $\tau^2 - 4\Delta = 0$ and so $(0,0)$ is a degenerate stable node.

14. From $x' = x(1 + y - 3x) = 0$, either $x = 0$ or $1 + y - 3x = 0$. If $x = 0$, then, from $y(4 - 2x - y) = 0$ we obtain $y(4 - y) = 0$. It follows that $(0,0)$ and $(0,4)$ are critical points. If $1 + y - 3x = 0$, then $y(5 - 5x) = 0$. Therefore $(1/3, 0)$ and $(1,2)$ are the remaining critical points. We will use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{bmatrix} 1 + y - 6x & x \\ -2y & 4 - 2x - 2y \end{bmatrix}$$

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to classify these four critical points. The results are as follows:

$\mathbf{X}$	τ	Δ	$\tau^2 - 4\Delta$	Conclusion
$(0,0)$	5	4	9	unstable node
$(0,4)$	–	-20	–	saddle point
$(\frac{1}{3},0)$	–	$-\frac{10}{3}$	–	saddle point
$(1,2)$	-5	10	-15	stable spiral point

If $\delta(x,y) = \dfrac{1}{xy}$, $\delta P = \dfrac{1}{y} + 1 - 3\dfrac{x}{y}$ and $\delta Q = \dfrac{4}{x} - 2 - \dfrac{y}{x}$. It follows that

$$\frac{\partial}{\partial x}(\delta P) + \frac{\partial}{\partial y}(\delta Q) = -\frac{3}{y} - \frac{1}{x} < 0$$

in quadrant one. Therefore there are no periodic solutions in the first quadrant.

15. The corresponding plane autonomous system is $x' = y$, $y' = \mu(1 - x^2) - x$ and so the Jacobian at the critical point $(0,0)$ is

$$\mathbf{g}'((0,0)) = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}.$$

Therefore $\tau = \mu$, $\Delta = 1$ and $\tau^2 - 4\Delta = \mu^2 - 4$. Now $\mu^2 - 4 < 0$ if and only if $-2 < \mu < 2$. We may therefore conclude that $(0,0)$ is a stable node for $\mu < -2$, a stable spiral point for $-2 < \mu < 0$, an unstable spiral point for $0 < \mu < 2$, and an unstable node for $\mu > 2$.

16. Critical points occur at $x = \pm 1$. Since

$$g'(x) = -\frac{1}{2}\, e^{-x/2}(x^2 - 4x - 1),$$

$g'(1) > 0$ and $g'(-1) < 0$. Therefore $x = 1$ is unstable and $x = -1$ is asymptotically stable.

17. $\dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{-2x\sqrt{y^2+1}}{y}$. We may separate variables to show that $\sqrt{y^2+1} = -x^2 + c$. But $x(0) = x_0$ and $y(0) = x'(0) = 0$. It follows that $c = 1 + x_0^2$ so that

$$y^2 = (1 + x_0^2 - x^2)^2 - 1.$$

Note that $1 + x_0^2 - x^2 > 1$ for $-x_0 < x < x_0$ and $y = 0$ for $x = \pm x_0$. Each x with $-x_0 < x < x_0$ has two corresponding values of y and so the solution $\mathbf{X}(t)$ with $\mathbf{X}(0) = (x_0, 0)$ is periodic.

18. The corresponding plane autonomous system is

$$x' = y, \quad y' = -\frac{\beta}{m}\, y - \frac{k}{m}(s + x)^3 + g$$

and so the Jacobian is

$$\mathbf{g}'(\mathbf{X}) = \begin{bmatrix} 0 & 1 \\ -\frac{3k}{m}(s+x)^2 & -\frac{\beta}{m} \end{bmatrix}.$$

For $\mathbf{X} = (0,0)$, $\tau = -\dfrac{\beta}{m} < 0$, $\Delta = \dfrac{3k}{m}\, s^2 > 0$. Therefore

$$\tau^2 - 4\Delta = \frac{\beta^2}{m^2} - \frac{12k}{m}\, s^2 = \frac{1}{m^2}(\beta^2 - 12kms^2).$$

Therefore $(0,0)$ is a stable node if $\beta^2 > 12kms^2$ and a stable spiral point provided $\beta^2 < 12kms^2$, where $ks^3 = mg$.

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19. For $P = 4x + 2y - 2x^2$ and $Q = 4x - 3y + 4xy$.

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 4 - 4x - 3 + 4x = 1 > 0.$$

Therefore there are no periodic solutions by Theorem 11.5.

20. If (x, y) is a critical point,

$$P = \epsilon x + y - x(x^2 + y^2) = 0$$
$$Q = -x + \epsilon y - y(x^2 + y^2) = 0.$$

But $yP - xQ = y^2 + x^2$. Therefore $x^2 + y^2 = 0$ and so $x = y = 0$. It follows that $(0, 0)$ is the only critical point and

$$\mathbf{g}'((0,0)) = \begin{bmatrix} \epsilon & 1 \\ -1 & \epsilon \end{bmatrix}$$

so that $\tau = 2\epsilon$, $\Delta = \epsilon^2 + 1 > 0$, and so $\tau^2 - 4\Delta = -4$. Note that if $\mathbf{n} = (-2x, -2y)$, $\mathbf{n} \cdot \mathbf{V} = 2r^2(r^2 - \epsilon)$. If $\epsilon > 0$, then $\tau > 0$ and so $(0, 0)$ is an unstable spiral point. If $r = 2\epsilon$, $\mathbf{n} \cdot \mathbf{V} \geq 0$ and so $x^2 + y^2 \leq 4\epsilon^2$ is an invariant region. It follows from Theorem 11.8(i) that there is at least one periodic solution in this region. If $\epsilon < 0$, $\tau < 0$ and so $(0, 0)$ is a stable spiral point. Note that $\mathbf{n} \cdot \mathbf{V} = 2r^2(r^2 - \epsilon) \geq 0$ for any r and so $x^2 + y^2 \leq r^2$ is an invariant region. It follows from Theorem 11.9(ii) that $\lim_{t \to +\infty} \mathbf{X}(t) = (0, 0)$ for any choice of initial position $\mathbf{X}_0$.

21. **(a)** If $x = \theta$ and $y = x' = \theta'$, the corresponding plane autonomous system is

$$x' = y, \quad y' = \omega^2 \sin x \cos x - \frac{g}{l} \sin x - \frac{\beta}{ml}\, y.$$

Therefore $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} = -\dfrac{\beta}{ml} < 0$ and so there are no periodic solutions.

(b) If (x, y) is a critical point, $y = 0$ and so $\sin x(\omega^2 \cos x - g/l) = 0$. Either $\sin x = 0$ (in which case $x = 0$) of $\cos x = g/\omega^2 l$. But if $\omega^2 < g/l$, $g/\omega^2 l > 1$ and so the latter equation has no real solutions. Therefore $(0, 0)$ is the only critical point if $\omega^2 < g/l$. The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{bmatrix} 0 & 1 \\ \omega^2 \cos 2x - \frac{g}{l} \cos x & -\frac{\beta}{ml} \end{bmatrix}$$

and so $\tau = -\beta/ml < 0$ and $\Delta = g/l - \omega^2 > 0$ for $\mathbf{X} = (0, 0)$. It follows that $(0, 0)$ is asymptotically stable and so after a small displacement, the pendulum will return to $\theta = 0$, $\theta' = 0$.

(c) If $\omega^2 > g/l$, $\cos x = g/\omega^2 l$ will have two solutions $x = \pm\hat{x}$ that satisfy $-\pi < x < \pi$. Therefore $(\pm\hat{x}, 0)$ are two additional critical points. If $\mathbf{X}_1 = (0, 0)$, $\Delta = g/l - \omega^2 < 0$ and so $(0, 0)$ is a saddle point. If $\mathbf{X}_1 = (\pm\hat{x}, 0)$, $\tau = -\beta/ml < 0$ and

$$\Delta = \frac{g}{l} \cos \hat{x} - \omega^2 \cos 2\hat{x} = \frac{g^2}{\omega^2 l^2} - \omega^2 \left(2\, \frac{g^2}{\omega^4 l^2} - 1 \right) = \omega^2 - \frac{g^2}{\omega^2 l^2} > 0.$$

Therefore $(\hat{x}, 0)$ and $(-\hat{x}, 0)$ are each stable. When $\theta(0) = \theta_0$, $\theta'(0) = 0$ and θ_0 is small we expect the pendulum to reach one of these two stable equilibrium positions.

(d) In (**b**), $(0, 0)$ is a stable spiral point provided

$$\tau^2 - 4\Delta = \frac{\beta^2}{m^2 l^2} - 4 \left(\frac{g}{l} - \omega^2 \right) < 0.$$

This condition is equivalent to $\beta < 2ml\sqrt{g/l - \omega^2}$. In (**c**), $(\pm\hat{x}, 0)$ are stable spiral points provided that

$$\tau^2 - 4\Delta = \frac{\beta^2}{m^2 l^2} - 4 \left(\omega^2 - \frac{g^2}{\omega^2 l^2} \right) < 0.$$

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This condition is equivalent to $\beta < 2ml\sqrt{\omega^2 - g^2/(\omega^2 l^2)}$.

22. The corresponding plane autonomous system is

$$x' = y, \quad y' = 2ky - cy^3 - \omega^2 x$$

and so $(0, 0)$ is the only critical point. Since

$$\mathbf{g}'(\mathbf{X}) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 2k - 3cy^2 \end{bmatrix}.$$

$\tau = 2k$ and $\Delta = \omega^2 > 0$. Since $k > 0$, $(0, 0)$ is an unstabale critical point. Assuming that a Type I invariant region exists that contains $(0, 0)$ in its interior, we may apply Theorem 11.8(i) to conclude that there is at least one periodic solution.

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12 Orthogonal Functions and Fourier Series

Exercises 12.1

1. $592\displaystyle\int_{-2}^{2} xx^2dx = \frac{1}{4}x^4\,\Big|_{-2}^{2} = 0$

2. $\displaystyle\int_{-1}^{1} x^3(x^2+1)dx = \frac{1}{6}x^6\,\Big|_{-1}^{1} + \frac{1}{4}x^4\,\Big|_{-1}^{1} = 0$

3. $\displaystyle\int_0^2 e^x(xe^{-x} - e^{-x})dx = \int_0^2 (x-1)dx = \left(\frac{1}{2}x^2 - x\right)\Big|_0^2 = 0$

4. $\displaystyle\int_0^{\pi} \cos x \sin^2 x\, dx = \frac{1}{3}\sin^3 x\,\Big|_0^{\pi} = 0$

5. $\displaystyle\int_{-\pi/2}^{\pi/2} x\cos 2x\, dx = \frac{1}{2}\left(\frac{1}{2}\cos 2x + x\sin 2x\right)\Big|_{-\pi/2}^{\pi/2} = 0$

6. $\displaystyle\int_{\pi/4}^{5\pi/4} e^x \sin x\, dx = \left(\frac{1}{2}e^x\sin x - \frac{1}{2}e^x\cos x\right)\Big|_{\pi/4}^{5\pi/4} = 0$

7. For $m \neq n$

$$\int_0^{\pi/2} \sin(2n+1)x\, \sin(2m+1)x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} [\cos 2(n-m)x - \cos 2(n+m+1)x]\, dx$$

$$= \frac{1}{4(n-m)}\sin 2(n-m)x\,\Big|_0^{\pi/2} - \frac{1}{4(n+m+1)}\sin 2(n+m+1)x\,\Big|_0^{\pi/2}$$

$$= 0.$$

For $m = n$

$$\int_0^{\pi/2} \sin^2(2n+_1)x\, dx = \int_0^{\pi/2}\left(\frac{1}{2} - \frac{1}{2}\cos 2(2n+1)x\right)dx$$

$$= \frac{1}{2}x\,\Big|_0^{\pi/2} - \frac{1}{4(2n+1)}\sin 2(2n+1)x\,\Big|_0^{\pi/2}$$

$$= \frac{\pi}{4}$$

so that

$$\|\sin(2n+1)x\| = \frac{1}{2}\sqrt{\pi}\,.$$

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8. For $m \neq n$

$$\int_0^{\pi/2} \cos(2n+1)x \, \cos(2m+1)x \, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} [\cos 2(n-m)x + \cos 2(n+m+1)x] \, dx$$

$$= \frac{1}{4(n-m)} \sin 2(n-m)x \, \Big|_0^{\pi/2} + \frac{1}{4(n+m+1)} \sin 2(n+m+1)x \, \Big|_0^{\pi/2}$$

$$= 0.$$

For $m = n$

$$\int_0^{\pi/2} \cos^2(2n+1)x \, dx = \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2(2n+1)x\right) dx$$

$$= \frac{1}{2}x \, \Big|_0^{\pi/2} + \frac{1}{4(2n+1)} \sin 2(2n+1)x \, \Big|_0^{\pi/2}$$

$$= \frac{\pi}{4}$$

so that

$$\| \cos(2n+1)x \| = \frac{1}{2}\sqrt{\pi} \, .$$

9. For $m \neq n$

$$\int_0^{\pi} \sin nx \sin mx \, dx = \frac{1}{2}\int_0^{\pi} [\cos(n-m)x - \cos(n+m)x] \, dx$$

$$= \frac{1}{2(n-m)} \sin(n-m)x \, \Big|_0^{\pi} - \frac{1}{2(n+m)} \sin 2(n+m)x \, \Big|_0^{\pi}$$

$$= 0.$$

For $m = n$

$$\int_0^{\pi} \sin^2 nx \, dx = \int_0^{\pi} \left[\frac{1}{2} - \frac{1}{2}\cos 2nx\right] dx = \frac{1}{2}x \, \Big|_0^{\pi} - \frac{1}{4n} \sin 2nx \, \Big|_0^{\pi} = \frac{\pi}{2}$$

so that

$$\| \sin nx \| = \sqrt{\frac{\pi}{2}} \, .$$

10. For $m \neq n$

$$\int_0^{p} \sin \frac{n\pi}{p}x \, \sin \frac{m\pi}{p}x \, dx = \frac{1}{2}\int_0^{p} \left(\cos \frac{(n-m)\pi}{p}x - \cos \frac{(n+m)\pi}{p}x\right) dx$$

$$= \frac{p}{2(n-m)\pi} \sin \frac{(n-m)\pi}{p}x \, \Big|_0^{p} - \frac{p}{2(n+m)\pi} \sin \frac{(n+m)\pi}{p}x \, \Big|_0^{p}$$

$$= 0.$$

For $m = n$

$$\int_0^{p} \sin^2 \frac{n\pi}{p}x \, dx = \int_0^{p} \left[\frac{1}{2} - \frac{1}{2}\cos \frac{2n\pi}{p}x\right] dx = \frac{1}{2}x \, \Big|_0^{p} - \frac{p}{4n\pi} \sin \frac{2n\pi}{p}x \, \Big|_0^{p} = \frac{p}{2}$$

so that

$$\left\| \sin \frac{n\pi}{p}x \right\| = \sqrt{\frac{p}{2}} \, .$$

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11. For $m \neq n$

$$\int_0^p \cos\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = \frac{1}{2}\int_0^p \left(\cos\frac{(n-m)\pi}{p}x + \cos\frac{(n+m)\pi}{p}x\right)\,dx$$

$$= \frac{p}{2(n-m)\pi}\sin\frac{(n-m)\pi}{p}x\,\Big|_0^p + \frac{p}{2(n+m)\pi}\sin\frac{(n+m)\pi}{p}x\,\Big|_0^p$$

$$= 0.$$

For $m = n$

$$\int_0^p \cos^2\frac{n\pi}{p}x\,dx = \int_0^p \left(\frac{1}{2} + \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx = \frac{1}{2}x\,\Big|_0^p + \frac{p}{4n\pi}\sin\frac{2n\pi}{p}x\,\Big|_0^p = \frac{p}{2}\,.$$

Also

$$\int_0^p 1\cdot\cos\frac{n\pi}{p}x\,dx = \frac{p}{n\pi}\sin\frac{n\pi}{p}x\,\Big|_0^p = 0 \quad\text{and}\quad \int_0^p 1^2 dx = p$$

so that

$$\|1\| = \sqrt{p} \quad\text{and}\quad \left\|\cos\frac{n\pi}{p}x\right\| = \sqrt{\frac{p}{2}}\,.$$

12. For $m \neq n$, we use Problems 11 and 10:

$$\int_{-p}^p \cos\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = 2\int_0^p \cos\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = 0$$

$$\int_{-p}^p \sin\frac{n\pi}{p}x\,\sin\frac{m\pi}{p}x\,dx = 2\int_0^p \sin\frac{n\pi}{p}x\,\sin\frac{m\pi}{p}x\,dx = 0.$$

Also

$$\int_{-p}^p \sin\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = \frac{1}{2}\int_{-p}^p \left(\sin\frac{(n-m)\pi}{p}x + \sin\frac{(n+m)\pi}{p}x\right)dx = 0,$$

$$\int_{-p}^p 1\cdot\cos\frac{n\pi}{p}x\,dx = \frac{p}{n\pi}\sin\frac{n\pi}{p}x\,\Big|_{-p}^p = 0,$$

$$\int_{-p}^p 1\cdot\sin\frac{n\pi}{p}x\,dx = -\frac{p}{n\pi}\cos\frac{n\pi}{p}x\,\Big|_{-p}^p = 0,$$

and

$$\int_{-p}^p \sin\frac{n\pi}{p}x\,\cos\frac{n\pi}{p}x\,dx = \int_{-p}^p \frac{1}{2}\sin\frac{2n\pi}{p}x\,dx = -\frac{p}{4n\pi}\cos\frac{2n\pi}{p}x\,\Big|_{-p}^p = 0.$$

For $m = n$

$$\int_{-p}^p \cos^2\frac{n\pi}{p}x\,dx = \int_{-p}^p \left(\frac{1}{2} + \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx = p,$$

$$\int_{-p}^p \sin^2\frac{n\pi}{p}x\,dx = \int_{-p}^p \left(\frac{1}{2} - \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx = p,$$

and

$$\int_{-p}^p 1^2 dx = 2p$$

so that

$$\|1\| = \sqrt{2p}\,, \quad \left\|\cos\frac{n\pi}{p}x\right\| = \sqrt{p}\,, \quad\text{and}\quad \left\|\sin\frac{n\pi}{p}x\right\| = \sqrt{p}\,.$$

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13. Since

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot 1 \cdot 2x\,dx = -e^{-x^2}\Big|_{-\infty}^{0} - e^{-x^2}\Big|_{0}^{\infty} = 0,$$

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2} \cdot 1 \cdot (4x^2-2)\,dx &= 2\int_{-\infty}^{\infty} x\left(2xe^{-x^2}\right)dx - 2\int_{-\infty}^{\infty} e^{-x^2}dx \\ &= 2\left(-xe^{-x^2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2}dx\right) - 2\int_{-\infty}^{\infty} e^{-x^2}dx \\ &= 2\left(-xe^{-x^2}\Big|_{-\infty}^{0} - xe^{-x^2}\Big|_{0}^{\infty}\right) = 0,\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2} \cdot 2x \cdot (4x^2-2)\,dx &= 4\int_{-\infty}^{\infty} x^2\left(2xe^{-x^2}\right)dx - 4\int_{-\infty}^{\infty} xe^{-x^2}dx \\ &= 4\left(-x^2e^{-x^2}\Big|_{-\infty}^{\infty} + 2\int_{-\infty}^{\infty} xe^{-x^2}dx\right) - 4\int_{-\infty}^{\infty} xe^{-x^2}dx \\ &= 4\left(-x^2e^{-x^2}\Big|_{-\infty}^{0} - x^2e^{-x^2}\Big|_{0}^{\infty}\right) + 2\int_{-\infty}^{\infty} 2xe^{-x^2}dx = 0,\end{aligned}$$

the functions are orthogonal.

14. Since

$$\int_{0}^{\infty} e^{-x^2} \cdot 1(1-x)\,dx = (x-1)e^{-x}\Big|_{0}^{\infty} - \int_{0}^{\infty} e^{-x}dx = 0,$$

$$\begin{aligned}\int_{0}^{\infty} e^{-x} \cdot 1 \cdot \left(\frac{1}{2}x^2 - 2x + 1\right)dx &= \left(2x - 1 - \frac{1}{2}x^2\right)e^{-x}\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-x}(x-2)\,dx \\ &= 1 + (2-x)e^{-x}\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-x}dx = 0,\end{aligned}$$

and

$$\begin{aligned}&\int_{0}^{\infty} e^{-x} \cdot (1-x)\left(\frac{1}{2}x^2 - 2x + 1\right)dx \\ &\qquad = \int_{0}^{\infty} e^{-x}\left(-\frac{1}{2}x^3 + \frac{5}{2}x^2 - 3x + 1\right)dx \\ &\qquad = e^{-x}\left(\frac{1}{2}x^3 - \frac{5}{2}x^2 + 3x - 1\right)\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-x}\left(-\frac{3}{2}x^2 + 5x - 3\right)dx \\ &\qquad = 1 + e^{-x}\left(\frac{3}{2}x^2 - 5x + 3\right)\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-x}(5-3x)\,dx \\ &\qquad = 1 - 3 + e^{-x}(3x-5)\Big|_{0}^{\infty} - 3\int_{0}^{\infty} e^{-x}dx = 0,\end{aligned}$$

the functions are orthogonal.

15. By orthogonality $\int_a^b \phi_0(x)\phi_n(x)dx = 0$ for $n = 1, 2, 3, \ldots$; that is, $\int_a^b \phi_n(x)dx = 0$ for $n = 1, 2, 3, \ldots$.

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16. Using the facts that ϕ_0 and ϕ_1 are orthogonal to ϕ_n for $n > 1$, we have

$$\begin{aligned}\int_a^b (\alpha x + \beta)\phi_n(x)\,dx &= \alpha \int_a^b x\phi_n(x)\,dx + \beta \int_a^b 1 \cdot \phi_n(x)\,dx \\ &= \alpha \int_a^b \phi_1(x)\phi_n(x)\,dx + \beta \int_a^b \phi_0(x)\phi_n(x)\,dx \\ &= \alpha \cdot 0 + \beta \cdot 0 = 0\end{aligned}$$

for $n = 2, 3, 4, \ldots$.

17. Using the fact that ϕ_n and ϕ_m are orthogonal for $n \neq m$ we have

$$\begin{aligned}\|\phi_m(x) + \phi_n(x)\|^2 &= \int_a^b [\phi_m(x) + \phi_n(x)]^2 dx \int_a^b \left[\phi_m^2(x) + 2\phi_m(x)\phi_n(x) + \phi_n^2(x)\right] dx \\ &= \int_a^b \phi_m^2(x)dx + 2\int_a^b \phi_m(x)\phi_n(x)dx + \int_a^b \phi_n^2(x)\,dx \\ &= \|\phi_m(x)\|^2 + \|\phi_n(x)\|^2 .\end{aligned}$$

18. Setting

$$0 = \int_{-2}^{2} f_3(x)f_1(x)\,dx = \int_{-2}^{2} \left(x^2 + c_1x^3 + c_2x^4\right) dx = \frac{16}{3} + \frac{64}{5}c_2$$

and

$$0 = \int_{-2}^{2} f_3(x)f_2(x)\,dx = \int_{-2}^{2} \left(x^3 + c_1x^4 + c_2x^5\right) dx = \frac{64}{5}c_1$$

we obtain $c_1 = 0$ and $c_2 = -5/12$.

19. Since $\sin nx$ is an odd function on $[-\pi, \pi]$,

$$(1, \sin nx) = \int_{-\pi}^{\pi} \sin nx\,dx = 0$$

and $f(x) = 1$ is orthogonal to every member of $\{\sin nx\}$. Thus $\{\sin nx\}$ is not complete.

20. $(f_1 + f_2, f_3) = \displaystyle\int_a^b [f_1(x) + f_2(x)]f_3(x)\,dx = \int_a^b f_1(x)f_3(x)\,dx + \int_a^b f_2(x)f_3(x)\,dx = (f_1, f_3) + (f_2, f_3)$

21. **(a)** The fundamental period is $2\pi/2\pi = 1$.

(b) The fundamental period is $2\pi/(4/L) = \frac{1}{2}\pi L$.

(c) The fundamental period of $\sin x + \sin 2x$ is 2π.

(d) The fundamental period of $\sin 2x + \cos 4x$ is $2\pi/2 = \pi$.

(e) The fundamental period of $\sin 3x + \cos 4x$ is 2π since the smallest integer multiples of $2\pi/3$ and $2\pi/4 = \pi/2$ that are equal are 3 and 4, respectively.

(f) The fundamental period of $f(x)$ is $2\pi/(n\pi/p) = 2p/n$.

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Exercises 12.2

1. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = 1$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi}{\pi}x\,dx = \frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx = 0$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi}{\pi}x\,dx = \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx = \frac{1}{n\pi}(1-\cos n\pi) = \frac{1}{n\pi}[1-(-1)^n]$

$f(x) = \dfrac{1}{2} + \dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$

2. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -1\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\,dx = 1$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\cos nx\,dx = 0$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\sin nx\,dx = \frac{3}{n\pi}[1-(-1)^n]$

$f(x) = \dfrac{1}{2} + \dfrac{3}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$

3. $a_0 = \displaystyle\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_{0}^{1} x\,dx = \frac{3}{2}$

$a_n = \displaystyle\int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0}\cos n\pi x\,dx + \int_{0}^{1} x\cos n\pi x\,dx = \frac{1}{n^2\pi^2}[(-1)^n - 1]$

$b_n = \displaystyle\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0}\sin n\pi x\,dx + \int_{0}^{1} x\sin n\pi x\,dx = -\frac{1}{n\pi}$

$f(x) = \dfrac{3}{4} + \displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x - \frac{1}{n\pi}\sin n\pi x\right]$

4. $a_0 = \displaystyle\int_{-1}^{1} f(x)\,dx = \int_{0}^{1} x\,dx = \frac{1}{2}$

$a_n = \displaystyle\int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{0}^{1} x\cos n\pi x\,dx = \frac{1}{n^2\pi^2}[(-1)^n - 1]$

$b_n = \displaystyle\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{0}^{1} x\sin n\pi x\,dx = \frac{(-1)^{n+1}}{n\pi}$

$f(x) = \dfrac{3}{4} + \displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x + \frac{(-1)^{n+1}}{n\pi}\sin n\pi x\right]$

5. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} x^2\,dx = \frac{1}{3}\pi^2$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx = \frac{1}{\pi}\left(\frac{x^2}{\pi}\sin nx\,\Big|_0^{\pi} - \frac{2}{n}\int_{0}^{\pi} x\sin nx\,dx\right) = \frac{2(-1)^n}{n^2}$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{0}^{\pi} x^2\sin nx\,dx = \frac{1}{\pi}\left(-\frac{x^2}{n}\cos nx\,\Big|_0^{\pi} + \frac{2}{n}\int_{0}^{\pi} x\cos nx\,dx\right) = \frac{\pi}{n}(-1)^{n+1} + \frac{2}{n^3\pi}[(-1)^n - 1]$

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$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos nx + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2[(-1)^n - 1]}{n^3\pi}\right)\sin nx\right]$$

6. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2\,dx + \frac{1}{\pi}\int_{0}^{\pi}\left(\pi^2 - x^2\right)\,dx = \frac{5}{3}\pi^2$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\left(\pi^2 - x^2\right)\cos nx\,dx$

$= \dfrac{1}{\pi}\left(\dfrac{\pi^2 - x^2}{n}\sin nx\,\Big|_0^{\pi} + \frac{2}{n}\displaystyle\int_0^{\pi} x\sin nx\,dx\right) = \frac{2}{n^2}(-1)^{n+1}$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\left(\pi^2 - x^2\right)\sin nx\,dx$

$= \dfrac{\pi}{n}[(-1)^n - 1] + \dfrac{1}{\pi}\left(\dfrac{x^2 - \pi^2}{n}\cos nx\,\Big|_0^{\pi} - \frac{2}{n}\displaystyle\int_0^{\pi} x\cos nx\,dx\right) = \frac{\pi}{n}(-1)^n + \frac{2}{n^3\pi}[1 - (-1)^n]$

$$f(x) = \frac{5\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}(-1)^{n+1}\cos nx + \left(\frac{\pi}{n}(-1)^n + \frac{2[1 - (-1^n]}{n^3\pi}\right)\sin nx\right]$$

7. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + \pi)\,dx = 2\pi$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + \pi)\cos nx\,dx = 0$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{n}(-1)^{n+1}$

$$f(x) = \pi + \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx$$

8. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(3 - 2x)\,dx = 6$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(3 - 2x)\cos nx\,dx = 0$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}(3 - 2x)\sin nx\,dx = \frac{4}{n}(-1)^n$

$$f(x) = 3 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

9. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi}\sin x\,dx = \frac{2}{\pi}$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}\sin x\,\cos nx\,dx = \frac{1}{2\pi}\int_{0}^{\pi}[\sin(n+1)x + \sin(1-n)x]\,dx$

$= \dfrac{1 + (-1)^n}{\pi(1 - n^2)}$ for $n = 2, 3, 4, \ldots$

$a_1 = \dfrac{1}{2\pi}\displaystyle\int_0^{\pi}\sin 2x\,dx = 0$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}\sin x\,\sin nx\,dx$

$= \dfrac{1}{2\pi}\displaystyle\int_0^{\pi}[\cos(1-n)x - \cos(1+n)x]\,dx = 0$ for $n = 2, 3, 4, \ldots$

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$$b_1 = \frac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{1}{2}$$

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=2}^{\infty}\frac{1+(-1)^n}{\pi(1-n^2)}\cos nx$$

10. $$a_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/2}\cos x\,dx = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\cos 2nx\,dx = \frac{2}{\pi}\int_0^{\pi/2}\cos x\,\cos 2nx\,dx = \frac{1}{\pi}\int_0^{\pi/2}[\cos(2n-1)x + \cos(2n+1)x]\,dx$$

$$= \frac{2(-1)^{n+1}}{\pi(4n^2-1)}$$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\sin 2nx\,dx = \frac{2}{\pi}\int_0^{\pi/2}\cos x\,\sin 2nx\,dx = \frac{1}{\pi}\int_0^{\pi/2}[\sin(2n-1)x + \sin(2n+1)x]\,dx$$

$$= \frac{4n}{\pi(4n^2-1)}$$

$$f(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{\pi(4n^2-1)}\cos 2nx + \frac{4n}{\pi(4n^2-1)}\sin 2nx\right]$$

11. $$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left(\int_{-1}^{0} -2\,dx + \int_0^1 1\,dx\right) = -\frac{1}{2}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_{-1}^{0}\left[-2\cos\frac{n\pi}{2}x\right]\,dx + \int_0^1\cos\frac{n\pi}{2}x\,dx\right) = -\frac{1}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_{-1}^{0}\left[-2\sin\frac{n\pi}{2}x\right]\,dx + \int_0^1\sin\frac{n\pi}{2}x\,dx\right) = \frac{3}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$

$$f(x) = -\frac{1}{4} + \sum_{n=1}^{\infty}\left[-\frac{1}{n\pi}\sin\frac{n\pi}{2}\,\cos\frac{n\pi}{2}x + \frac{3}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}x\right]$$

12. $$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left(\int_0^1 x\,dx + \int_1^2 1\,dx\right) = \frac{3}{4}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_0^1 x\cos\frac{n\pi}{2}x\,dx + \int_1^2\cos\frac{n\pi}{2}x\,dx\right) = \frac{2}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_0^1 x\sin\frac{n\pi}{2}x\,dx + \int_1^2\sin\frac{n\pi}{2}x\,dx\right) = \frac{2}{n^2\pi^2}\left(\sin\frac{n\pi}{2} + \frac{n\pi}{2}(-1)^{n+1}\right)$$

$$f(x) = \frac{3}{8} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)\cos\frac{n\pi}{2}x + \frac{2}{n^2\pi^2}\left(\sin\frac{n\pi}{2} + \frac{n\pi}{2}(-1)^{n+1}\right)\sin\frac{n\pi}{2}x\right]$$

13. $$a_0 = \frac{1}{5}\int_{-5}^{5} f(x)\,dx = \frac{1}{5}\left(\int_{-5}^{0} 1\,dx + \int_0^5 (1+x)\,dx\right) = \frac{9}{2}$$

$$a_n = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi}{5}x\,dx = \frac{1}{5}\left(\int_{-5}^{0}\cos\frac{n\pi}{5}x\,dx + \int_0^5 (1+x)\cos\frac{n\pi}{5}x\,dx\right) = \frac{5}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi}{5}x\,dx = \frac{1}{5}\left(\int_{-5}^{0}\sin\frac{n\pi}{5}x\,dx + \int_0^5 (1+x)\cos\frac{n\pi}{5}x\,dx\right) = \frac{5}{n\pi}(-1)^{n+1}$$

$$f(x) = \frac{9}{4} + \sum_{n=1}^{\infty}\left[\frac{5}{n^2\pi^2}[(-1)^n - 1]\cos\frac{n\pi}{5}x + \frac{5}{n\pi}(-1)^{n+1}\sin\frac{n\pi}{5}x\right]$$

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14. $a_0 = \dfrac{1}{2}\displaystyle\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left(\int_{-2}^{0}(2+x)\,dx + \int_{0}^{2} 2\,dx\right) = 3$

$a_n = \dfrac{1}{2}\displaystyle\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_{-2}^{0}(2+x)\cos\frac{n\pi}{2}x\,dx + \int_{0}^{2} 2\cos\frac{n\pi}{2}x\,dx\right) = \frac{2}{n^2\pi^2}[1-(-1)^n]$

$b_n = \dfrac{1}{2}\displaystyle\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_{-2}^{0}(2+x)\sin\frac{n\pi}{2}x\,dx + \int_{0}^{2} 2\sin\frac{n\pi}{2}x\,dx\right) = \frac{2}{n\pi}(-1)^{n+1}$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}[1-(-1)^n]\cos\frac{n\pi}{2}x + \frac{2}{n\pi}(-1)^{n+1}\sin\frac{n\pi}{2}x\right]$$

15. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}(e^{\pi}-e^{-\pi})$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{(-1)^n(e^{\pi}-e^{-\pi})}{\pi(1+n^2)}$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin nx\,dx = \frac{(-1)^n n(e^{-\pi}-e^{\pi})}{\pi(1+n^2)}$

$$f(x) = \frac{e^{\pi}-e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n(e^{\pi}-e^{-\pi})}{\pi(1+n^2)}\cos nx + \frac{(-1)^n n(e^{-\pi}-e^{\pi})}{\pi(1+n^2)}\sin nx\right]$$

16. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi}(e^x-1)\,dx = \frac{1}{\pi}(e^{\pi}-\pi-1)$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}(e^x-1)\cos nx\,dx = \frac{[e^{\pi}(-1)^n-1]}{\pi(1+n^2)}$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}(e^x-1)\sin nx\,dx = \frac{1}{\pi}\left(\frac{ne^{\pi}(-1)^{n+1}}{1+n^2} + \frac{n}{1+n^2} + \frac{(-1)^n}{n} - \frac{1}{n}\right)$

$$f(x) = \frac{e^{\pi}-\pi-1}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{e^{\pi}(-1)^n-1}{\pi(1+n^2)}\cos nx + \left(\frac{n}{1+n^2}\left[e^{\pi}(-1)^{n+1}+1\right] + \frac{(-1)^n-1}{n}\right)\sin nx\right]$$

17. The function in Problem 5 is discontinuous at $x=\pi$, so the corresponding Fourier series converges to $\pi^2/2$ at $x=\pi$. That is,

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos n\pi + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2[(-1)^n-1]}{n^3\pi}\right)\sin n\pi\right]$$

$$= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}(-1)^n = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2}{n^2} = \frac{\pi^2}{6} + 2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

and

$$\frac{\pi^2}{6} = \frac{1}{2}\left(\frac{\pi^2}{2} - \frac{\pi^2}{6}\right) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$$

At $x=0$ the series converges to 0 and

$$0 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2} = \frac{\pi^2}{6} + 2\left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right)$$

so

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

18. From Problem 17

$$\frac{\pi^2}{8} = \frac{1}{2}\left(\frac{\pi^2}{6} + \frac{\pi^2}{12}\right) = \frac{1}{2}\left(2 + \frac{2}{3^2} + \frac{2}{5^2} + \cdots\right) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

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19. The function in Problem 7 is continuous at $x = \pi/2$ so

$$\frac{3\pi}{2} = f\left(\frac{\pi}{2}\right) = \pi + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin\frac{n\pi}{2} = \pi + 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

and

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

20. The function in Problem 9 is continuous at $x = \pi/2$ so

$$1 = f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1 + (-1)^n}{\pi(1 - n^2)} \cos\frac{n\pi}{2}$$

$$= \frac{1}{\pi} + \frac{1}{2} + \frac{2}{3\pi} - \frac{2}{3 \cdot 5\pi} + \frac{2}{5 \cdot 7\pi} - \cdots$$

and

$$\pi = 1 + \frac{\pi}{2} + \frac{2}{3} - \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} - \cdots$$

or

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \cdots.$$

21. Writing

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi}{p}x + \cdots + a_n \cos\frac{n\pi}{p}x + \cdots + b_1 \sin\frac{\pi}{p}x + \cdots + b_n \sin\frac{n\pi}{p}x + \cdots$$

we see that $f^2(x)$ consists exclusively of squared terms of the form

$$\frac{a_0^2}{4}, \qquad a_n^2 \cos^2\frac{n\pi}{p}x, \qquad b_n^2 \sin^2\frac{n\pi}{p}x$$

and cross-product terms, with $m \neq n$, of the form

$$a_0 a_n \cos\frac{n\pi}{p}x, \qquad a_0 b_n \sin\frac{n\pi}{p}x, \qquad 2a_m a_n \cos\frac{m\pi}{p}x \cos\frac{n\pi}{p}x,$$

$$2a_m b_n \cos\frac{m\pi}{p}x \sin\frac{n\pi}{p}x, \qquad 2b_m b_n \sin\frac{m\pi}{p}x \sin\frac{n\pi}{p}x.$$

The integral of each cross-product term taken over the interval $(-p, p)$ is zero by orthogonality. For the squared terms we have

$$\frac{a_0^2}{4}\int_{-p}^{p} dx = \frac{a_0^2 p}{2}, \qquad a_n^2 \int_{-p}^{p} \cos^2\frac{n\pi}{p}x\,dx = a_n^2 p, \qquad b_n^2 \int_{-p}^{p} \sin^2\frac{n\pi}{p}x\,dx = b_n^2 p.$$

Thus

$$RMS(f) = \sqrt{\frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}.$$

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Exercises 12.3

1. Since $f(-x) = \sin(-3x) = -\sin 3x = -f(x)$, $f(x)$ is an odd function.

2. Since $f(-x) = -x\cos(-x) = -x\cos x = -f(x)$, $f(x)$ is an odd function.

3. Since $f(-x) = (-x)^2 - x = x^2 - x$, $f(x)$ is neither even nor odd.

4. Since $f(-x) = (-x)^3 + 4x = -(x^3 - 4x) = -f(x)$, $f(x)$ is an odd function.

5. Since $f(-x) = e^{|-x|} = e^{|x|} = f(x)$, $f(x)$ is an even function.

6. Since $f(-x) = e^{-x} - e^{x} = -f(x)$, $f(x)$ is an odd function.

7. For $0 < x < 1$, $f(-x) = (-x)^2 = x^2 = -f(x)$, $f(x)$ is an odd function.

8. For $0 \le x < 2$, $f(-x) = -x + 5 = f(x)$, $f(x)$ is an even function.

9. Since $f(x)$ is not defined for $x < 0$, it is neither even nor odd.

10. Since $f(-x) = \left|(-x)^5\right| = \left|x^5\right| = f(x)$, $f(x)$ is an even function.

11. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin nx\, dx = \frac{2}{n\pi}\left[1 - (-1)^n\right].$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - (-1)^n\right]\sin nx.$$

12. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \int_1^2 1\, dx = 1$$

$$a_n = \int_1^2 \cos\frac{n\pi}{2}x\, dx = -\frac{2}{n\pi}\sin\frac{n\pi}{2}.$$

Thus

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi}{2}x.$$

13. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\, dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx = \frac{2}{n^2\pi}[(-1)^n - 1].$$

Thus

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[(-1)^n - 1]\cos nx.$$

14. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\, dx = \frac{2}{n}(-1)^{n+1}.$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx.$$

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15. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = 2\int_0^1 x^2\,dx = \frac{2}{3}$$

$$a_n = 2\int_0^1 x^2\cos n\pi x\,dx = 2\left(\frac{x^2}{n\pi}\sin n\pi x\,\Big|_0^1 - \frac{2}{n\pi}\int_0^1 x\sin n\pi x\,dx\right) = \frac{4}{n^2\pi^2}(-1)^n.$$

Thus

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(-1)^n\cos n\pi x.$$

16. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = 2\int_0^1 x^2\sin n\pi x\,dx = 2\left(-\frac{x^2}{n\pi}\cos n\pi x\,\Big|_0^1 + \frac{2}{n\pi}\int_0^1 x\cos n\pi x\,dx\right)$$

$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}[(-1)^n - 1].$$

Thus

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}[(-1)^n - 1]\right)\sin n\pi x.$$

17. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\,dx = \frac{4}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\cos nx\,dx = \frac{2}{\pi}\left(\frac{\pi^2 - x^2}{n}\sin nx\,\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\sin nx\,dx\right) = \frac{4}{n^2}(-1)^{n+1}.$$

Thus

$$f(x) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n+1}\cos nx\,dx.$$

18. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^3\sin nx\,dx = \frac{2}{\pi}\left(-\frac{x^3}{n}\cos nx\,\Big|_0^{\pi} + \frac{3}{n}\int_0^{\pi} x^2\cos nx\,dx\right)$$

$$= \frac{2\pi^2}{n}(-1)^{n+1} - \frac{12}{n^2\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2\pi^2}{n}(-1)^{n+1} - \frac{12}{n^2\pi}\left(-\frac{x}{n}\cos nx\,\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right) = \frac{2\pi^2}{n}(-1)^{n+1} + \frac{12}{n^3}(-1)^n.$$

Thus

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2\pi^2}{n}(-1)^{n+1} + \frac{12}{n^3}(-1)^n\right)\sin nx.$$

19. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^{\pi}(x+1)\sin nx\,dx = \frac{2(\pi+1)}{n\pi}(-1)^{n+1} + \frac{2}{n\pi}.$$

Thus

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2(\pi+1)}{n\pi}(-1)^{n+1} + \frac{2}{n\pi}\right)\sin nx.$$

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20. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = 2\int_0^1 (x-1)\sin n\pi x\,dx = 2\left[\int_0^1 x\sin n\pi x\,dx - \int_0^1 \sin n\pi x\,dx\right]$$

$$= 2\left[\frac{1}{n^2\pi^2}\sin n\pi x - \frac{x}{n\pi}\cos n\pi x + \frac{1}{n\pi}\cos n\pi x\right]_0^1 = -\frac{2}{n\pi}.$$

Thus

$$f(x) = -\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin n\pi x.$$

21. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \int_0^1 x\,dx + \int_1^2 1\,dx = \frac{3}{2}$$

$$a_n = \int_0^1 x\cos\frac{n\pi}{2}x\,dx + \int_1^2 \cos\frac{n\pi}{2}x\,dx = \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right).$$

Thus

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos\frac{n\pi}{2}x.$$

22. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin\frac{n}{2}x\,dx + \int_{\pi}^{2\pi}\pi\sin\frac{n}{2}x\,dx = \frac{4}{n^2\pi}\sin\frac{n\pi}{2} + \frac{2}{n}(-1)^{n+1}.$$

Thus

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi}\sin\frac{n\pi}{2} + \frac{2}{n}(-1)^{n+1}\right)\sin\frac{n}{2}x.$$

23. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\sin x\,dx = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\sin x\,\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x + \sin(1-n)x]\,dx$$

$$= \frac{2}{\pi(1-n^2)}[1 + (-1)^n] \quad \text{for } n = 2, 3, 4, \ldots$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx = 0.$$

Thus

$$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty}\frac{2[1+(-1)^n]}{\pi(1-n^2)}\cos nx.$$

24. Since $f(x)$ is an even function, we expand in a cosine series. [See the solution of Problem 10 in Exercise 12.2 for the computation of the integrals.]

$$a_0 = \frac{2}{\pi/2}\int_0^{\pi/2}\cos x\,dx = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi/2}\int_0^{\pi/2}\cos x\cos\frac{n\pi}{\pi/2}x\,dx = \frac{4(-1)^{n+1}}{\pi(4n^2-1)}$$

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Thus

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi\,(4n^2-1)} \cos 2nx.$$

25. $a_0 = 2\displaystyle\int_0^{1/2} 1\,dx = 1$

$a_n = 2\displaystyle\int_0^{1/2} 1 \cdot \cos n\pi x\,dx = \frac{2}{n\pi}\sin\frac{n\pi}{2}$

$b_n = 2\displaystyle\int_0^{1/2} 1 \cdot \sin n\pi x\,dx = \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)$

$f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos n\pi x$

$f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)\sin n\pi x$

26. $a_0 = 2\displaystyle\int_{1/2}^{1} 1\,dx = 1$

$a_n = 2\displaystyle\int_{1/2}^{1} 1 \cdot \cos n\pi x\,dx = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$

$b_n = 2\displaystyle\int_{1/2}^{1} 1 \cdot \sin n\pi x\,dx = \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)$

$f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\sin\frac{n\pi}{2}\,\cos n\pi x\right)$

$f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)\sin n\pi x$

27. $a_0 = \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2} \cos x\,dx = \frac{4}{\pi}$

$a_n = \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2} \cos x\,\cos 2nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} [\cos(2n+1)x + \cos(2n-1)x]\,dx = \frac{4(-1)^n}{\pi(1-4n^2)}$

$b_n = \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2} \cos x\,\sin 2nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} [\sin(2n+1)x + \sin(2n-1)x]\,dx = \frac{8n}{\pi(4n^2-1)}$

$f(x) = \dfrac{2}{\pi} + \displaystyle\sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi(1-4n^2)}\,\cos 2nx$

$f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{8n}{\pi(4n^2-1)}\,\sin 2nx$

28. $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin x\,dx = \frac{4}{\pi}$

$a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin x\,\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\,dx = \frac{2[(-1)^n+1]}{\pi(1-n^2)}$ for $n = 2, 3, 4, \ldots$

$b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin x\,\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} [\cos(n-1)x - \cos(n+1)x]\,dx = 0$ for $n = 2, 3, 4, \ldots$

$a_1 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \sin 2x\,dx = 0$

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$$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin^2 x\,dx = 1$$

$$f(x) = \sin x$$

$$f(x) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{(-1)^n + 1}{1 - n^2}\cos nx$$

29. $$a_0 = \frac{2}{\pi}\left(\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right) = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\left(\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right) = \frac{2}{n^2\pi}\left(2\cos\frac{n\pi}{2} + (-1)^{n+1} - 1\right)$$

$$b_n = \frac{2}{\pi}\left(\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\,dx\right) = \frac{4}{n^2\pi}\sin\frac{n\pi}{2}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left(2\cos\frac{n\pi}{2} + (-1)^{n+1} - 1\right)\cos nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi}\sin\frac{n\pi}{2}\sin nx$$

30. $$a_0 = \frac{1}{\pi}\int_{\pi}^{2\pi} (x - \pi)\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{\pi}^{2\pi} (x - \pi)\cos\frac{n}{2}x\,dx = \frac{4}{n^2\pi}\left[(-1)^n - \cos\frac{n\pi}{2}\right]$$

$$b_n = \frac{1}{\pi}\int_{\pi}^{2\pi} (x - \pi)\sin\frac{n}{2}x\,dx = \frac{2}{n}(-1)^{n+1} - \frac{4}{n^2\pi}\sin\frac{n\pi}{2}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi}\left[(-1)^n - \cos\frac{n\pi}{2}\right]\cos\frac{n}{2}x$$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{2}{n}(-1)^{n+1} - \frac{4}{n^2\pi}\sin\frac{n\pi}{2}\right)\sin\frac{n}{2}x$$

31. $$a_0 = \int_0^1 x\,dx + \int_1^2 1\,dx = \frac{3}{2}$$

$$a_n = \int_0^1 x\cos\frac{n\pi}{2}x\,dx = \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

$$b_n = \int_0^1 x\sin\frac{n\pi}{2}x\,dx + \int_1^2 1\cdot\sin\frac{n\pi}{2}x\,dx = \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}(-1)^{n+1}$$

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos\frac{n\pi}{2}x$$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}(-1)^{n+1}\right]\sin\frac{n\pi}{2}x$$

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32. $a_0 = \displaystyle\int_0^1 1\,dx + \int_1^2 (2-x)\,dx = \frac{3}{2}$

$a_n = \displaystyle\int_0^1 1\cdot\cos\frac{n\pi}{2}x\,dx + \int_1^2 (2-x)\cos\frac{n\pi}{2}x\,dx = \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)$

$b_n = \displaystyle\int_0^1 1\cdot\sin\frac{n\pi}{2}x\,dx + \int_1^2 (2-x)\sin\frac{n\pi}{2}x\,dx = \frac{2}{n\pi} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$

$f(x) = \dfrac{3}{4} + \displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)\cos\frac{n\pi}{2}x$

$f(x) = \displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{n\pi} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}x$

33. $a_0 = 2\displaystyle\int_0^1 (x^2+x)\,dx = \frac{5}{3}$

$a_n = 2\displaystyle\int_0^1 (x^2+x)\cos n\pi x\,dx = \frac{2(x^2+x)}{n\pi}\sin n\pi x\,\Big|_0^1 - \frac{2}{n\pi}\int_0^1 (2x+1)\sin n\pi x\,dx = \frac{2}{n^2\pi^2}[3(-1)^n - 1]$

$b_n = 2\displaystyle\int_0^1 (x^2+x)\sin n\pi x\,dx = -\frac{2(x^2+x)}{n\pi}\cos n\pi x\,\Big|_0^1 + \frac{2}{n\pi}\int_0^1 (2x+1)\cos n\pi x\,dx$

$= \dfrac{4}{n\pi}(-1)^{n+1} + \dfrac{4}{n^3\pi^3}[(-1)^n - 1]$

$f(x) = \dfrac{5}{6} + \displaystyle\sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}[3(-1)^n - 1]\cos n\pi x$

$f(x) = \displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{n\pi}(-1)^{n+1} + \frac{4}{n^3\pi^3}[(-1)^n - 1]\right)\sin n\pi x$

34. $a_0 = \displaystyle\int_0^2 (2x - x^2)\,dx = \frac{4}{3}$

$a_n = \displaystyle\int_0^2 (2x-x^2)\cos\frac{n\pi}{2}x\,dx = \frac{8}{n^2\pi^2}[(-1)^{n+1} - 1]$

$b_n = \displaystyle\int_0^2 (2x-x^2)\sin\frac{n\pi}{2}x\,dx = \frac{16}{n^3\pi^3}[1 - (-1)^n]$

$f(x) = \dfrac{2}{3} + \displaystyle\sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}[(-1)^{n+1} - 1]\cos\frac{n\pi}{2}x$

$f(x) = \displaystyle\sum_{n=1}^{\infty}\frac{16}{n^3\pi^3}[1 - (-1)^n]\sin\frac{n\pi}{2}x$

35. $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} x^2\,dx = \frac{8}{3}\pi^2$

$a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} x^2\cos nx\,dx = \frac{4}{n^2}$

$b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} x^2\sin nx\,dx = -\frac{4\pi}{n}$

$f(x) = \dfrac{4}{3}\pi^2 + \displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$

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36. $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x\,dx = \pi$

$a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x\cos 2nx\,dx = 0$

$b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x\sin 2nx\,dx = -\dfrac{1}{n}$

$f(x) = \dfrac{\pi}{2} + \displaystyle\sum_{n=1}^{\infty}\left(-\frac{1}{n}\sin 2nx\right)$

37. $a_0 = 2\displaystyle\int_0^{1}(x+1)\,dx = 3$

$a_n = 2\displaystyle\int_0^{1}(x+1)\cos 2n\pi x\,dx = 0$

$b_n = 2\displaystyle\int_0^{1}(x+1)\sin 2n\pi x\,dx = -\dfrac{1}{n\pi}$

$f(x) = \dfrac{3}{2} - \displaystyle\sum_{n=1}^{\infty}\frac{1}{n\pi}\sin 2n\pi x$

38. $a_0 = 2\displaystyle\int_0^{2}(2-x)\,dx = 2$

$a_n = 2\displaystyle\int_0^{2}(2-x)\cos n\pi x\,dx = 0$

$b_n = 2\displaystyle\int_0^{2}(2-x)\sin n\pi x\,dx = \dfrac{2}{n\pi}$

$f(x) = 1 + \displaystyle\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin n\pi x$

39. We have

$$b_n = \frac{2}{\pi}\int_0^{\pi} 5\sin nt\,dt = \frac{10}{n\pi}[1-(-1)^n]$$

so that

$$f(t) = \sum_{n=1}^{\infty}\frac{10[1-(-1)^n]}{n\pi}\sin nt.$$

Substituting the assumption $x_p(t) = \displaystyle\sum_{n=1}^{\infty} B_n \sin nt$ into the differential equation then gives

$$x_p'' + 10x_p = \sum_{n=1}^{\infty} B_n(10-n^2)\sin nt = \sum_{n=1}^{\infty}\frac{10[1-(-1)^n]}{n\pi}\sin nt$$

and so $B_n = \dfrac{10[1-(-1)^n]}{n\pi(10-n^2)}$. Thus

$$x_p(t) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n(10-n^2)}\sin nt.$$

40. We have

$$b_n = \frac{2}{\pi}\int_0^{1}(1-t)\sin n\pi t\,dt = \frac{2}{n\pi}$$

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so that

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi t.$$

Substituting the assumption $x_p(t) = \sum_{n=1}^{\infty} B_n \sin n\pi t$ into the differential equation then gives

$$x_p'' + 10x_p = \sum_{n=1}^{\infty} B_n(10 - n^2\pi^2) \sin n\pi t = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi t$$

and so $B_n = \dfrac{2}{n\pi(10 - n^2\pi^2)}$. Thus

$$x_p(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n(10 - n^2\pi^2)} \sin n\pi t.$$

41. We have

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (2\pi t - t^2)\, dt = \frac{4}{3}\pi^2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (2\pi t - t^2) \cos nt\, dt = -4n^2$$

so that

$$f(t) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-4)}{n^2} \cos nt.$$

Substituting the assumption

$$x_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nt$$

into the differential equation then gives

$$\frac{1}{4}\, x_p'' + 12x_p = 6A_0 + \sum_{n=1}^{\infty} A_n \left(-\frac{1}{4}n^2 + 12\right) \cos nt = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-4)}{n^2} \cos nt$$

and $A_0 = \dfrac{\pi^2}{9}$, $A_n = \dfrac{16}{n^2(n^2 - 48)}$. Thus

$$x_p(t) = \frac{\pi^2}{18} + 16 \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)} \cos nt.$$

42. We have

$$a_0 = \frac{2}{(1/2)} \int_0^{1/2} t\, dt = \frac{1}{2}$$

$$a_n = \frac{2}{(1/2)} \int_0^{1/2} t \cos 2n\pi t\, dt = \frac{1}{n^2\pi^2} \left[(-1)^n - 1\right]$$

so that

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2} \cos 2n\pi t.$$

Substituting the assumption

$$x_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos 2n\pi t$$

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into the differential equation then gives

$$\frac{1}{4}\,x_p'' + 12x_p = 6A_0 + \sum_{n=1}^{\infty} A_n(12 - n^2\pi^2)\cos 2n\pi t = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2}\,\cos 2n\pi t$$

and $A_0 = \dfrac{1}{24}$, $A_n = \dfrac{(-1)^n - 1}{n^2\pi^2(12 - n^2\pi^2)}$. Thus

$$x_p(t) = \frac{1}{48} + \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2(12 - n^2\pi^2)}\,\cos 2n\pi t.$$

43. (a) The general solution is $x(t) = c_1 \cos\sqrt{10}t + c_2 \sin\sqrt{10}t + x_p(t)$, where

$$x_p(t) = \frac{10}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n(10 - n^2)}\sin nt.$$

The initial condition $x(0) = 0$ implies $c_1 + x_p(0) = 0$. Since $x_p(0) = 0$, we have $c_1 = 0$ and $x(t) = c_2 \sin\sqrt{10}t + x_p(t)$. Then $x'(t) = c_2\sqrt{10}\cos\sqrt{10}t + x_p'(t)$ and $x'(0) = 0$ implies

$$c_2\sqrt{10} + \frac{10}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{10 - n^2}\cos 0 = 0.$$

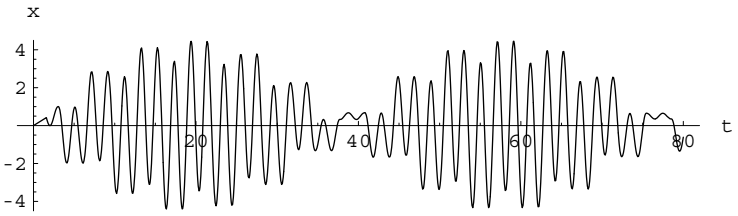
Thus

$$c_2 = -\frac{\sqrt{10}}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{10 - n^2}$$

and

$$x(t) = \frac{10}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{10 - n^2}\left[\frac{1}{n}\sin nt - \frac{1}{\sqrt{10}}\sin\sqrt{10}t\right].$$

(b) The graph is plotted using eight nonzero terms in the series expansion of $x(t)$.



44. (a) The general solution is $x(t) = c_1 \cos 4\sqrt{3}t + c_2 \sin 4\sqrt{3}t + x_p(t)$, where

$$x_p(t) = \frac{\pi^2}{18} + 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)}\cos nt.$$

The initial condition $x(0) = 0$ implies $c_1 + x_p(0) = 1$ or

$$c_1 = 1 - x_p(0) = 1 - \frac{\pi^2}{18} - 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)}.$$

Now $x'(t) = -4\sqrt{3}c_1 \sin 4\sqrt{3}t + 4c_2 \cos 4\sqrt{3}t + x_p'(t)$, so $x'(0) = 0$ implies $4c_2 + x_p'(0) = 0$. Since $x_p'(0) = 0$, we have $c_2 = 0$ and

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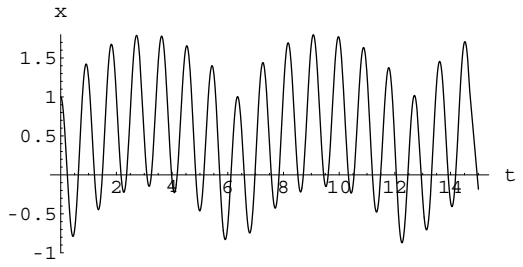
$$x(t) = \left(1 - \frac{\pi^2}{18} - 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2-48)}\right)\cos 4\sqrt{3}t + \frac{\pi^2}{18} + 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2-48)}\cos nt$$

$$= \frac{\pi^2}{18} + \left(1 - \frac{\pi^2}{18}\right)\cos 4\sqrt{3}t + 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2-48)}\left[\cos nt - \cos 4\sqrt{3}t\right].$$

(b) The graph is plotted using five nonzero terms in the series expansion of $x(t)$.



45. We have

$$b_n = \frac{2}{L}\int_0^L \frac{w_0 x}{L}\sin\frac{n\pi}{L}x\,dx = \frac{2w_0}{n\pi}(-1)^{n+1}$$

so that

$$w(x) = \sum_{n=1}^{\infty} \frac{2w_0}{n\pi}(-1)^{n+1}\sin\frac{n\pi}{L}x.$$

If we assume $y(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x$ then

$$y^{(4)} = \sum_{n=1}^{\infty} \frac{n^4\pi^4}{L^4} B_n \sin\frac{n\pi}{L}x$$

and so the differential equation $EIy^{(4)} = w(x)$ gives

$$B_n = \frac{2w_0(-1)^{n+1}L^4}{EIn^5\pi^5}.$$

Thus

$$y(x) = \frac{2w_0L^4}{EI\pi^5}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}\sin\frac{n\pi}{L}x.$$

46. We have

$$b_n = \frac{2}{L}\int_{L/3}^{2L/3} w_0 \sin\frac{n\pi}{L}x\,dx = \frac{2w_0}{n\pi}\left[\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right]$$

so that

$$w(x) = \sum_{n=1}^{\infty} \frac{2w_0}{n\pi}\left[\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right]\sin\frac{n\pi}{L}x.$$

If we assume $y(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x$ then

$$y^{(4)} = \sum_{n=1}^{\infty} \frac{n^4\pi^4}{L^4} B_n \sin\frac{n\pi}{L}x$$

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and so the differential equation $EIy^{(4)} = w(x)$ gives

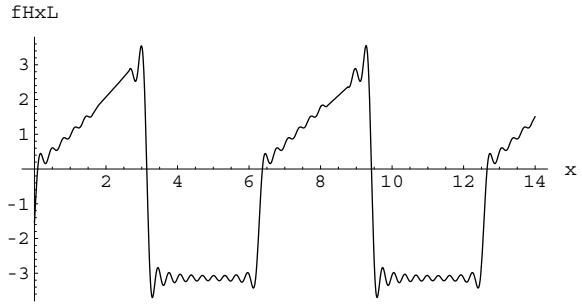
$$B_n = 2w_0L^4\frac{\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}}{EIn^5\pi^5}.$$

Thus

$$y(x) = \frac{2w_0L^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}}{n^5}\sin\frac{n\pi}{L}x.$$

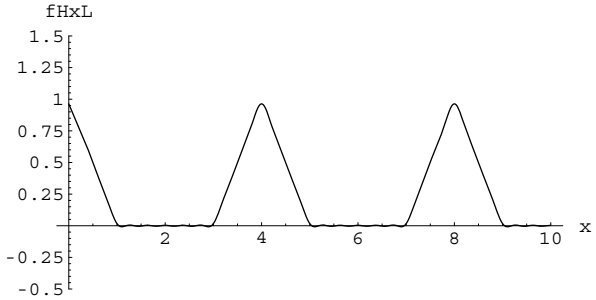
47. The graph is obtained by summing the series from $n = 1$ to 20. It appears that

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ -\pi, & \pi < x < 2\pi. \end{cases}$$



48. The graph is obtained by summing the series from $n = 1$ to 10. It appears that

$$f(x) = \begin{cases} 1 - x, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$



49. **(a)** If f and g are even and $h(x) = f(x)g(x)$ then

$$h(-x) = f(-x)g(-x) = f(x)g(x) = h(x)$$

and h is even.

(c) If f is even and g is odd and $h(x) = f(x)g(x)$ then

$$h(-x) = f(-x)g(-x) = f(x)[-g(x)] = -h(x)$$

and h is odd.

(d) Let $h(x) = f(x) \pm g(x)$ where f and g are even. Then

$$h(-x) = f(-x) \pm g(-x) = f(x) \pm g(x),$$

and h is an even function.

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(f) If f is even then

$$\int_{-a}^{a} f(x)\,dx = -\int_{a}^{0} f(-u)\,du + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

(g) If f is odd then

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= -\int_{-a}^{0} f(-x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{a}^{0} f(u)\,du + \int_{0}^{a} f(x)\,dx \\ &= -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0.\end{aligned}$$

50. The answers are not unique. The function in Problem 47 could also be

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ 1, & x = \pi \\ -\pi, & \pi < x < 2\pi. \end{cases}$$

In Problem 48 the function could also be

$$f(x) = \begin{cases} 0, & -2 < x < -1 \\ x+1, & -1 < x < 0 \\ -x+1, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$

Exercises 12.4

In this section we make use of the following identities due to Euler's formula:

$$e^{in\pi} = e^{-in\pi} = (-1)^n, \qquad e^{-2in\pi} = 1, \qquad e^{-in\pi/2} = (-i)^n.$$

1. Identifying $p = 2$ we have

$$\begin{aligned}c_n &= \frac{1}{4}\int_{-2}^{2} f(x)e^{-in\pi x/2}dx = \frac{1}{4}\left[\int_{-2}^{0}(-1)e^{-in\pi x/2}dx + \int_{0}^{2} e^{-in\pi x/2}dx\right] \\ &= \frac{i}{2n\pi}\left[-1 + e^{in\pi} + e^{-in\pi} - 1\right] = \frac{i}{2n\pi}\left[-1 + (-1)^n + (-1)^n - 1\right] = \frac{1-(-1)^n}{n\pi i}\end{aligned}$$

and

$$c_0 = \frac{1}{4}\int_{-2}^{2} f(x)dx = 0.$$

Thus

$$f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1-(-1)^n}{in\pi} e^{in\pi x/2}.$$

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2. Identifying $2p = 2$ or $p = 1$ we have

$$c_n = \frac{1}{2}\int_0^2 f(x)e^{-in\pi x}dx = \frac{1}{2}\int_1^2 e^{-in\pi x}dx = -\frac{1}{2in\pi}e^{-in\pi x}\Big|_1^2$$
$$= -\frac{1}{2in\pi}\left(e^{-2in\pi} - e^{-in\pi}\right) = -\frac{1}{2in\pi}[1-(-1)^n] = \frac{i}{2n\pi}[1-(-1)^n]$$

and

$$c_0 = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\int_1^2 dx = \frac{1}{2}\,.$$

Thus

$$f(x) = \frac{1}{2} + \frac{i}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1-(-1)^n}{n}e^{in\pi x}.$$

3. Identifying $p = 1/2$ we have

$$c_n = \int_{-1/2}^{1/2} f(x)e^{-2in\pi x}dx = \int_0^{1/4} e^{-2in\pi x}dx = -\frac{1}{2in\pi}e^{-2in\pi x}\Big|_0^{1/4}$$
$$= -\frac{1}{2in\pi}\left[e^{-in\pi/2} - 1\right] = -\frac{1}{2in\pi}[(-i)^n - 1] = \frac{i}{2n\pi}[(-i)^n - 1]$$

and

$$c_0 = \int_0^{1/4} dx = \frac{1}{4}\,.$$

Thus

$$f(x) = \frac{1}{4} + \frac{i}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-i)^n - 1}{n}e^{2in\pi x}.$$

4. Identifying $p = \pi$ we have

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx/\pi}dx = \frac{1}{2\pi}\int_0^{\pi} xe^{-inx/\pi}dx$$
$$= \frac{1}{2}\left(\frac{\pi}{n^2} + \frac{ix}{n}\right)e^{-inx/\pi}\Big|_0^{\pi} = \frac{\pi(1+in)}{2n^2}e^{-in} - \frac{\pi}{2n^2}$$

and

$$c_0 = \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{\pi}{4}\,.$$

Thus

$$f(x) = \frac{\pi}{4} + \frac{\pi}{2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n^2}\Big[(1+in)e^{-in} - 1\Big]e^{inx}.$$

5. Identifying $2p = 2\pi$ or $p = \pi$ we have

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_0^{2\pi} xe^{-inx}dx$$
$$= \frac{1}{2\pi}\left(\frac{1}{n^2} + \frac{ix}{n}\right)e^{-inx}\Big|_0^{2\pi} = \frac{1+2in\pi}{2n^2\pi} - \frac{1}{2n^2\pi} = \frac{i}{n}$$

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and

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \pi.$$

Thus

$$f(x) = \pi + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i}{n} e^{inx}.$$

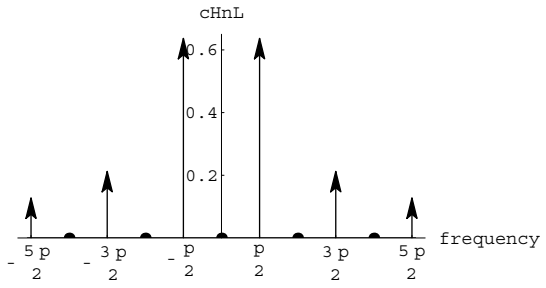
6. Identifying $p = 1$ we have

$$\begin{aligned} c_n &= \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\pi x}dx = \frac{1}{2}\left[\int_{-1}^{0} e^{x}e^{-in\pi x}dx + \int_{0}^{1} e^{-x}e^{-in\pi x}dx\right] \\ &= \frac{1}{2}\left[-\frac{1}{1-in\pi}e^{(1-in\pi)x}\Big|_{-1}^{0} - \frac{1}{1+in\pi}e^{-(1+in\pi)x}\Big|_{0}^{1}\right] \\ &= \frac{e-(-1)^n}{e(1-in\pi)} + \frac{1-e^{-1}(-1)^n}{1+in\pi} = \frac{2[e-(-1)^n]}{e(1+n^2\pi^2)}. \end{aligned}$$

Thus

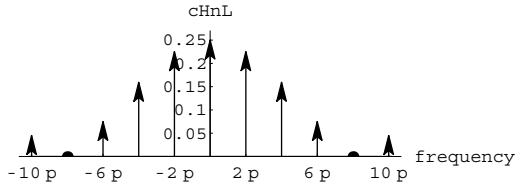
$$f(x) = \sum_{n=-\infty}^{\infty} \frac{2[e-(-1)^n]}{e(1+n^2\pi^2)} e^{in\pi x}.$$

7. The fundamental period is $T = 4$, so $\omega = 2\pi/4 = \pi/2$ and the values of $n\omega$ are $0, \pm\pi/2, \pm\pi, \pm 3\pi/2, \ldots$. From Problem 1, $c_0 = 0$ and $|c_n| = (1-(-1)^n)/n\pi$. The table shows some values of n with corresponding values of $|c_n|$. The graph is a portion of the frequency spectrum.



n	-5	-4	-3	-2	-1	0	1	2	3	4	5
c_n	0.1273	0.0000	0.2122	0.0000	0.6366	0.0000	0.6366	0.0000	0.2122	0.0000	0.1273

8. The fundamental period is $T = 1$, so $\omega = 2\pi$ and the values of $n\omega$ are $0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$. From Problem 3, $c_0 = \frac{1}{4}$ and $|c_n| = |(-i)^n - 1|/2n\pi$, or $c_1 = c_{-1} = \sqrt{2}/2\pi$, $c_2 = c_{-2} = 1/2\pi$, $c_3 = c_{-3} = \sqrt{2}/6\pi$, $c_4 = c_{-4} = 0$, $c_5 = c_{-5} = \sqrt{2}/10\pi$, $c_6 = c_{-6} = 1/6\pi$, $c_7 = c_{-7} = \sqrt{2}/14\pi$, $c_8 = c_{-8} = 0, \ldots$. The table shows some values of n with corresponding values of $|c_n|$. The graph is a portion of the frequency spectrum.



n	-5	-4	-3	-2	-1	0	1	2	3	4	5
c_n	0.0450	0.0000	0.0750	0.1592	0.2251	0.2500	0.2251	0.1592	0.0750	0.0000	0.0450

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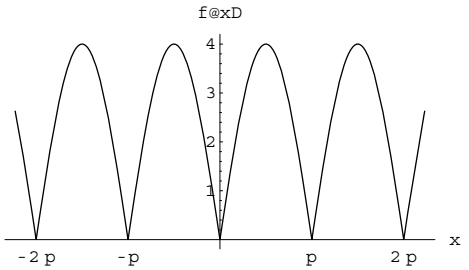
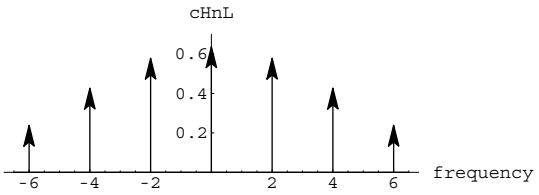
9. Identifying $2p = \pi$ or $p = \pi/2$, and using $\sin x = (e^{ix} - e^{-ix})/2i$, we have

$$\begin{aligned} c_n &= \frac{1}{\pi}\int_0^{\pi} f(x)e^{-2inx/\pi}\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} (\sin x)e^{-2inx/\pi}\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} \frac{1}{2i}(e^{ix} - e^{-ix})e^{-2inx/\pi}\,dx \\ &= \frac{1}{2\pi i}\int_0^{\pi} \left(e^{(1-2n/\pi)ix} - e^{-(1+2n/\pi)ix}\right)dx \\ &= \frac{1}{2\pi i}\left[\frac{1}{i(1-2n/\pi)}e^{(1-2n/\pi)ix}\right. \\ &\qquad \left. + \frac{1}{i(1+2n/\pi)}e^{-(1+2n/\pi)ix}\right]_0^{\pi} \\ &= \frac{\pi(1+e^{-2in})}{\pi^2 - 4n^2}\,. \end{aligned}$$



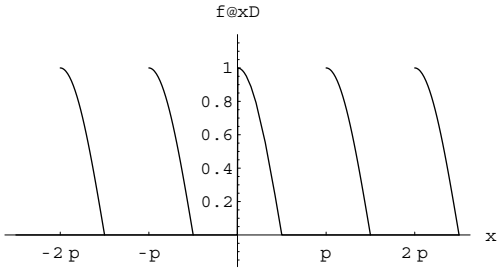


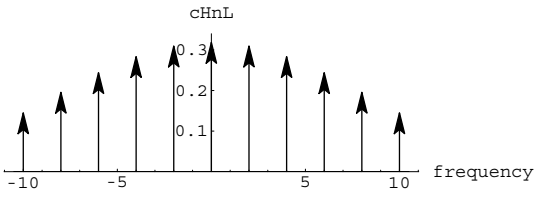
The fundamental period is $T = \pi$, so $\omega = 2\pi/\pi = 2$ and the values of $n\omega$ are 0, ± 2, ± 4, ± 6, ... Values of $|c_n|$ for $n = 0$, ± 1, ± 2, ± 3, ± 4, and ± 5 are shown in the table. The bottom graph is a portion of the frequency spectrum.

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
c_n	0.0198	0.0759	0.2380	0.4265	0.5784	0.6366	0.5784	0.4265	0.2380	0.0759	0.0198

10. Identifying $2p = \pi$ or $p = \pi/2$, and using $\cos x = (e^{ix} - e^{-ix})/2$, we have

$$\begin{aligned} c_n &= \frac{1}{\pi}\int_0^{\pi} f(x)e^{-2inx/\pi}\,dx \\ &= \frac{1}{\pi}\int_0^{\pi/2} (\cos x)e^{-2inx/\pi}\,dx \\ &= \frac{1}{\pi}\int_0^{\pi/2} \frac{1}{2}(e^{ix} - e^{-ix})e^{-2inx/\pi}\,dx \\ &= \frac{1}{2\pi}\int_0^{\pi/2} \left(e^{(1-2n/\pi)ix} - e^{-(1+2n/\pi)ix}\right)dx \\ &= \frac{1}{2\pi}\left[\frac{1}{i(1-2n/\pi)}e^{(1-2n/\pi)ix}\right. \\ &\qquad \left. + \frac{1}{i(1+2n/\pi)}e^{-(1+2n/\pi)ix}\right]_0^{\pi/2} \\ &= \frac{2ne^{-in} + i\pi}{\pi^2 - 4n^2}\,. \end{aligned}$$





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The fundamental period is $T = \pi$, so $\omega = 2\pi/\pi = 2$ and the values of $n\omega$ are 0, ± 2, ± 4, ± 6, $\ldots$. Values of $|c_n|$ for $n = 0, \pm 1, \pm 2, \pm 3, \pm 4$, and ± 5 are shown in the table. The bottom graph is a portion of the frequency spectrum.

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
c_n	0.1447	0.1954	0.2437	0.2833	0.3093	0.3183	0.3093	0.2833	0.2437	0.1954	0.1447

11. (a) Adding $c_n = \frac{1}{2}(a_n - ib_n)$ and $c_{-n} = \frac{1}{2}(a_n + ib_n)$ we get $c_n + c_{-n} = a_n$. Subtracting, we get $c_n - c_{-n} = -ib_n$. Multiplying both sides by i we obtain $i(c_n - c_{-n}) = b_n$.

(b) From

$$a_n = c_n + c_{-n} = (-1)^n \frac{\sinh \pi}{\pi}\left[\frac{1 - in}{n^2 + 1} + \frac{1 + in}{n^2 + 1}\right] = \frac{2(-1)^n \sinh \pi}{\pi(n^2 + 1)}, \quad n = 0, 1, 2, \ldots$$

and

$$b_n = i(c_n - c_{-n}) = i(-1)^n \frac{\sinh \pi}{\pi}\left[\frac{1 - in}{n^2 + 1} - \frac{1 + in}{n^2 + 1}\right] = i(-1)^n \frac{\sinh \pi}{\pi}\left[-\frac{2in}{n^2 + 1}\right] = \frac{2(-1)^n n \sinh \pi}{\pi(n^2 + 1)},$$

the Fourier series of f is

$$f(x) = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2 + 1} \cos nx + \frac{n(-1)^n}{n^2 + 1} \sin nx\right].$$

12. From Problem 11 and the fact that f is odd, $c_n + c_{-n} = a_n = 0$, so $c_{-n} = -c_n$. Then $b_n = i(c_n - c_{-n}) = 2ic_n$. From Problem 1, $b_n = 2i[1 - (-1)^n]/n\pi i = 2[1 - (-1)^n]/n\pi$, and the Fourier sine series of f is

$$f(x) = \sum_{i=1}^{\infty} \frac{2[1 - (-1)^n]}{n\pi} \sin \frac{n\pi x}{2}.$$

Exercises 12.5

1. For $\lambda \leq 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda > 0$ we have

$$y = c_1 \cos \sqrt{\lambda}\, x + c_2 \sin \sqrt{\lambda}\, x.$$

Now

$$y'(x) = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda}\, x + c_2 \sqrt{\lambda} \cos \sqrt{\lambda}\, x$$

and $y'(0) = 0$ implies $c_2 = 0$, so

$$y(1) + y'(1) = c_1(\cos \sqrt{\lambda} - \sqrt{\lambda} \sin \sqrt{\lambda}\,) = 0 \quad \text{or} \quad \cot \sqrt{\lambda} = \sqrt{\lambda}.$$

The eigenvalues are $\lambda_n = x_n^2$ where $x_1, x_2, x_3, \ldots$ are the consecutive positive solutions of $\cot \sqrt{\lambda} = \sqrt{\lambda}$. The corresponding eigenfunctions are $\cos \sqrt{\lambda_n}\, x = \cos x_n x$ for $n = 1, 2, 3, \ldots$. Using a CAS we find that the first four eigenvalues are 0.7402, 11.7349, 41.4388, and 90.8082 with corresponding eigenfunctions $\cos 0.8603x$, $\cos 3.4256x$, $\cos 6.4373x$, and $\cos 9.5293x$.

2. For $\lambda < 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = 0$ we have $y = c_1 x + c_2$. Now $y' = c_1$ and the boundary conditions both imply $c_1 + c_2 = 0$. Thus, $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $y_0 = x - 1$.

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For $\lambda > 0$ we have

$$y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$$

and

$$y' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x.$$

The boundary conditions imply

$$c_1 + c_2\sqrt{\lambda} = 0$$
$$c_1 \cos\sqrt{\lambda} + c_2 \sin\sqrt{\lambda} = 0$$

which gives

$$-c_2\sqrt{\lambda}\cos\sqrt{\lambda} + c_2 \sin\sqrt{\lambda} = 0 \quad \text{or} \quad \tan\sqrt{\lambda} = \sqrt{\lambda}\,.$$

The eigenvalues are $\lambda_n = x_n^2$ where x_1, x_2, x_3, ... are the consecutive positive solutions of $\tan\sqrt{\lambda} = \sqrt{\lambda}$. The corresponding eigenfunctions are $\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\,x - \sin\sqrt{\lambda_n}\,x$ (obtained by taking $c_2 = -1$ in the first equation of the system.) Using a CAS we find that the first four positive eigenvalues are 20.1907, 59.6795, 118.9000, and 197.858 with corresponding eigenfunctions $4.4934\cos 4.4934x - \sin 4.4934x$, $7.7253\cos 7.7253x - \sin 7.7253x$, $10.9041\cos 10.9041x - \sin 10.9041x$, and $14.0662\cos 14.0662x - \sin 14.0662x$.

3. For $\lambda = 0$ the solution of $y'' = 0$ is $y = c_1x + c_2$. The condition $y'(0) = 0$ implies $c_1 = 0$, so $\lambda = 0$ is an eigenvalue with corresponding eigenfunction 1. For $\lambda < 0$ we have $y = c_1\cosh\sqrt{-\lambda}\,x + c_2\sin\sqrt{-\lambda}\,x$ and $y' = c_1\sqrt{-\lambda}\sinh\sqrt{-\lambda}\,x + c_2\sqrt{-\lambda}\cosh\sqrt{-\lambda}\,x$. The condition $y'(0) = 0$ implies $c_2 = 0$ and so $y = c_1\cosh\sqrt{-\lambda}\,x$. Now the condition $y'(L) = 0$ implies $c_1 = 0$. Thus $y = 0$ and there are no negative eigenvalues. For $\lambda > 0$ we have $y = c_1\cos\sqrt{\lambda}\,x + c_2\sin\sqrt{\lambda}\,x$ and $y' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x$. The condition $y'(0) = 0$ implies $c_2 = 0$ and so $y = c_1\cos\sqrt{\lambda}\,x$. Now the condition $y'(L) = 0$ implies $-c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,L = 0$. For $c_1 \neq 0$ this condition will hold when $\sqrt{\lambda}\,L = n\pi$ or $\lambda = n^2\pi^2/L^2$, where $n = 1, 2, 3, \ldots$. These are the positive eigenvalues with corresponding eigenfunctions $\cos(n\pi/L)x$, $n = 1, 2, 3, \ldots$.

4. For $\lambda < 0$ we have

$$y = c_1\cosh\sqrt{-\lambda}\,x + c_2\sinh\sqrt{-\lambda}\,x$$
$$y' = c_1\sqrt{-\lambda}\sinh\sqrt{-\lambda}\,x + c_2\sqrt{-\lambda}\cosh\sqrt{-\lambda}\,x.$$

Using the fact that $\cosh x$ is an even function and $\sinh x$ is odd we have

$$\begin{aligned} y(-L) &= c_1\cosh(-\sqrt{-\lambda}\,L) + c_2\sinh(-\sqrt{-\lambda}\,L) \\ &= c_1\cosh\sqrt{-\lambda}\,L - c_2\sinh\sqrt{-\lambda}\,L \end{aligned}$$

and

$$\begin{aligned} y'(-L) &= c_1\sqrt{-\lambda}\sinh(-\sqrt{-\lambda}\,L) + c_2\sqrt{-\lambda}\cosh(-\sqrt{-\lambda}\,L) \\ &= -c_1\sqrt{-\lambda}\sinh\sqrt{-\lambda}\,L + c_2\sqrt{-\lambda}\cosh\sqrt{-\lambda}\,L. \end{aligned}$$

The boundary conditions imply

$$c_1\cosh\sqrt{-\lambda}\,L - c_2\sinh\sqrt{-\lambda}\,L = c_1\cosh\sqrt{-\lambda}\,L + c_2\sinh\sqrt{-\lambda}\,L$$

or

$$2c_2\sinh\sqrt{-\lambda}\,L = 0$$

and

$$-c_1\sqrt{-\lambda}\sinh\sqrt{-\lambda}\,L + c_2\sqrt{-\lambda}\cosh\sqrt{-\lambda}\,L = c_1\sqrt{-\lambda}\sinh\sqrt{-\lambda}\,L + c_2\sqrt{-\lambda}\cosh\sqrt{-\lambda}\,L$$

or

$$2c_1\sqrt{-\lambda}\sinh\sqrt{-\lambda}\,L = 0.$$

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Since $\sqrt{-\lambda}\,L \neq 0$, $c_1 = c_2 = 0$ and the only solution of the boundary-value problem in this case is $y = 0$.
For $\lambda = 0$ we have

$$y = c_1 x + c_2$$
$$y' = c_1.$$

From $y(-L) = y(L)$ we obtain

$$-c_1 L + c_2 = c_1 L + c_2.$$

Then $c_1 = 0$ and $y = 1$ is an eigenfunction corresponding to the eigenvalue $\lambda = 0$.
For $\lambda > 0$ we have

$$y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$$
$$y' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x.$$

The first boundary condition implies

$$c_1 \cos\sqrt{\lambda}\,L - c_2 \sin\sqrt{\lambda}\,L = c_1 \cos\sqrt{\lambda}\,L + c_2 \sin\sqrt{\lambda}\,L$$

or

$$2c_2 \sin\sqrt{\lambda}\,L = 0.$$

Thus, if $c_1 = 0$ and $c_2 \neq 0$,

$$\sqrt{\lambda}\,L = n\pi \quad \text{or} \quad \lambda = \frac{n^2\pi^2}{L^2},\ n = 1, 2, 3, \ldots .$$

The corresponding eigenfunctions are $\sin\dfrac{n\pi}{L}x$, for $n = 1,\ 2,\ 3,\ \ldots$. Similarly, the second boundary condition implies

$$2c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,L = 0.$$

If $c_2 = 0$ and $c_1 \neq 0$,

$$\sqrt{\lambda}\,L = n\pi \quad \text{or} \quad \lambda = \frac{n^2\pi^2}{L^2},\ n = 1, 2, 3, \ldots,$$

and the corresponding eigenfunctions are $\cos\dfrac{n\pi}{L}x$, for $n = 1,\ 2,\ 3,\ \ldots$.

5. The eigenfunctions are $\cos\sqrt{\lambda_n}\,x$ where $\cot\sqrt{\lambda_n} = \sqrt{\lambda_n}$. Thus

$$\begin{aligned}
\|\cos\sqrt{\lambda_n}\,x\|^2 &= \int_0^1 \cos^2\sqrt{\lambda_n}\,x\,dx = \frac{1}{2}\int_0^1 \left(1 + \cos 2\sqrt{\lambda_n}\,x\right)\,dx \\
&= \frac{1}{2}\left(x + \frac{1}{2\sqrt{\lambda_n}}\sin 2\sqrt{\lambda_n}\,x\right)\Bigg|_0^1 = \frac{1}{2}\left(1 + \frac{1}{2\sqrt{\lambda_n}}\sin 2\sqrt{\lambda_n}\right) \\
&= \frac{1}{2}\left[1 + \frac{1}{2\sqrt{\lambda_n}}\left(2\sin\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\right)\right] \\
&= \frac{1}{2}\left[1 + \frac{1}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}\cot\sqrt{\lambda_n}\sin\sqrt{\lambda_n}\right] \\
&= \frac{1}{2}\left[1 + \frac{1}{\sqrt{\lambda_n}}\left(\sin\sqrt{\lambda_n}\right)\sqrt{\lambda_n}\left(\sin\sqrt{\lambda_n}\right)\right] = \frac{1}{2}\left(1 + \sin^2\sqrt{\lambda_n}\right).
\end{aligned}$$

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6. The eigenfunctions are $\sin\sqrt{\lambda_n}\,x$ where $\tan\sqrt{\lambda_n} = -\lambda_n$. Thus

$$\begin{aligned}\|\sin\sqrt{\lambda_n}\,x\|^2 &= \int_0^1 \sin^2\sqrt{\lambda_n}\,x\,dx = \frac{1}{2}\int_0^1 \left(1-\cos 2\sqrt{\lambda_n}\,x\right)dx \\ &= \frac{1}{2}\left(x - \frac{1}{2\sqrt{\lambda_n}}\sin 2\sqrt{\lambda_n}\,x\right)\Big|_0^1 = \frac{1}{2}\left(1 - \frac{1}{2\sqrt{\lambda_n}}\sin 2\sqrt{\lambda_n}\right) \\ &= \frac{1}{2}\left[1 - \frac{1}{2\sqrt{\lambda_n}}\left(2\sin\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\right)\right] \\ &= \frac{1}{2}\left[1 - \frac{1}{\sqrt{\lambda_n}}\tan\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\right] \\ &= \frac{1}{2}\left[1 - \frac{1}{\sqrt{\lambda_n}}\left(-\sqrt{\lambda_n}\cos^2\sqrt{\lambda_n}\right)\right] = \frac{1}{2}\left(1+\cos^2\sqrt{\lambda_n}\right).\end{aligned}$$

7. **(a)** If $\lambda \le 0$ the initial conditions imply $y = 0$. For $\lambda > 0$ the general solution of the Cauchy-Euler differential equation is $y = c_1\cos(\sqrt{\lambda}\,\ln x) + c_2\sin(\sqrt{\lambda}\,\ln x)$. The condition $y(1) = 0$ implies $c_1 = 0$, so that $y = c_2\sin(\sqrt{\lambda}\,\ln x)$. The condition $y(5) = 0$ implies $\sqrt{\lambda}\,\ln 5 = n\pi$, $n = 1, 2, 3, \ldots$. Thus, the eigenvalues are $n^2\pi^2/(\ln 5)^2$ for $n = 1, 2, 3, \ldots$, with corresponding eigenfunctions $\sin[(n\pi/\ln 5)\ln x]$.

(b) The self-adjoint form is

$$\frac{d}{dx}[xy'] + \frac{\lambda}{x}y = 0.$$

(c) An orthogonality relation is

$$\int_1^5 \sin\left(\frac{m\pi}{\ln 5}\ln x\right)\sin\left(\frac{n\pi}{\ln 5}\ln x\right)dx = 0.$$

8. **(a)** The roots of the auxiliary equation $m^2 + m + \lambda = 0$ are $\frac{1}{2}(-1 \pm \sqrt{1-4\lambda}\,)$. When $\lambda = 0$ the general solution of the differential equation is $c_1 + c_2e^{-x}$. The initial conditions imply $c_1 + c_2 = 0$ and $c_1 + c_2e^{-2} = 0$. Since the determinant of the coefficients is not 0, the only solution of this homogeneous system is $c_1 = c_2 = 0$, in which case $y = 0$. Similarly, if $0 < \lambda < \frac{1}{4}$, the general solution is

$$y = c_1e^{\frac{1}{2}(-1+\sqrt{1-4\lambda})x} + c_2e^{\frac{1}{2}(-1-\sqrt{1-4\lambda})x}.$$

In this case the initial conditions again imply $c_1 = c_2 = 0$, and so $y = 0$. Now, for $\lambda > \frac{1}{4}$, the general solution of the differential equation is

$$y = c_1e^{-x/2}\cos\sqrt{4\lambda-1}\,x + c_2e^{-x/2}\sin\sqrt{4\lambda-1}\,x.$$

The condition $y(0) = 0$ implies $c_1 = 0$ so $y = c_2e^{-x/2}\sin\sqrt{4\lambda-1}\,x$. From

$$y(2) = c_2e^{-1}\sin 2\sqrt{4\lambda-1} = 0$$

we see that the eigenvalues are determined by $2\sqrt{4\lambda-1} = n\pi$ for $n = 1, 2, 3, \ldots$. Thus, the eigenvalues are $n^2\pi^2/4^2 + 1/4$ for $n = 1, 2, 3, \ldots$, with corresponding eigenfunctions $e^{-x/2}\sin\dfrac{n\pi}{2}x$.

(b) The self-adjoint form is

$$\frac{d}{dx}[e^xy'] + \lambda e^xy = 0.$$

(c) An orthogonality relation is

$$\int_0^2 e^x\left(e^{-x/2}\sin\frac{m\pi}{2}x\right)\left(e^{-x/2}\cos\frac{n\pi}{2}x\right)dx = \int_0^2\left(\sin\frac{m\pi}{2}x\right)\left(\cos\frac{n\pi}{2}x\right)dx = 0.$$

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9. **(a)** An orthogonality relation is

$$\int_0^1 \cos x_m x \cos x_n x = 0$$

where $x_m \neq x_n$ are positive solutions of $\cot x = x$.

(b) Referring to Problem 1 we use a CAS to compute

$$\int_0^1 (\cos 0.8603x)(\cos 3.4256x)\,dx = -1.8771 \times 10^{-6}.$$

10. **(a)** An orthogonality relation is

$$\int_0^1 (x_m \cos x_m x - \sin x_m x)(x_n \cos x_n x - \sin x_n x)\,dx = 0$$

where $x_m \neq x_n$ are positive solutions of $\tan x = x$.

(b) Referring to Problem 2 we use a CAS to compute

$$\int_0^1 (4.4934 \cos 4.4934x - \sin 4.4934x)(7.7253 \cos 7.7253x - \sin 7.7253x)\,dx = -2.5650 \times 10^{-4}.$$

11. To obtain the self-adjoint form we note that an integrating factor is $(1/x)e^{\int (1-x)dx/x} = e^{-x}$. Thus, the differential equation is

$$xe^{-x}y'' + (1-x)e^{-x}y' + ne^{-x}y = 0$$

and the self-adjoint form is

$$\frac{d}{dx}\left[xe^{-x}y'\right] + ne^{-x}y = 0.$$

Identifying the weight function $p(x) = e^{-x}$ and noting that since $r(x) = xe^{-x}$, $r(0) = 0$ and $\lim_{x\to\infty} r(x) = 0$, we have the orthogonality relation

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\,dx = 0,\ m \neq n.$$

12. To obtain the self-adjoint form we note that an integrating factor is $e^{\int -2x\,dx} = e^{-x^2}$. Thus, the differential equation is

$$e^{-x^2}y'' - 2xe^{-x^2}y' + 2ne^{-x^2}y = 0$$

and the self-adjoint form is

$$\frac{d}{dx}\left[e^{-x^2}y'\right] + 2ne^{-x^2}y = 0.$$

Identifying the weight function $p(x) = 2e^{-x^2}$ and noting that since $r(x) = e^{-x^2}$, $\lim_{x\to-\infty} r(x) = \lim_{x\to\infty} r(x) = 0$, we have the orthogonality relation

$$\int_\infty^\infty 2e^{-x^2} H_n(x) H_m(x)\,dx = 0,\ m \neq n.$$

13. **(a)** The differential equation is

$$(1+x^2)y'' + 2xy' + \frac{\lambda}{1+x^2}\,y = 0.$$

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Letting $x = \tan\theta$ we have $\theta = \tan^{-1} x$ and

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{1}{1+x^2}\frac{dy}{d\theta}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1}{1+x^2}\frac{dy}{d\theta}\right] = \frac{1}{1+x^2}\left(\frac{d^2y}{d\theta^2}\frac{d\theta}{dx}\right) - \frac{2x}{(1+x^2)^2}\frac{dy}{d\theta}$$

$$= \frac{1}{(1+x^2)^2}\frac{d^2y}{d\theta^2} - \frac{2x}{(1+x^2)^2}\frac{dy}{d\theta}.$$

The differential equation can then be written in terms of $y(\theta)$ as

$$(1+x^2)\left[\frac{1}{(1+x^2)^2}\frac{d^2y}{d\theta^2} - \frac{2x}{(1+x^2)^2}\frac{dy}{d\theta}\right] + 2x\left[\frac{1}{1+x^2}\frac{dy}{d\theta}\right] + \frac{\lambda}{1+x^2}y$$

$$= \frac{1}{1+x^2}\frac{d^2y}{d\theta^2} + \frac{\lambda}{1+x^2}y = 0$$

or

$$\frac{d^2y}{d\theta^2} + \lambda y = 0.$$

The boundary conditions become $y(0) = y(\pi/4) = 0$. For $\lambda \le 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda > 0$ the general solution of the differential equation is $y = c_1\cos\sqrt{\lambda}\,\theta + c_2\sin\sqrt{\lambda}\,\theta$. The condition $y(0) = 0$ implies $c_1 = 0$ so $y = c_2\sin\sqrt{\lambda}\,\theta$. Now the condition $y(\pi/4) = 0$ implies $c_2\sin\sqrt{\lambda}\,\pi/4 = 0$. For $c_2 \neq 0$ this condition will hold when $\sqrt{\lambda}\,\pi/4 = n\pi$ or $\lambda = 16n^2$, where $n = 1, 2, 3, \ldots$. These are the eigenvalues with corresponding eigenfunctions $\sin 4n\theta = \sin(4n\tan^{-1}x)$, for $n = 1, 2, 3, \ldots$.

(b) An orthogonality relation is

$$\int_0^1 \frac{1}{x^2+1}\sin(4m\tan^{-1}x)\sin(4n\tan^{-1}x)\,dx = 0.$$

14. (a) This is the parametric Bessel equation with $\nu = 1$. The general solution is

$$y = c_1J_1(\lambda x) + c_2Y_1(\lambda x).$$

Since Y is unbounded at 0 we must have $c_2 = 0$, so that $y = c_1J_1(\lambda x)$. The condition $J_1(3\lambda) = 0$ defines the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$. (When $\lambda = 0$ the differential equation is Cauchy-Euler and the only solution satisfying the boundary condition is $y = 0$, so $\lambda = 0$ is not an eigenvalue.)

(b) From Table 5.2 in the text we see that eigenvalues are determined by $3\lambda_1 = 3.832$, $3\lambda_2 = 7.016$, $3\lambda_3 = 10.173$, and $3\lambda_4 = 13.323$. The first four eigenvalues are thus $\lambda_1 = 1.2773$, $\lambda_2 = 2.3387$, $\lambda_3 = 3.391$, and $\lambda_4 = 4.441$.

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Exercises 12.6

1. Identifying $b = 3$, the first four eigenvalues are

$$\lambda_1 = \frac{3.832}{3} \approx 1.277$$
$$\lambda_2 = \frac{7.016}{3} \approx 2.339$$
$$\lambda_3 = \frac{10.173}{3} = 3.391$$
$$\lambda_4 = \frac{13.323}{3} \approx 4.441.$$

2. We first note from Case III in the text that 0 is an eigenvalue. Now, since $J_0'(2\lambda) = 0$ is equivalent to $J_1(2\lambda) = 0$, the next three eigenvalues are

$$\lambda_2 = \frac{3.832}{2} = 1.916$$
$$\lambda_3 = \frac{7.016}{2} = 3.508$$
$$\lambda_4 = \frac{10.173}{2} = 5.087.$$

3. The boundary condition indicates that we use (15) and (16) of Section 12.6. With $b = 2$ we obtain

$$c_i = \frac{2}{4J_1^2(2\lambda_i)} \int_0^2 xJ_0(\lambda_i x)\,dx \qquad \boxed{t = \lambda_i x \qquad dt = \lambda_i\,dx}$$
$$= \frac{1}{2J_1^2(2\lambda_i)} \cdot \frac{1}{\lambda_i^2} \int_0^{2\lambda_i} tJ_0(t)\,dt$$
$$= \frac{1}{2\lambda_i^2 J_1^2(2\lambda_i)} \int_0^{2\lambda_i} \frac{d}{dt}[tJ_1(t)]\,dt \qquad \text{[From (4) in the text]}$$
$$= \frac{1}{2\lambda_i^2 J_1^2(2\lambda_i)} tJ_1(t)\Big|_0^{2\lambda_i}$$
$$= \frac{1}{\lambda_i J_1(2\lambda_i)}.$$

Thus

$$f(x) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i J_1(2\lambda_i)} J_0(\lambda_i x).$$

4. The boundary condition indicates that we use (19) and (20) of Section 12.6. With $b = 2$ we obtain

$$c_1 = \frac{2}{4} \int_0^2 x\,dx = \frac{2}{4} \frac{x^2}{2}\Big|_0^2 = 1,$$

$$c_i = \frac{2}{4J_0^2(2\lambda_i)} \int_0^2 xJ_0(\lambda_i x)\,dx \qquad \boxed{t = \lambda_i x \qquad dt = \lambda_i\,dx}$$
$$= \frac{1}{2J_0^2(2\lambda_i)} \cdot \frac{1}{\lambda_i^2} \int_0^{2\lambda_i} tJ_0(t)\,dt$$

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$$= \frac{1}{2\lambda_i^2 J_0^2(2\lambda_i)} \int_0^{2\lambda_i} \frac{d}{dt}[tJ_1(t)]\,dt \qquad \text{[From (4) in the text]}$$

$$= \frac{1}{2\lambda_i^2 J_0^2(2\lambda_i)} tJ_1(t)\Big|_0^{2\lambda_i}$$

$$= \frac{J_1(2\lambda_i)}{\lambda_i J_0^2(2\lambda_i)}.$$

Now since $J_0'(2\lambda_i) = 0$ is equivalent to $J_1(2\lambda_i) = 0$ we conclude $c_i = 0$ for $i = 2, 3, 4, \ldots$. Thus the expansion of f on $0 < x < 2$ consists of a series with one nontrivial term:

$$f(x) = c_1 = 1.$$

5. The boundary condition indicates that we use (17) and (18) of Section 12.6. With $b = 2$ and $h = 1$ we obtain

$$c_i = \frac{2\lambda_i^2}{(4\lambda_i^2 + 1)J_0^2(2\lambda_i)} \int_0^2 xJ_0(\lambda_i x)\,dx \qquad \boxed{t = \lambda_i x \quad dt = \lambda_i\,dx}$$

$$= \frac{2\lambda_i^2}{(4\lambda_i^2 + 1)J_0^2(2\lambda_i)} \cdot \frac{1}{\lambda_i^2} \int_0^{2\lambda_i} tJ_0(t)\,dt$$

$$= \frac{2}{(4\lambda_i^2 + 1)J_0^2(2\lambda_i)} \int_0^{2\lambda_i} \frac{d}{dt}[tJ_1(t)]\,dt \qquad \text{[From (4) in the text]}$$

$$= \frac{2}{(4\lambda_i^2 + 1)J_0^2(2\lambda_i)} tJ_1(t)\Big|_0^{2\lambda_i}$$

$$= \frac{4\lambda_i J_1(2\lambda_i)}{(4\lambda_i^2 + 1)J_0^2(2\lambda_i)}.$$

Thus

$$f(x) = 4\sum_{i=1}^{\infty} \frac{\lambda_i J_1(2\lambda_i)}{(4\lambda_i^2 + 1)J_0^2(2\lambda_i)} J_0(\lambda_i x).$$

6. Writing the boundary condition in the form

$$2J_0(2\lambda) + 2\lambda J_0'(2\lambda) = 0$$

we identify $b = 2$ and $h = 2$. Using (17) and (18) of Section 12.6 we obtain

$$c_i = \frac{2\lambda_i^2}{(4\lambda_i^2 + 4)J_0^2(2\lambda_i)} \int_0^2 xJ_0(\lambda_i x)\,dx \qquad \boxed{t = \lambda_i x \quad dt = \lambda_i\,dx}$$

$$= \frac{\lambda_i^2}{2(\lambda_i^2 + 1)J_0^2(2\lambda_i)} \cdot \frac{1}{\lambda_i^2} \int_0^{2\lambda_i} tJ_0(t)\,dt$$

$$= \frac{1}{2(\lambda_i^2 + 1)J_0^2(2\lambda_i)} \int_0^{2\lambda_i} \frac{d}{dt}[tJ_1(t)]\,dt \qquad \text{[From (4) in the text]}$$

$$= \frac{1}{2(\lambda_i^2 + 1)J_0^2(2\lambda_i)} tJ_1(t)\Big|_0^{2\lambda_i}$$

$$= \frac{\lambda_i J_1(2\lambda_i)}{(\lambda_i^2 + 1)J_0^2(2\lambda_i)}.$$

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Thus

$$f(x) = \sum_{i=1}^{\infty} \frac{\lambda_i J_1(2\lambda_i)}{(\lambda_i^2 + 1)J_0^2(2\lambda_i)} J_0(\lambda_i x).$$

7. The boundary condition indicates that we use (17) and (18) of Section 12.6. With $n = 1$, $b = 4$, and $h = 3$ we obtain

$$c_i = \frac{2\lambda_i^2}{(16\lambda_i^2 - 1 + 9)J_1^2(4\lambda_i)} \int_0^4 xJ_1(\lambda_i x)5x\,dx \qquad \boxed{t = \lambda_i x \qquad dt = \lambda_i\,dx}$$

$$= \frac{5\lambda_i^2}{4(2\lambda_i^2 + 1)J_1^2(4\lambda_i)} \cdot \frac{1}{\lambda_i^3} \int_0^{4\lambda_i} t^2 J_1(t)\,dt$$

$$= \frac{5}{4\lambda_i(2\lambda_i^2 + 1)J_1^2(4\lambda_i)} \int_0^{4\lambda_i} \frac{d}{dt}[t^2 J_2(t)]\,dt \qquad \text{[From (4) in the text]}$$

$$= \frac{5}{4\lambda_i(2\lambda_i^2 + 1)J_1^2(4\lambda_i)} t^2 J_2(t)\Big|_0^{4\lambda_i}$$

$$= \frac{20\lambda_i J_2(4\lambda_i)}{(2\lambda_i^2 + 1)J_1^2(4\lambda_i)}.$$

Thus

$$f(x) = 20\sum_{i=1}^{\infty} \frac{\lambda_i J_2(4\lambda_i)}{(2\lambda_i^2 + 1)J_1^2(4\lambda_i)} J_1(\lambda_i x).$$

8. The boundary condition indicates that we use (15) and (16) of Section 12.6. With $n = 2$ and $b = 1$ we obtain

$$c_1 = \frac{2}{J_3^2(\lambda_i)} \int_0^1 xJ_2(\lambda_i x)x^2\,dx \qquad \boxed{t = \lambda_i x \qquad dt = \lambda_i\,dx}$$

$$= \frac{2}{J_3^2(\lambda_i)} \cdot \frac{1}{\lambda_i^4} \int_0^{\lambda_i} t^3 J_2(t)\,dt$$

$$= \frac{2}{\lambda_i^4 J_3^2(\lambda_i)} \int_0^{\lambda_i} \frac{d}{dt}[t^3 J_3(t)]\,dt \qquad \text{[From (4) in the text]}$$

$$= \frac{2}{\lambda_i^4 J_3^2(\lambda_i)} t^3 J_3(t)\Big|_0^{\lambda_i}$$

$$= \frac{2}{\lambda_i J_3(\lambda_i)}.$$

Thus

$$f(x) = 2\sum_{i=1}^{\infty} \frac{1}{\lambda_i J_3(\lambda_i)} J_2(\lambda_i x).$$

9. The boundary condition indicates that we use (19) and (20) of Section 12.6. With $b = 3$ we obtain

$$c_1 = \frac{2}{9}\int_0^3 xx^2\,dx = \frac{2}{9}\,\frac{x^4}{4}\,\Big|_0^3 = \frac{9}{2},$$

$$c_i = \frac{2}{9J_0^2(3\lambda_i)} \int_0^3 xJ_0(\lambda_i x)x^2\,dx$$

$$\boxed{t = \lambda_i x \qquad dt = \lambda_i\,dx}$$

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$$= \frac{2}{9J_0^2(3\lambda_i)} \cdot \frac{1}{\lambda_i^4} \int_0^{3\lambda_i} t^3 J_0(t)\, dt$$

$$= \frac{2}{9\lambda_i^4 J_0^2(3\lambda_i)} \int_0^{3\lambda_i} t^2 \frac{d}{dt}\left[tJ_1(t)\right]\, dt$$

$u = t^2$	$dv = \frac{d}{dt}\left[tJ_1(t)\right] dt$
$du = 2t\, dt$	$v = tJ_1(t)$

$$= \frac{2}{9\lambda_i^4 J_0^2(3\lambda_i)} \left(t^3 J_1(t) \Big|_0^{3\lambda_i} - 2\int_0^{3\lambda_i} t^2 J_1(t)\, dt \right)$$

With $n = 0$ in equation (5) in Section 12.6 in the text we have $J_0'(x) = -J_1(x)$, so the boundary condition $J_0'(3\lambda_i) = 0$ implies $J_1(3\lambda_i) = 0$. Then

$$c_i = \frac{2}{9\lambda_i^4 J_0^2(3\lambda_i)} \left(-2\int_0^{3\lambda_i} \frac{d}{dt}\left[t^2 J_2(t)\right]\, dt \right) = \frac{2}{9\lambda_i^4 J_0^2(3\lambda_i)} \left(-2t^2 J_2(t) \Big|_0^{3\lambda_i} \right)$$

$$= \frac{2}{9\lambda_i^4 J_0^2(3\lambda_i)} \left[-18\lambda_i^2 J_2(3\lambda_i) \right] = \frac{-4J_2(3\lambda_i)}{\lambda_i^2 J_0^2(3\lambda_i)}.$$

Thus

$$f(x) = \frac{9}{2} - 4\sum_{i=1}^{\infty} \frac{J_2(3\lambda_i)}{\lambda_i^2 J_0^2(3\lambda_i)} J_0(\lambda_i x).$$

10. The boundary condition indicates that we use (15) and (16) of Section 12.6. With $b = 1$ it follows that

$$c_i = \frac{2}{J_1^2(\lambda_i)} \int_0^1 x\left(1 - x^2\right) J_0(\lambda_i x)\, dx$$

$$= \frac{2}{J_1^2(\lambda_i)} \left[\int_0^1 x J_0(\lambda_i x)\, dx - \int_0^1 x^3 J_0(\lambda_i x)\, dx \right]$$

$t = \lambda_i x$	$dt = \lambda_i\, dx$

$$= \frac{2}{J_1^2(\lambda_i)} \left[\frac{1}{\lambda_i^2} \int_0^{\lambda_i} tJ_0(t)\, dt - \frac{1}{\lambda_i^4} \int_0^{\lambda_i} t^3 J_0(t)\, dt \right]$$

$$= \frac{2}{J_1^2(\lambda_i)} \left[\frac{1}{\lambda_i^2} \int_0^{\lambda_i} \frac{d}{dt}\left(tJ_1(t)\right)\, dt - \frac{1}{\lambda_i^4} \int_0^{\lambda_i} t^2 \frac{d}{dt}\left[tJ_1(t)\right] dt \right]$$

$u = t^2$	$dv = \frac{d}{dt}\left[tJ_1(t)\right] dt$
$du = 2t\, dt$	$v = tJ_1(t)$

$$= \frac{2}{J_1^2(\lambda_i)} \left[\frac{1}{\lambda_i^2} tJ_1(t) \Big|_0^{\lambda_i} - \frac{1}{\lambda_i^4} \left(t^3 J_1(t) \Big|_0^{\lambda_i} - 2\int_0^{\lambda_i} t^2 J_1(t)\, dt \right) \right]$$

$$= \frac{2}{J_1^2(\lambda_i)} \left[\frac{J_1(\lambda_i)}{\lambda_i} - \frac{J_1(\lambda_i)}{\lambda_i} + \frac{2}{\lambda_i^4} \int_0^{\lambda_i} \frac{d}{dt}\left[t^2 J_2(t)\right]\, dt \right]$$

$$= \frac{2}{J_1^2(\lambda_i)} \left[\frac{2}{\lambda_i^4} t^2 J_2(t) \Big|_0^{\lambda_i} \right] = \frac{4J_2(\lambda_i)}{\lambda_i^2 J_1^2(\lambda_i)}.$$

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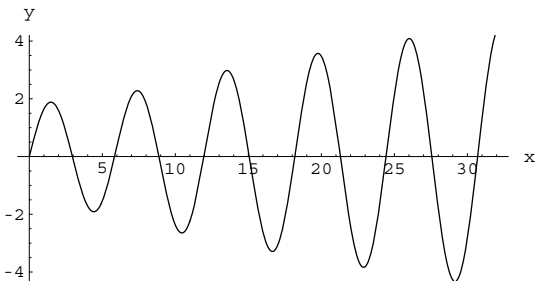
Thus

$$f(x) = 4\sum_{i=1}^{\infty} \frac{J_2(\lambda_i)}{\lambda_i^2 J_1^2(\lambda_i)}\, j_0(\lambda_i x).$$

11. (a)



(b) Using **FindRoot** in *Mathematica* we find the roots $x_1 = 2.9496$, $x_2 = 5.84113$, $x_3 = 8.87273$, $x_4 = 11.9561$, and $x_5 = 15.0624$.

(c) Dividing the roots in part (b) by 4 we find the eigenvalues $\lambda_1 = 0.7374$, $\lambda_2 = 1.46028$, $\lambda_3 = 2.21818$, $\lambda_4 = 2.98904$, and $\lambda_5 = 3.76559$.

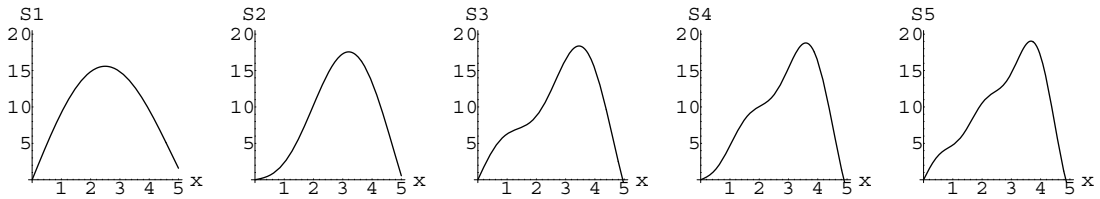
(d) The next five eigenvalues are $\lambda_6 = 4.54508$, $\lambda_7 = 5.32626$, $\lambda_8 = 6.1085$, $\lambda_9 = 6.89145$, and $\lambda_{10} = 7.6749$.

12. (a) From Problem 7, the coefficients of the Fourier-Bessel series are

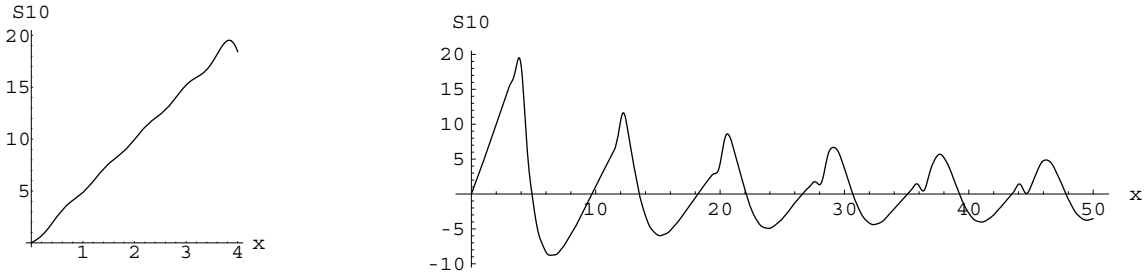
$$c_1 = \frac{20\lambda_i J_2(4\lambda_i)}{(2\lambda_i^2 + 1)J_1^2(4\lambda_i)}\,.$$

Using a CAS we find $c_1 = 26.7896$, $c_2 = -12.4624$, $c_3 = 7.1404$, $c_4 = -4.68705$, and $c_5 = 3.35619$.

(b)



(c)



13. Since f is expanded as a series of Bessel functions, $J_1(\lambda_i x)$ and J_1 is an odd function, the series should represent an odd function.

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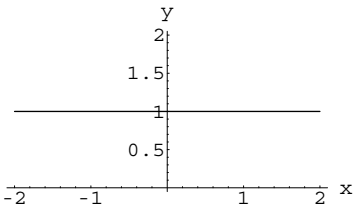
14. **(a)** Since J_0 is an even function, a series expansion of a function defined on $(0, 2)$ would converge to the even extension of the function on $(-2, 0)$.



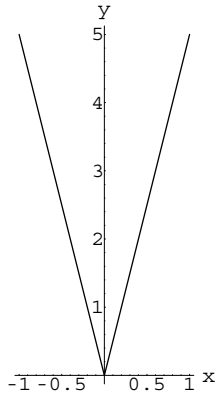
(b) In Section 5.3 we saw that $J_2'(x) = 2J_2(x)/x - J_3(x)$. Since J_2 is even and J_3 is odd we see that

$$\begin{aligned} J_2'(-x) &= 2J_2(-x)/(-x) - J_3(-x) \\ &= -2J_2(x)/x + J_3(x) = -J_2'(x), \end{aligned}$$

so that J_2' is an odd function. Now, if $f(x) = 3J_2(x) + 2xJ_2'(x)$, we see that

$$\begin{aligned} f(-x) &= 3J_2(-x) - 2xJ_2'(-x) \\ &= 3J_2(x) + 2xJ_2'(x) = f(x), \end{aligned}$$

so that f is an even function. Thus, a series expansion of a function defined on $(0, 1)$ would converge to the even extension of the function on $(-1, 0)$.



15. We compute

$$c_0 = \frac{1}{2}\int_0^1 xP_0(x)\,dx = \frac{1}{2}\int_0^1 x\,dx = \frac{1}{4}$$

$$c_1 = \frac{3}{2}\int_0^1 xP_1(x)\,dx = \frac{3}{2}\int_0^1 x^2\,dx = \frac{1}{2}$$

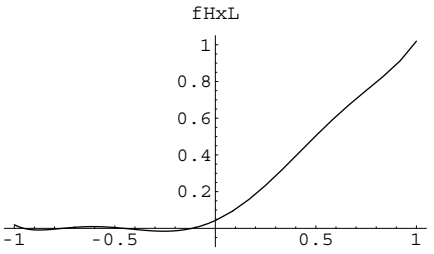
$$c_2 = \frac{5}{2}\int_0^1 xP_2(x)\,dx = \frac{5}{2}\int_0^1 \frac{1}{2}(3x^3 - x)dx = \frac{5}{16}$$

$$c_3 = \frac{7}{2}\int_0^1 xP_3(x)\,dx = \frac{7}{2}\int_0^1 \frac{1}{2}(5x^4 - 3x^2)dx = 0$$

$$c_4 = \frac{9}{2}\int_0^1 xP_4(x)\,dx = \frac{9}{2}\int_0^1 \frac{1}{8}(35x^5 - 30x^3 + 3x)dx = -\frac{3}{32}$$

$$c_5 = \frac{11}{2}\int_0^1 xP_5(x)\,dx = \frac{11}{2}\int_0^1 \frac{1}{8}(63x^6 - 70x^4 + 15x^2)dx = 0$$

$$c_6 = \frac{13}{2}\int_0^1 xP_6(x)\,dx = \frac{13}{2}\int_0^1 \frac{1}{16}(231x^7 - 315x^5 + 105x^3 - 5x)dx = \frac{13}{256}\,.$$



Thus

$$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + \frac{13}{256}P_6(x) + \cdots.$$

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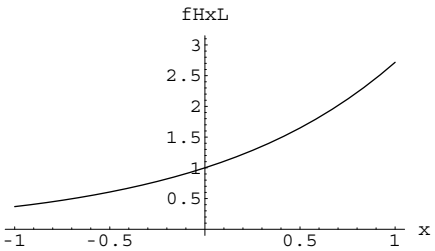
16. We compute



$$c_0 = \frac{1}{2}\int_{-1}^{1} e^x P_0(x)\,dx = \frac{1}{2}\int_{-1}^{1} e^x dx = \frac{1}{2}(e - e^{-1})$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} e^x P_1(x)\,dx = \frac{3}{2}\int_{-1}^{1} xe^x\,dx = 3e^{-1}$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} e^x P_2(x)\,dx = \frac{5}{2}\int_{-1}^{1} \frac{1}{2}(3x^2e^x - e^x)dx = \frac{5}{2}(e - 7e^{-1})$$

$$c_3 = \frac{7}{2}\int_{-1}^{1} e^x P_3(x)\,dx = \frac{7}{2}\int_{-1}^{1} \frac{1}{2}(5x^3e^x - 3xe^x)dx = \frac{7}{2}(-5e + 37e^{-1})$$

$$c_4 = \frac{9}{2}\int_{-1}^{1} e^x P_4(x)\,dx = \frac{9}{2}\int_{-1}^{1} \frac{1}{8}(35x^4e^x - 30x^2e^x + 3e^x)dx = \frac{9}{2}(36e - 266e^{-1}).$$

Thus

$$f(x) = \frac{1}{2}(e - e^{-1})P_0(x) + 3e^{-1}P_1(x) + \frac{5}{2}(e - 7e^{-1})P_2(x)$$
$$+ \frac{7}{2}(-5e + 37e^{-1})P_3(x) + \frac{9}{2}(36e - 266e^{-1})P_4(x) + \cdots.$$

17. Using $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$ we have

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$$
$$= \frac{3}{4}(\cos 2\theta + 1) - \frac{1}{2} = \frac{3}{4}\cos 2\theta + \frac{1}{4} = \frac{1}{4}(3\cos 2\theta + 1).$$

18. From Problem 17 we have

$$P_2(\cos 2\theta) = \frac{1}{4}(3\cos 2\theta + 1)$$

or

$$\cos 2\theta = \frac{4}{3}P_2(\cos\theta) - \frac{1}{3}.$$

Then, using $P_0(\cos\theta) = 1$,

$$F(\theta) = 1 - \cos 2\theta = 1 - \left[\frac{4}{3}P_2(\cos\theta) - \frac{1}{3}\right]$$
$$= \frac{4}{3} - \frac{4}{3}P_2(\cos\theta) = \frac{4}{3}P_0(\cos\theta) - \frac{4}{3}P_2(\cos\theta).$$

19. If f is an even function on $(-1, 1)$ then

$$\int_{-1}^{1} f(x)P_{2n}(x)\,dx = 2\int_{0}^{1} f(x)P_{2n}(x)\,dx$$

and

$$\int_{-1}^{1} f(x)P_{2n+1}(x)\,dx = 0.$$

Thus

$$c_{2n} = \frac{2(2n)+1}{2}\int_{-1}^{1} f(x)P_{2n}(x)\,dx = \frac{4n+1}{2}\left(2\int_{0}^{1} f(x)P_{2n}(x)\,dx\right)$$
$$= (4n+1)\int_{0}^{1} f(x)P_{2n}(x)\,dx,$$

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$c_{2n+1} = 0$, and

$$f(x) = \sum_{n=0}^{\infty} c_{2n} P_{2n}(x).$$

20. If f is an odd function on $(-1, 1)$ then

$$\int_{-1}^{1} f(x)P_{2n}(x)\,dx = 0$$

and

$$\int_{-1}^{1} f(x)P_{2n+1}(x)\,dx = 2\int_{0}^{1} f(x)P_{2n+1}(x)\,dx.$$

Thus

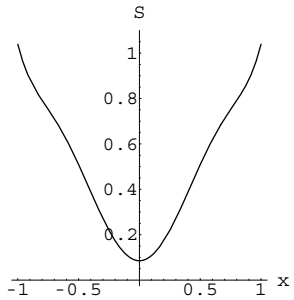
$$c_{2n+1} = \frac{2(2n+1)+1}{2}\int_{-1}^{1} f(x)P_{2n+1}(x)\,dx = \frac{4n+3}{2}\left(2\int_{0}^{1} f(x)P_{2n+1}(x)\,dx\right)$$

$$= (4n+1)\int_{0}^{1} f(x)P_{2n+1}(x)\,dx,$$

$c_{2n} = 0$, and

$$f(x) = \sum_{n=0}^{\infty} c_{2n+1} P_{2n+1}(x).$$

21. From (26) in Problem 19 in the text we find



$$c_0 = \int_0^1 xP_0(x)\,dx = \int_0^1 x\,dx = \frac{1}{2},$$

$$c_2 = 5\int_0^1 xP_2(x)\,dx = 5\int_0^1 \frac{1}{2}(3x^3 - x)dx = \frac{5}{8},$$

$$c_4 = 9\int_0^1 xP_4(x)\,dx = 9\int_0^1 \frac{1}{8}(35x^5 - 30x^3 + 3x)dx = -\frac{3}{16},$$

and

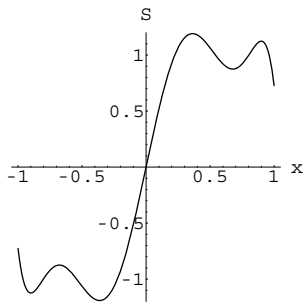
$$c_6 = 13\int_0^1 xP_6(x)\,dx = 13\int_0^1 \frac{1}{16}(231x^7 - 315x^5 + 105x^3 - 5x)dx = \frac{13}{128}.$$

Hence, from (25) in the text,

$$f(x) = \frac{1}{2}P_0(x) + \frac{5}{8}P_2(x) - \frac{3}{16}P_4(x) + \frac{13}{128}P_6 + \cdots.$$

On the interval $-1 < x < 1$ this series represents the function $f(x) = |x|$.

22. From (28) in Problem 20 in the text we find



$$c_1 = 3\int_0^1 P_1(x)\,dx = 3\int_0^1 x\,dx = \frac{3}{2},$$

$$c_3 = 7\int_0^1 P_3(x)\,dx = 7\int_0^1 \frac{1}{2}\left(5x^3 - 3x\right)dx = -\frac{7}{8},$$

$$c_5 = 11\int_0^1 P_5(x)\,dx = 11\int_0^1 \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)dx = \frac{11}{16}$$

and

$$c_7 = 15\int_0^1 P_7(x)\,dx = 15\int_0^1 \frac{1}{16}\left(429x^7 - 693x^5 + 315x^3 - 35x\right)dx = -\frac{75}{128}.$$

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Hence, from (27) in the text,

$$f(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - \frac{75}{128}P_7(x) + \cdots .$$

On the interval $-1 < x < 1$ this series represents the odd function

$$f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1. \end{cases}$$

Chapter 12 Review Exercises

1. True, since $\int_{-\pi}^{\pi}(x^2-1)x^5\,dx = 0$

2. Even, since if f and g are odd then $h(-x) = f(-x)g(-x) = -f(x)[-g(x)] = f(x)g(x) = h(x)$

3. Cosine, since f is even

4. True

5. False; the Sturm-Liouville problem,

$$\frac{d}{dx}[r(x)y'] + \lambda p(x)y = 0, \qquad y'(a) = 0, \quad y'(b) = 0,$$

on the interval $[a, b]$, has eigenvalue $\lambda = 0$.

6. $3/2$, the average of 3 and 0

7. The Fourier series will converge to 1, the cosine series to 1, and the sine series to 0 at $x = 0$. Respectively, this is because the rule $(x^2 + 1)$ defining $f(x)$ determines a continuous function on $(-3, 3)$, the even extension of f to $(-3, 0)$ is continuous at 0, and the odd extension of f to $(-3, 0)$ approaches -1 as x approaches 0 from the left.

8. $\cos 5x$, since the general solution is $y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$ and $y'(0) = 0$ implies $c_2 = 0$.

9. Since the coefficient of y in the differential equation is n^2, the weight function is the integrating factor

$$\frac{1}{a(x)}e^{\int (b/a)dx} = \frac{1}{1-x^2}e^{\int -\frac{x}{1-x^2}\,dx} = \frac{1}{1-x^2}e^{\frac{1}{2}\ln(1-x^2)} = \frac{\sqrt{1-x^2}}{1-x^2} = \frac{1}{\sqrt{1-x^2}}$$

on the interval $[-1, 1]$. The orthogonality relation is

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,T_m(x)T_n(x)\,dx = 0, \quad m \neq n.$$

10. Since $P_n(x)$ is orthogonal to $P_0(x) = 1$ for $n > 0$,

$$\int_{-1}^{1} P_n(x)\,dx = \int_{-1}^{1} P_0(x)P_n(x)\,dx = 0.$$

11. We know from a half-angle formula in trigonometry that $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$, which is a cosine series.

12. **(a)** For $m \neq n$

$$\int_0^L \sin\frac{(2n+1)\pi}{2L}x\,\sin\frac{(2m+1)\pi}{2L}x\,dx = \frac{1}{2}\int_0^L \left(\cos\frac{n-m}{L}\pi x - \cos\frac{n+m+\pi}{L}\pi x\right)dx = 0.$$

(b) From

$$\int_0^L \sin^2\frac{(2n+1)\pi}{2L}x\,dx = \int_0^L \left(\frac{1}{2} - \frac{1}{2}\cos\frac{(2n+1)\pi}{2L}x\right)dx = \frac{L}{2}$$

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we see that

$$\left\|\sin\frac{(2n+1)\pi}{2L}x\right\| = \sqrt{\frac{L}{2}}\,.$$

13. Since

$$A_0 = \int_{-1}^{0}(-2x)\,dx = 1,$$

$$A_n = \int_{-1}^{0}(-2x)\cos n\pi x\,dx = \frac{2}{n^2\pi^2}[(-1)^n - 1],$$

and

$$B_n = \int_{-1}^{0}(-2x)\sin n\pi x\,dx = \frac{4}{n\pi}(-1)^n$$

for $n = 1, 2, 3, \ldots$ we have

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi^2}[(-1)^n - 1]\cos n\pi x + \frac{4}{n\pi}(-1)^n \sin n\pi x\right).$$

14. Since

$$A_0 = \int_{-1}^{1}(2x^2 - 1)\,dx = -\frac{2}{3},$$

$$A_n = \int_{-1}^{1}(2x^2 - 1)\cos n\pi x\,dx = \frac{8}{n^2\pi^2}(-1)^n,$$

and

$$B_n = \int_{-1}^{1}(2x^2 - 1)\sin n\pi x\,dx = 0$$

for $n = 1, 2, 3, \ldots$ we have

$$f(x) = -\frac{1}{3} + \sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}(-1)^n \cos n\pi x.$$

15. Since

$$A_0 = 2\int_{0}^{1} e^{-x}dx$$

and

$$A_n = 2\int_{-1}^{1} e^{-x}\cos n\pi x\,dx = \frac{2}{1+n^2\pi^2}[(1-(-1)^n e^{-1}]$$

for $n = 1, 2, 3, \ldots$ we have

$$f(x) = 1 - e^{-1} + 2\sum_{n=1}^{\infty}\frac{1-(-1)^n e^{-1}}{1+n^2\pi^2}\cos n\pi x.$$

Since

$$B_n = 2\int_{0}^{1} e^{-x}\sin n\pi x\,dx = \frac{2n\pi}{1+n^2\pi^2}[(1-(-1)^n e^{-1}]$$

for $n = 1, 2, 3, \ldots$ we have

$$f(x) = \sum_{n=1}^{\infty}\frac{2n\pi}{1+n^2\pi^2}[(1-(-1)^n e^{-1}]\sin n\pi x.$$

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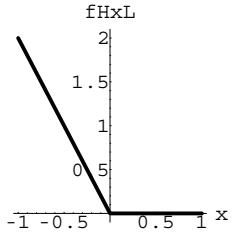
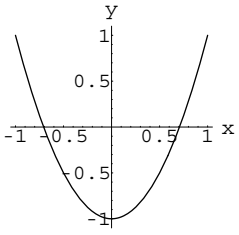
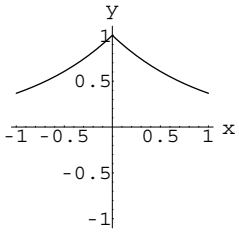
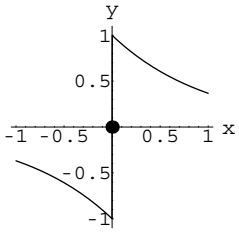
16.









$f(x) = |x| - 1$ $\qquad$ $f(x) = 2x^2 - 1$ $\qquad$ $f(x) = e^{-|x|}$ $\qquad$ $f(x) = \begin{cases} e^{-x}, & 0 < x < 1 \\ 0, & x = 0 \\ -e^{x}, & -1 < x < 0 \end{cases}$

17. For $\lambda > 0$ a general solution of the given differential equation is

$$y = c_1 \cos(3\sqrt{\lambda}\,\ln x) + c_2 \sin(3\sqrt{\lambda}\,\ln x)$$

and

$$y' = -\frac{3c_1\sqrt{\lambda}}{x}\sin(3\sqrt{\lambda}\,\ln x) + \frac{3c_2\sqrt{\lambda}}{x}\cos(3\sqrt{\lambda}\,\ln x).$$

Since $\ln 1 = 0$, the boundary condition $y'(1) = 0$ implies $c_2 = 0$. Therefore

$$y = c_1 \cos(3\sqrt{\lambda}\,\ln x).$$

Using $\ln e = 1$ we find that $y(e) = 0$ implies $c_1 \cos 3\sqrt{\lambda} = 0$ or $3\sqrt{\lambda} = \dfrac{2n-1}{2}\pi$, for $n = 1, 2, 3, \ldots$. The eigenvalues are $\lambda = (2n-1)^2\pi^2/36$ with corresponding eigenfunctions $\cos\left(\dfrac{2n-1}{2}\pi \ln x\right)$ for $n = 1, 2, 3, \ldots$.

18. To obtain the self-adjoint form of the differential equation in Problem 17 we note that an integrating factor is $(1/x^2)e^{\int dx/x} = 1/x$. Thus the weight function is $9/x$ and an orthogonality relation is

$$\int_1^e \frac{9}{x}\cos\left(\frac{2n-1}{2}\pi\ln x\right)\cos\left(\frac{2m-1}{2}\pi\ln x\right)dx = 0,\ m \neq n.$$

19. The boundary condition indicates that we use (15) and (16) of Section 12.6. With $b = 4$ we obtain

$$c_i = \frac{2}{16J_1^2(4\lambda_i)}\int_0^4 xJ_0(\lambda_i x)f(x)\,dx$$

$$= \frac{1}{8J_1^2(4\lambda_i)}\int_0^2 xJ_0(\lambda_i x)\,dx \qquad \boxed{t = \lambda_i x \qquad dt = \lambda_i\,dx}$$

$$= \frac{1}{8J_1^2(4\lambda_i)}\cdot\frac{1}{\lambda_i^2}\int_0^{2\lambda_i} tJ_0(t)\,dt$$

$$= \frac{1}{8J_1^2(4\lambda_i)}\int_0^{2\lambda_i}\frac{d}{dt}[tJ_1(t)]\,dt \qquad \text{[From (4) in 12.6 in the text]}$$

$$= \frac{1}{8J_1^2(4\lambda_i)}tJ_1(t)\Big|_0^{2\lambda_i} = \frac{J_1(2\lambda_i)}{4\lambda_i J_1^2(4\lambda_i)}.$$

Thus

$$f(x) = \frac{1}{4}\sum_{i=1}^{\infty}\frac{J_1(2\lambda_i)}{\lambda_i J_1^2(4\lambda_i)}J_0(\lambda_i x).$$

20. Since $f(x) = x^4$ is a polynomial in x, an expansion of f in polynomials in x must terminate with the term having the same degree as f. Using the fact that $x^4P_1(x)$ and $x^4P_3(x)$ are odd functions, we see immediately

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that $c_1 = c_3 = 0$. Now

$$c_0 = \frac{1}{2}\int_{-1}^{1} x^4 P_0(x)\,dx = \frac{1}{2}\int_{-1}^{1} x^4 dx = \frac{1}{5}$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} x^4 P_2(x)\,dx = \frac{5}{2}\int_{-1}^{1} \frac{1}{2}(3x^6 - x^4)dx = \frac{4}{7}$$

$$c_4 = \frac{9}{2}\int_{-1}^{1} x^4 P_4(x)\,dx = \frac{9}{2}\int_{-1}^{1} \frac{1}{8}(35x^8 - 30x^6 + 3x^4)dx = \frac{8}{35}\,.$$

Thus

$$f(x) = \frac{1}{5}P_0(x) + \frac{4}{7}P_2(x) + \frac{8}{35}P_4(x).$$

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13 Boundary-Value Problems in Rectangular Coordinates

Exercises 13.1

1. If $u = XY$ then

$$u_x = X'Y,$$
$$u_y = XY',$$
$$X'Y = XY',$$

and

$$\frac{X'}{X} = \frac{Y'}{Y} = \pm\lambda^2.$$

Then

$$X' \mp \lambda^2 X = 0 \quad \text{and} \quad Y' \mp \lambda^2 Y = 0$$

so that

$$X = A_1 e^{\pm\lambda^2 x},$$
$$Y = A_2 e^{\pm\lambda^2 y},$$

and

$$u = XY = c_1 e^{c_2(x+y)}.$$

2. If $u = XY$ then

$$u_x = X'Y,$$
$$u_y = XY',$$
$$X'Y = -3XY',$$

and

$$\frac{X'}{-3X} = \frac{Y'}{Y} = \pm\lambda^2.$$

Then

$$X' \pm 3\lambda^2 X = 0 \quad \text{and} \quad Y' \mp \lambda^2 Y = 0$$

so that

$$X = A_1 e^{\mp 3\lambda^2 x},$$
$$Y = A_2 e^{\pm\lambda^2 y},$$

and

$$u = XY = c_1 e^{c_2(y-3x)}.$$

3. If $u = XY$ then

$$u_x = X'Y,$$
$$u_y = XY',$$
$$X'Y = X(Y - Y'),$$

and

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$$\frac{X'}{X} = \frac{Y - Y'}{Y} = \pm\lambda^2.$$

Then

$$X' \mp \lambda^2 X = 0 \quad \text{and} \quad Y' - (1 \mp \lambda^2)Y = 0$$

so that

$$X = A_1 e^{\pm\lambda^2 x},$$
$$Y = A_2 e^{(1\mp\lambda^2)y},$$

and

$$u = XY = c_1 e^{y + c_2(x-y)}.$$

4. If $u = XY$ then

$$u_x = X'Y,$$
$$u_y = XY',$$
$$X'Y = X(Y + Y'),$$

and

$$\frac{X'}{X} = \frac{Y + Y'}{Y} = \pm\lambda^2.$$

Then

$$X' \mp \lambda^2 X = 0 \quad \text{and} \quad Y' - (-1 \pm \lambda^2)Y = 0$$

so that

$$X = A_1 e^{\pm\lambda^2 x},$$
$$Y = A_2 e^{(-1\pm\lambda^2)y},$$

and

$$u = XY = c_1 e^{-y + c_2(x+y)}.$$

5. If $u = XY$ then

$$u_x = X'Y,$$
$$u_y = XY',$$
$$xX'Y = yXY',$$

and

$$\frac{xX'}{X} = \frac{yY'}{Y} = \pm\lambda^2.$$

Then

$$X \mp \frac{1}{x}\lambda^2 X = 0 \quad \text{and} \quad Y' \mp \frac{1}{y}\lambda^2 Y = 0$$

so that

$$X = A_1 x^{\pm\lambda^2},$$
$$Y = A_2 y^{\pm\lambda^2},$$

and

$$u = XY = c_1 (xy)^{c_2}.$$

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6. If $u = XY$ then

$$u_x = X'Y,$$
$$u_y = XY',$$
$$yX'Y = xXY',$$

and

$$\frac{X'}{xX} = \frac{Y'}{-yY} = \pm\lambda^2.$$

Then

$$X \mp \lambda^2 xX = 0 \quad \text{and} \quad Y' \pm \lambda^2 yY = 0$$

so that

$$X = A_1 e^{\pm\lambda^2 x^2/2},$$
$$Y = A_2 e^{\mp\lambda^2 y^2/2},$$

and

$$u = XY = c_1 e^{c_2(x^2 - y^2)}.$$

7. If $u = XY$ then

$$u_{xx} = X''Y, \quad u_{yy} = XY'', \quad u_{yx} = X'Y',$$

and

$$X''Y + X'Y' + XY'' = 0,$$

which is not separable.

8. If $u = XY$ then

$$u_{yx} = X'Y',$$
$$yX'Y' + XY = 0,$$

and

$$\frac{X'}{-X} = \frac{Y}{yY'} = \pm\lambda^2.$$

Then

$$X' \mp \lambda^2 X = 0 \quad \text{and} \quad \pm\lambda^2 yY' - Y = 0$$

so that

$$X = A_1 e^{\mp\lambda^2 x},$$
$$Y = A_2 y^{\pm 1/\lambda^2},$$

and

$$u = XY = c_1 e^{-c_2 x} y^{1/c_2}.$$

9. If $u = XT$ then

$$u_t = XT',$$
$$u_{xx} = X''T,$$
$$kX''T - XT = XT',$$

and we choose

$$\frac{T'}{T} = \frac{kX'' - X}{X} = -1 \pm k\lambda^2$$

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so that

$$T' - (-1 \pm k\lambda^2)T = 0 \quad \text{and} \quad X'' - (\pm\lambda^2)X = 0.$$

For $\lambda^2 > 0$ we obtain

$$X = A_1 \cosh \lambda x + A_2 \sinh \lambda x \quad \text{and} \quad T = A_3 e^{(-1+k\lambda^2)t}$$

so that

$$u = XT = e^{(-1+k\lambda^2)t} \left(c_1 \cosh \lambda x + c_2 \sinh \lambda x\right).$$

For $-\lambda^2 < 0$ we obtain

$$X = A_1 \cos \lambda x + A_2 \sin \lambda x \quad \text{and} \quad T = A_3 e^{(-1-k\lambda^2)t}$$

so that

$$u = XT = e^{(-1-k\lambda^2)t}(c_3 \cos \lambda x + c_4 \sin \lambda x).$$

If $\lambda^2 = 0$ then

$$X'' = 0 \quad \text{and} \quad T' + T = 0,$$

and we obtain

$$X = A_1 x + A_2 \quad \text{and} \quad T = A_3 e^{-t}.$$

In this case

$$u = XT = e^{-t}(c_5 x + c_6)$$

10. If $u = XT$ then

$$u_t = XT',$$
$$u_{xx} = X''T,$$
$$kX''T = XT',$$

and

$$\frac{X''}{X} = \frac{T'}{kT} = \pm\lambda^2$$

so that

$$X'' \mp \lambda^2 X = 0 \quad \text{and} \quad T' \mp \lambda^2 kT = 0.$$

For $\lambda^2 > 0$ we obtain

$$X = A_1 \cos \lambda x + A_2 \sin \lambda x \quad \text{and} \quad T = A_3 e^{\lambda^2 kt},$$

so that

$$u = XT = e^{-c_3^2 kt}(c_1 \cos c_3 x + c_2 \sin c_3 x).$$

For $-\lambda^2 < 0$ we obtain

$$X = A_1 e^{-\lambda x} + A_2 e^{\lambda x},$$
$$T = A_3 e^{\lambda^2 kt},$$

and

$$u = XT = e^{c_3^2 kt}\left(c_1 e^{-c_3 x} + c_2 e^{c_3 x}\right).$$

For $\lambda^2 = 0$ we obtain

$$T = A_3, \qquad X = A_1 x + A_2, \quad \text{and} \quad u = XT = c_1 x + c_2.$$

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11. If $u = XT$ then

$$u_{xx} = X''T,$$
$$u_{tt} = XT'',$$
$$a^2X''T = XT'',$$

and

$$\frac{X''}{X} = \frac{T''}{a^2T} = \pm\lambda^2$$

so that

$$X'' \mp \lambda^2 X = 0 \quad \text{and} \quad T'' \mp a^2\lambda^2 T = 0.$$

For $\lambda^2 > 0$ we obtain

$$X = A_1 \cos\lambda x + A_2 \sin\lambda x,$$
$$T = A_3 \cos a\lambda t + A_4 \sin a\lambda t,$$

and

$$u = XT = (A_1 \cos\lambda x + A_2 \sin\lambda x)(A_3 \cos a\lambda t + A_4 \sin a\lambda t).$$

For $-\lambda^2 < 0$ we obtain

$$X = A_1 e^{\lambda x} + A_2 e^{-\lambda x},$$
$$T = A_3 e^{a\lambda t} + A_4 e^{-a\lambda t},$$

and

$$u = XT = \left(A_1 e^{\lambda x} + A_2 e^{-\lambda x}\right)\left(A_3 e^{a\lambda t} + A_4 e^{-a\lambda t}\right).$$

For $\lambda^2 = 0$ we obtain

$$X = A_1 x + A_2,$$
$$T = A_3 t + A_4,$$

and

$$u = XT = (A_1 x + A_2)(A_3 t + A_4).$$

12. If $u = XT$ then

$$u_t = XT',$$
$$u_{tt} = XT'',$$
$$u_{xx} = X''T,$$
$$a^2X''T = XT'' + 2kXT',$$

and

$$\frac{X''}{X} = \frac{T'' + 2kT'}{a^2T} = \pm\lambda^2$$

so that

$$X'' \mp \lambda^2 X = 0 \quad \text{and} \quad T'' + 2kT' \mp a^2\lambda^2 T = 0.$$

For $\lambda^2 > 0$ we obtain

$$X = A_1 e^{\lambda x} + A_2 e^{-\lambda x},$$
$$T = A_3 e^{(-k+\sqrt{k^2+a^2\lambda^2})t} + A_4 e^{(-k-\sqrt{k^2+a^2\lambda^2})t},$$

and

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$$u = XT = \left(A_1 e^{\lambda x} + A_2 e^{-\lambda x}\right)\left(A_3 e^{(-k+\sqrt{k^2+a^2\lambda^2})t} + A_4 e^{(-k-\sqrt{k^2+a^2\lambda^2})t}\right).$$

For $-\lambda^2 < 0$ we obtain

$$X = A_1 \cos \lambda x + A_2 \sin \lambda x.$$

If $k^2 - a^2\lambda^2 > 0$ then

$$T = A_3 e^{(-k+\sqrt{k^2-a^2\lambda^2})t} + A_4 e^{(-k-\sqrt{k^2-a^2\lambda^2})t}.$$

If $k^2 - a^2\lambda^2 < 0$ then

$$T = e^{-kt}\left(A_3 \cos \sqrt{a^2\lambda^2 - k^2}\, t + A_4 \sin \sqrt{a^2\lambda^2 - k^2}\, t\right).$$

If $k^2 - a^2\lambda^2 = 0$ then

$$T = A_3 e^{-kt} + A_4 t e^{-kt}$$

so that

$$\begin{aligned} u = XT &= (A_1 \cos \lambda x + A_2 \sin \lambda x)\left(A_3 e^{(-k+\sqrt{k^2-a^2\lambda^2})t} + A_4 e^{(-k-\sqrt{k^2-a^2\lambda^2})t}\right) \\ &= (A_1 \cos \lambda x + A_2 \sin \lambda x) e^{-kt}\left(A_3 \cos \sqrt{a^2\lambda^2 - k^2}\, t + A_4 \sin \sqrt{a^2\lambda^2 - k^2}\, t\right) \\ &= \left(A_1 \cos \frac{k}{a}x + A_2 \sin \frac{k}{a}x\right)\left(A_3 e^{-kt} + A_4 t e^{-kt}\right). \end{aligned}$$

For $\lambda^2 = 0$ we obtain

$$X A_1 x + A_2,$$
$$T = A_3 + A_4 e^{-2kt},$$

and

$$u = XT = (A_1 x + A_2)(A_3 + A_4 e^{-2kt}).$$

13. If $u = XY$ then

$$u_{xx} = X''Y,$$
$$u_{yy} = XY'',$$
$$X''Y + xY'' = 0,$$

and

$$\frac{X''}{-X} = \frac{Y''}{Y} = \pm\lambda^2$$

so that

$$X'' \pm \lambda^2 X = 0 \quad \text{and} \quad Y'' \mp \lambda^2 Y = 0.$$

For $\lambda^2 > 0$ we obtain

$$X = A_1 \cos \lambda x + A_2 \sin \lambda x,$$
$$Y = A_3 e^{\lambda y} + A_4 e^{-\lambda y},$$

and

$$u = XY = (A_1 \cos \lambda x + A_2 \sin \lambda x)(A_3 e^{\lambda y} + A_4 e^{-\lambda y}).$$

For $-\lambda^2 < 0$ we obtain

$$X = A_1 e^{\lambda x} + A_2 e^{-\lambda x},$$
$$Y = A_3 \cos \lambda y + A_4 \sin \lambda y,$$

and

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$$u = XY = (A_1 e^{\lambda x} + A_2 e^{-\lambda x})(A_3 \cos \lambda y + A_4 \sin \lambda y).$$

For $\lambda^2 = 0$ we obtain

$$u = XY = (A_1 x + A_2)(A_3 y + A_4).$$

14. If $u = XY$ then

$$u_{xx} = X''Y,$$
$$u_{yy} = XY'',$$
$$x^2 X''Y + xY'' = 0,$$

and

$$\frac{x^2 X''}{-X} = \frac{Y''}{Y} = \pm\lambda^2$$

so that

$$x^2 X'' \pm \lambda^2 X = 0 \quad \text{and} \quad Y'' \mp \lambda^2 Y = 0.$$

For $\lambda^2 > 0$ we obtain

$$Y = A_3 e^{\lambda y} + A_4 e^{-\lambda y}.$$

If $1 - 4\lambda^2 > 0$ then

$$X = A_1 x^{(1/2+\sqrt{1-4\lambda^2}/2)} + A_2 x^{(1/2-\sqrt{1-4\lambda^2}/2)}.$$

If $1 - 4\lambda^2 < 0$ then

$$X = x^{1/2}\left(A_1 \cos \frac{1}{2}\sqrt{4\lambda^2 - 1}\, \ln x + A_2 \sin \frac{1}{2}\sqrt{4\lambda^2 - 1}\, \ln x\right).$$

If $1 - 4\lambda^2 = 0$ then

$$X = A_1 x^{1/2} + A_2 x^{1/2} \ln x$$

so that

$$\begin{aligned} u = XY &= \left(A_1 x^{(1/2+\sqrt{1-4\lambda^2}/2)} + A_2 x^{(1/2-\sqrt{1-4\lambda^2}/2)}\right)\left(A_3 e^{\lambda y} + A_4 e^{-\lambda y}\right) \\ &= x^{1/2}\left(A_1 \cos \frac{1}{2}\sqrt{4\lambda^2 - 1}\, \ln x + A_2 \sin \frac{1}{2}\sqrt{4\lambda^2 - 1}\, \ln x\right)\left(A_3 e^{y/2} + A_4 e^{-y/2}\right) \\ &= \left(A_1 x^{1/2} + A_2 x^{1/2} \ln x\right)\left(A_3 e^{y/2} + A_4 e^{-y/2}\right). \end{aligned}$$

For $-\lambda^2 < 0$ we obtain

$$X = A_1 e^{(1/2+\sqrt{1+4\lambda^2}/2)x} + A_2 e^{(1/2-\sqrt{1+4\lambda^2}/2)x},$$
$$Y = A_3 \cos \lambda y + A_4 \sin \lambda y,$$

and

$$u = XY = \left(A_1 e^{(1/2+\sqrt{1+4\lambda^2}/2)x} + A_2 e^{(1/2-\sqrt{1+4\lambda^2}/2)x}\right)(A_3 \cos \lambda y + A_4 \sin \lambda y).$$

For $\lambda^2 = 0$ we obtain

$$X = A_1 x + A_2,$$
$$Y = A_3 y + A_4,$$

and

$$u = XY = (A_1 x + A_2)(A_3 y + A_4).$$

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15. If $u = XY$ then

$$u_{xx} = X''Y,$$
$$u_{yy} = XY'',$$
$$X''Y + XY'' = XY,$$

and

$$\frac{X''}{X} = \frac{Y - Y''}{Y} = \pm\lambda^2$$

so that

$$X'' \mp \lambda^2 X = 0 \quad \text{and} \quad Y'' + (\pm\lambda^2 - 1)Y = 0.$$

For $\lambda^2 > 0$ we obtain

$$X = A_1 e^{\lambda x} + A_2 e^{-\lambda x}.$$

If $\lambda^2 - 1 > 0$ then

$$Y = A_3 \cos\sqrt{\lambda^2 - 1}\, y + A_4 \sin\sqrt{\lambda^2 - 1}\, y.$$

If $\lambda^2 - 1 < 0$ then

$$Y = A_3 e^{\sqrt{1-\lambda^2}\, y} + A_4 e^{-\sqrt{1-\lambda^2}\, y}.$$

If $\lambda^2 - 1 = 0$ then $Y = A_3 y + A_4$ so that

$$\begin{aligned} u = XY &= \left(A_1 e^{\lambda x} + A_2 e^{-\lambda x}\right)\left(A_3 \cos\sqrt{\lambda^2 - 1}\, y + A_4 \sin\sqrt{\lambda^2 - 1}\, y\right), \\ &= \left(A_1 e^{\lambda x} + A_2 e^{-\lambda x}\right)\left(A_3 e^{\sqrt{1-\lambda^2}\, y} + A_4 e^{-\sqrt{1-\lambda^2}\, y}\right) \\ &= (A_1 e^x + A_2 e^{-x})(A_3 y + A_4). \end{aligned}$$

For $-\lambda^2 < 0$ we obtain

$$X = A_1 \cos\lambda x + A_2 \sin\lambda x,$$
$$Y = A_3 e^{\sqrt{1+\lambda^2}\, y} + A_4 e^{-\sqrt{1+\lambda^2}\, y},$$

and

$$u = XY = (A_1 \cos\lambda x + A_2 \sin\lambda x)\left(A_3 e^{\sqrt{1+\lambda^2}\, y} + A_4 e^{-\sqrt{1+\lambda^2}\, y}\right).$$

For $\lambda^2 = 0$ we obtain

$$X = A_1 x + A_2,$$
$$Y = A_3 e^y + A_4 e^{-y},$$

and

$$u = XY = (A_1 x + A_2)(A_3 e^y + A_4 e^{-y}).$$

16. If $u = XT$ then

$$u_{tt} = XT'', \qquad u_{xx} = X''T, \quad \text{and} \quad a^2 X''T - g = XT'',$$

which is not separable.

17. Identifying $A = B = C = 1$, we compute $B^2 - 4AC = -3 < 0$. The equation is elliptic.

18. Identifying $A = 3$, $B = 5$, and $C = 1$, we compute $B^2 - 4AC = 13 > 0$. The equation is hyperbolic.

19. Identifying $A = 1$, $B = 6$, and $C = 9$, we compute $B^2 - 4AC = 0$. The equation is parabolic.

20. Identifying $A = 1$, $B = -1$, and $C = -3$, we compute $B^2 - 4AC = 13 > 0$. The equation is hyperbolic.

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21. Identifying $A = 1$, $B = -9$, and $C = 0$, we compute $B^2 - 4AC = 81 > 0$. The equation is hyperbolic.

22. Identifying $A = 0$, $B = 1$, and $C = 0$, we compute $B^2 - 4AC = 1 > 0$. The equation is hyperbolic.

23. Identifying $A = 1$, $B = 2$, and $C = 1$, we compute $B^2 - 4AC = 0$. The equation is parabolic.

24. Identifying $A = 1$, $B = 0$, and $C = 1$, we compute $B^2 - 4AC = -4 < 0$. The equation is elliptic.

25. Identifying $A = a^2$, $B = 0$, and $C = -1$, we compute $B^2 - 4AC = 4a^2 > 0$. The equation is hyperbolic.

26. Identifying $A = k > 0$, $B = 0$, and $C = 0$, we compute $B^2 - 4AC = -4k < 0$. The equation is elliptic.

27. If $u = RT$ then

$$u_r = R'T,$$
$$u_{rr} = R''T,$$
$$u_t = RT',$$
$$RT' = k\left(R''T + \frac{1}{r}R'T\right),$$

and

$$\frac{r^2R'' + rR'}{r^2R} = \frac{T'}{kT} = \pm\lambda^2.$$

If we use $-\lambda^2 < 0$ then

$$r^2R'' + rR' + \lambda^2 r^2 R = 0 \quad \text{and} \quad T'' \mp \lambda^2 kT = 0$$

so that

$$R = A_2 J_0(\lambda r) + A_3 Y_0(\lambda r),$$
$$T = A_1 e^{-k\lambda^2 t},$$

and

$$u = RT = e^{-k\lambda^2 t}[c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)]$$

28. If $u = R\Theta$ then

$$u_r = R'\Theta,$$
$$u_{rr} = R''\Theta,$$
$$u_t = R\Theta',$$
$$u_{tt} = R\Theta'',$$
$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0,$$

and

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2.$$

Then

$$r^2R'' + rR' - \lambda^2 R = 0 \quad \text{and} \quad \Theta'' + \lambda^2\Theta = 0$$

so that

$$R = c_3 r^\lambda + c_4 r^{-\lambda}$$
$$\Theta\theta = c_1 \cos\lambda\theta + c_2 \sin\lambda\theta,$$

and

$$u = R\Theta = (c_1 \cos\lambda\theta + c_2 \sin\lambda\theta)(c_3 r^\lambda + c_4 r^{-\lambda}).$$

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29. For $u = A_1 e^{\lambda^2 y}\cosh 2\lambda x + B_1 e^{\lambda^2 y}\sinh 2\lambda x$ we compute

$$\frac{\partial^2 u}{\partial x^2} = 4\lambda^2 A_1 e^{\lambda^2 y}\cosh 2\lambda x + 4\lambda^2 B_1 e^{\lambda^2 y}\sinh 2\lambda x$$

and

$$\frac{\partial u}{\partial y} = \lambda^2 A_1 e^{\lambda^2 y}\cosh 2\lambda x + \lambda^2 B_1 e^{\lambda^2 y}\sinh 2\lambda x.$$

Then $\partial^2 u/\partial x^2 = 4\partial u/\partial y$.

For $u = A_2 e^{-\lambda^2 y}\cos 2\lambda x + B_2 e^{-\lambda^2 y}\sin 2\lambda x$ we compute

$$\frac{\partial^2 u}{\partial x^2} = -4\lambda^2 A_2 e^{-\lambda^2 y}\cos 2\lambda x - 4\lambda^2 B_2 e^{-\lambda^2 y}\sin 2\lambda x$$

and

$$\frac{\partial u}{\partial y} = -\lambda^2 A_2\cos 2\lambda x - \lambda^2 B_2\sin 2\lambda x.$$

Then $\partial^2 u/\partial x^2 = 4\partial u/\partial y$.

For $u = A_3 x + B_3$ we compute $\partial^2 u/\partial x^2 = \partial u/\partial y = 0$. Then $\partial^2 u/\partial x^2 = 4\partial u/\partial y$.

30. We identify $A = xy + 1$, $B = x + 2y$, and $C = 1$. Then $B^2 - 4AC = x^2 + 4y^2 - 4$. The equation $x^2 + 4y^2 = 4$ defines an ellipse. The partial differential equation is hyperbolic outside the ellipse, parabolic on the ellipse, and elliptic inside the ellipse.

Exercises 13.2

1. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0$

$u(0,t) = 0, \quad \left.\dfrac{\partial u}{\partial x}\right|_{x=L} = 0, \quad t > 0$

$u(x,0) = f(x), \quad 0 < x < L$

2. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0$

$u(0,t) = u_0, \quad u(L,t) = u_1, \quad t > 0$

$u(x,0) = 0, \quad 0 < x < L$

3. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0$

$u(0,t) = 100, \quad \left.\dfrac{\partial u}{\partial x}\right|_{x=L} = -hu(L,t), \quad t > 0$

$u(x,0) = f(x), \quad 0 < x < L$

4. $k\dfrac{\partial^2 u}{\partial x^2} + h(u - 50) = \dfrac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0$

$\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\dfrac{\partial u}{\partial x}\right|_{x=L} \quad t > 0$

$u(x,0) = 100, \quad 0 < x < L$

5. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \quad 0 < x < L,\ t > 0$

$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$

$u(x,0) = x(L - x), \quad \left.\dfrac{\partial u}{\partial x}\right|_{t=0} = 0, \quad 0 < x < L$

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6. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \quad 0 < x < L,\ t > 0$

$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$

$u(x,0) = 0, \quad \left.\dfrac{\partial u}{\partial x}\right|_{t=0} = \sin\dfrac{\pi x}{L}, \quad 0 < x < L$

7. $a^2\dfrac{\partial^2 u}{\partial x^2} - 2\beta\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial t^2}, \quad 0 < x < L,\ t > 0$

$u(0,t) = 0, \quad u(L,t) = \sin \pi t, \quad t > 0$

$u(x,0) = f(x), \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < L$

8. $a^2\dfrac{\partial^2 u}{\partial x^2} + Ax = \dfrac{\partial^2 u}{\partial t^2}, \quad 0 < x < L,\ t > 0,\ A$ a constant

$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$

$u(x,0) = 0, \quad \left.\dfrac{\partial u}{\partial x}\right|_{t=0} = 0, \quad 0 < x < L$

9. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 4,\ 0 < y < 2$

$\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, \quad u(4,y) = f(y), \quad 0 < y < 2$

$\left.\dfrac{\partial u}{\partial y}\right|_{y=0} = 0, \quad u(x,2) = 0, \quad 0 < x < 4$

10. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi,\ y > 0$

$u(0,y) = e^{-y}, \quad u(\pi,y) = \begin{cases} 100, & 0 < y \le 1 \\ 0, & y > 1 \end{cases}$

$u(x,0) = f(x), \quad 0 < x < \pi$

Exercises 13.3

1. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(L) = 0,$$

and

$$T' + k\lambda^2 T = 0.$$

Then

$$X = c_1 \sin\frac{n\pi}{L}x \quad \text{and} \quad T = c_2 e^{-\frac{kn^2\pi^2}{L^2}t}$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n e^{-\frac{kn^2\pi^2}{L^2}t} \sin\frac{n\pi}{L}x.$$

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Imposing

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x$$

gives

$$A_n = \frac{2}{L} \int_0^{L/2} \sin \frac{n\pi}{L} x \, dx = \frac{2}{n\pi} \left(1 - \cos \frac{n\pi}{2}\right)$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos \frac{n\pi}{2}}{n} \, e^{-\frac{kn^2\pi^2}{L^2}t} \sin \frac{n\pi}{L} x.$$

2. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(L) = 0,$$

and

$$T' + k\lambda^2 T = 0.$$

Then

$$X = c_1 \sin \frac{n\pi}{L} x \quad \text{and} \quad T = c_2 e^{-\frac{kn^2\pi^2}{L^2}t}$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n e^{-\frac{kn^2\pi^2}{L^2}t} \sin \frac{n\pi}{L} x.$$

Imposing

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x$$

gives

$$A_n = \frac{2}{L} \int_0^{L} x(L - x) \sin \frac{n\pi}{L} x \, dx = \frac{4L^2}{n^3\pi^3}[1 - (-1)^n]$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x,t) = \frac{4L^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \, e^{-\frac{kn^2\pi^2}{L^2}t} \sin \frac{n\pi}{L} x.$$

3. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X'(0) = 0,$$
$$X'(L) = 0,$$

and

$$T' + k\lambda^2 T = 0.$$

Then

$$X = c_1 \cos \frac{n\pi}{L} x \quad \text{and} \quad T = c_2 e^{-\frac{kn^2\pi^2}{L^2}t}$$

for $n = 0, 1, 2, \ldots$ so that

$$u = \sum_{n=0}^{\infty} A_n e^{-\frac{kn^2\pi^2}{L^2}t} \cos \frac{n\pi}{L} x.$$

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Imposing

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi}{L}x$$

gives

$$u(x,t) = \frac{1}{L}\int_0^L f(x)\,dx + \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx\right)e^{-\frac{kn^2\pi^2}{L^2}t}\cos\frac{n\pi}{L}x.$$

4. If $L = 2$ and $f(x)$ is x for $0 < x < 1$ and $f(x)$ is 0 for $1 < x < 2$ then

$$u(x,t) = \frac{1}{4} + 4\sum_{n=1}^{\infty}\left[\frac{1}{2n\pi}\sin\frac{n\pi}{2} + \frac{1}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\right]e^{-\frac{kn^2\pi^2}{4}t}\cos\frac{n\pi}{2}x.$$

5. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X'(0) = 0,$$
$$X'(L) = 0,$$

and

$$T' + (h + k\lambda^2)T = 0.$$

Then

$$X = c_1\cos\frac{n\pi}{L}x \quad \text{and} \quad T = c_2 e^{-\left(h+\frac{kn^2\pi^2}{L^2}\right)t}$$

for $n = 0, 1, 2, \ldots$ so that

$$u = \sum_{n=0}^{\infty} A_n e^{-\left(h+\frac{kn^2\pi^2}{L^2}\right)t}\cos\frac{n\pi}{L}x.$$

Imposing

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi}{L}x$$

gives

$$u(x,t) = \frac{e^{-ht}}{L}\int_0^L f(x)\,dx + \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx\right)e^{-\left(h+\frac{kn^2\pi^2}{L^2}\right)t}\cos\frac{n\pi}{L}x.$$

6. In Problem 5 we instead find that $X(0) = 0$ and $X(L) = 0$ so that

$$X = c_1\sin\frac{n\pi}{L}x$$

and

$$u = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx\right)e^{-\left(h+\frac{kn^2\pi^2}{L^2}\right)t}\sin\frac{n\pi}{L}x.$$

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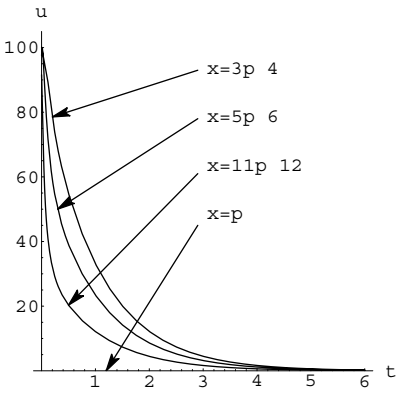
7.



8. (a) The solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k(n^2\pi^2/100^2)t} \sin \frac{n\pi}{100} x,$$

where

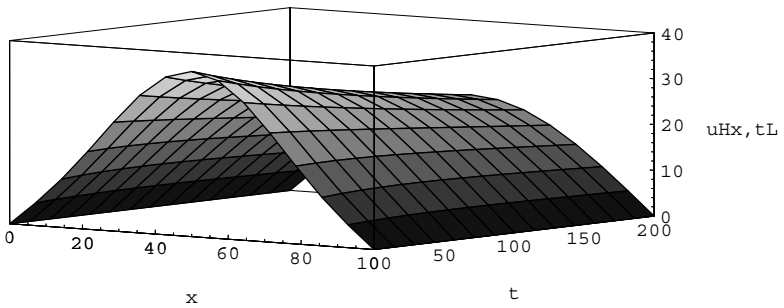
$$A_n = \frac{2}{100}\left[\int_0^{50} 0.8x \sin \frac{n\pi}{100} x\, dx + \int_{50}^{100} 0.8(100-x) \sin \frac{n\pi}{100} x\, dx\right] = \frac{320}{n^2\pi^2} \sin \frac{n\pi}{x}.$$

Thus,

$$u(x,t) = \frac{320}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\sin \frac{n\pi}{2}\right) e^{-k(n^2\pi^2/100^2)t} \sin \frac{n\pi}{100} x.$$

(b) Since $A_n = 0$ for n even, the first five nonzero terms correspond to $n = 1, 3, 5, 7, 9$. In this case $\sin(n\pi/2) = \sin(2p-1)/2 = (-1)^{p+1}$ for $p = 1, 2, 3, 4, 5$, and

$$u(x,t) = \frac{320}{\pi^2} \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{(2p-1)^2} e^{(-1.6352(2p-1)^2\pi^2/100^2)t} \sin \frac{(2p-1)\pi}{100} x.$$



Exercises 13.4

1. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(L) = 0,$$

and

$$T'' + \lambda^2 a^2 T = 0.$$

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Then

$$X = c_1 \sin \frac{n\pi}{L}x \quad \text{and} \quad T = c_2 \cos \frac{n\pi a}{L}t + c_3 \sin \frac{n\pi a}{L}t$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi a}{L}t + B_n \sin \frac{n\pi a}{L}t \right) \sin \frac{n\pi}{L}x.$$

Imposing

$$u(x, 0) = \frac{1}{4}x(L - x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L}x$$

and

$$u_t(x, 0) = 0 = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{L} \sin \frac{n\pi}{L}x$$

gives

$$A_n = \frac{L}{n^3\pi^3}[1 - (-1)^n] \quad \text{and} \quad B_n = 0$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x, t) = \frac{L^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \cos \frac{n\pi a}{L}t \sin \frac{n\pi}{L}x.$$

2. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(L) = 0,$$

and

$$T'' + \lambda^2 a^2 T = 0.$$

Then

$$X = c_1 \sin \frac{n\pi}{L}x \quad \text{and} \quad T = c_2 \cos \frac{n\pi a}{L}t + c_3 \sin \frac{n\pi a}{L}t$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi a}{L}t + B_n \sin \frac{n\pi a}{L}t \right) \sin \frac{n\pi}{L}x.$$

Imposing

$$u(x, 0) = 0 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L}x$$

and

$$u_t(x, 0) = x(L - x) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{L} \sin \frac{n\pi}{L}x$$

gives

$$A_n = 0 \quad \text{and} \quad B_n \frac{n\pi a}{L} = \frac{4L^2}{n^3\pi^3}[1 - (-1)^n]$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x, t) = \frac{4L^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^4} \sin \frac{n\pi a}{L}t \sin \frac{n\pi}{L}x.$$

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3. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(L) = 0,$$

and

$$T'' + \lambda^2 a^2 T = 0.$$

Then

$$X = c_1 \sin \frac{n\pi}{L}x \quad \text{and} \quad T = c_2 \cos \frac{n\pi a}{L}t + c_3 \sin \frac{n\pi a}{L}t$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi a}{L}t + B_n \sin \frac{n\pi a}{L}t \right) \sin \frac{n\pi}{L}x.$$

Imposing

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L}x$$

gives

$$A_n = \frac{2}{L} \left(\int_0^{L/3} \frac{3}{L}x \sin \frac{n\pi}{L}x \, dx + \int_{L/3}^{2L/3} \sin \frac{n\pi}{L}x \, dx + \int_{2L/3}^{L} \left(3 - \frac{3}{L}x \right) \sin \frac{n\pi}{L}x \, dx \right)$$

so that

$$A_1 = \frac{6\sqrt{3}}{\pi^2},$$
$$A_2 = A_3 = A_4 = 0,$$
$$A_5 = -\frac{6\sqrt{3}}{5^2\pi^2},$$
$$A_6 = 0,$$
$$a_7 = \frac{6\sqrt{3}}{7^2\pi^2} \ldots .$$

Imposing

$$u_t(x, 0) = 0 = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{L} \sin \frac{n\pi}{L}x$$

gives $B_n = 0$ for $n = 1, 2, 3, \ldots$ so that

$$u(x, t) = \frac{6\sqrt{3}}{\pi^2} \left(\cos \frac{\pi a}{L}t \sin \frac{\pi}{L}x - \frac{1}{5^2} \cos \frac{5\pi a}{L}t \sin \frac{5\pi}{L}x + \frac{1}{7^2} \cos \frac{7\pi a}{L}t \sin \frac{7\pi}{L}x - \cdots \right).$$

4. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(\pi) = 0,$$

and

$$T'' + \lambda^2 a^2 T = 0.$$

Then

$$X = c_1 \sin nx \quad \text{and} \quad T = c_2 \cos nat + c_3 \sin nat$$

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for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} (A_n \cos nat + B_n \sin nat) \sin nx.$$

Imposing

$$u(x, 0) = \frac{1}{6}x(\pi^2 - x^2) = \sum_{n=1}^{\infty} A_n \sin nx \quad \text{and} \quad u_t(x, 0) = 0$$

gives

$$B_n = 0 \quad \text{and} \quad A_n = \frac{2}{n^3}(-1)^{n+1}$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x, t) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \cos nat \, \sin nx.$$

5. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(\pi) = 0,$$

and

$$T'' + \lambda^2 a^2 T = 0.$$

Then

$$X = c_1 \sin nx \quad \text{and} \quad T = c_2 \cos nat + c_3 \sin nat$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} (A_n \cos nat + B_n \sin nat) \sin nx.$$

Imposing

$$u(x, 0) = 0 = \sum_{n=1}^{\infty} A_n \sin nx \quad \text{and} \quad u_t(x, 0) = \sin x = \sum_{n=1}^{\infty} B_n na \sin nx$$

gives

$$A_n = 0, \quad B_1 = \frac{1}{a^2}, \quad \text{and} \quad B_n = 0$$

for $n = 2, 3, 4, \ldots$ so that

$$u(x, t) = \frac{1}{a} \sin at \, \sin x.$$

6. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(1) = 0,$$

and

$$T'' + \lambda^2 a^2 T = 0.$$

Then

$$X = c_1 \sin n\pi x \quad \text{and} \quad T = c_2 \cos n\pi at + c_3 \sin n\pi at$$

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for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} (A_n \cos n\pi at + B_n \sin n\pi at) \sin n\pi x.$$

Imposing

$$u(x, 0) = 0.01 \sin 3\pi x = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

and

$$u_t(x, 0) = 0 = \sum_{n=1}^{\infty} B_n n\pi a \sin n\pi x$$

gives $B_n = 0$ for $n = 1, 2, 3, \ldots$, $A_3 = 0.01$, and $A_n = 0$ for $n = 1, 2, 4, 5, 6, \ldots$ so that

$$u(x, t) = 0.01 \sin 3\pi x \cos 3\pi at.$$

7. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(L) = 0,$$

and

$$T'' + \lambda^2 a^2 T = 0.$$

Then

$$X = c_1 \sin \frac{n\pi}{L} x \quad \text{and} \quad T = c_2 \cos \frac{n\pi a}{L} t + c_3 \sin \frac{n\pi a}{L} t$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi a}{L} t + B_n \sin \frac{n\pi a}{L} t \right) \sin \frac{n\pi}{L} x.$$

Imposing

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x$$

and

$$u_t(x, 0) = 0 = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{L} \sin \frac{n\pi}{L} x$$

gives

$$B_n = 0 \quad \text{and} \quad A_n = \frac{8h}{n^2 \pi^2} \sin \frac{n\pi}{2}$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x, t) = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \cos \frac{n\pi a}{L} t \sin \frac{n\pi}{L} x.$$

8. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X'(0) = 0,$$
$$X'(L) = 0,$$

and

$$T'' + a^2 \lambda^2 T = 0.$$

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Then

$$X = c_1 \cos \frac{n\pi}{L}x \quad \text{and} \quad T = c_2 \cos \frac{n\pi a}{L}t$$

for $n = 1, 2, 3, \ldots$. Since $\lambda = 0$ is also an eigenvalue with eigenfunction $X(x) = 1$ we have

$$u = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi a}{L}t \cos \frac{n\pi}{L}x.$$

Imposing

$$u(x, 0) = x = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi}{L}x$$

gives

$$A_0 = \frac{1}{L}\int_0^L x\, dx = \frac{L}{2}$$

and

$$A_n = \frac{2}{L}\int_0^L x \cos \frac{n\pi}{L}x\, dx = \frac{2L}{n^2\pi^2}[(-1)^n - 1]$$

for $n = 1, 2, 3, \ldots$, so that

$$u(x, t) = \frac{L}{2} + \frac{2L}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos \frac{n\pi a}{L}t \cos \frac{n\pi}{L}x.$$

9. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(\pi) = 0,$$

and

$$T'' + 2\beta T' + \lambda^2 T = 0.$$

Then

$$X = c_1 \sin nx \quad \text{and} \quad T = e^{-\beta t}\left(c_2 \cos \sqrt{n^2 - \beta^2}\, t + c_3 \sin \sqrt{n^2 - \beta^2}\, t\right)$$

so that

$$u = \sum_{n=1}^{\infty} e^{-\beta t}\left(A_n \cos \sqrt{n^2 - \beta^2}\, t + B_n \sin \sqrt{n^2 - \beta^2}\, t\right) \sin nx.$$

Imposing

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin nx$$

and

$$u_t(x, 0) = 0 = \sum_{n=1}^{\infty}\left(B_n\sqrt{n^2 - \beta^2} - \beta A_n\right) \sin nx$$

gives

$$u(x, t) = e^{-\beta t}\sum_{n=1}^{\infty} A_n \left(\cos \sqrt{n^2 - \beta^2}\, t + \frac{\beta}{\sqrt{n^2 - \beta^2}} \sin \sqrt{n^2 - \beta^2}\, t\right) \sin nx,$$

where

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx.$$

10. Using $u = XT$ and $-\lambda^2$ as a separation constant leads to $X'' + \lambda^2 X = 0$, $X(0) = 0$, $X(\pi) = 0$ and

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$T'' + (1 + \lambda^2)T = 0$, $T'(0) = 0$. Then $X = c_2 \sin nx$ and $T = c_3 \cos\sqrt{n^2 + 1}\, t$ for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n1}^{\infty} B_n \cos\sqrt{n^2 + 1}\, t \sin nx.$$

Imposing $u(x, 0) = \sum_{n=1}^{\infty} B_n \sin nx$ gives

$$B_n = \frac{2}{\pi}\int_0^{\pi/2} x \sin nx\, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

$$= \begin{cases} 0, & n \text{ even} \\ \frac{4}{\pi n^2}(-1)^{(n+3)/2}, & n = 2k - 1,\ k = 1, 2, 3, \ldots. \end{cases}$$

Thus with $n = 2k - 1$,

$$u(x, t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n^2} \cos\sqrt{n^2 + 1}\, t \sin nx = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \cos\sqrt{(2k-1)^2 + 1}\, t \sin(2k-1)x.$$

11. Separating variables in the partial differential equation gives

$$\frac{X^{(4)}}{X} = -\frac{T''}{a^2 T} = \lambda^4$$

so that

$$X^{(4)} - \lambda^4 X = 0$$
$$T'' + a^2\lambda^4 T = 0$$

and

$$X = c_1 \cosh\lambda x + c_2 \sinh\lambda x + c_3 \cos\lambda x + c_4 \sin\lambda x$$
$$T = c_5 \cos a\lambda^2 t + c_6 \sin a\lambda^2 t.$$

The boundary conditions translate into $X(0) = X(L) = 0$ and $X''(0) = X''(L) = 0$. From $X(0) = X''(0) = 0$ we find $c_1 = c_3 = 0$. From

$$X(L) = c_2 \sinh\lambda L + c_4 \sin\lambda L = 0$$
$$X''(L) = \lambda^2 c_2 \sinh\lambda L - \lambda^2 c_4 \sin\lambda L = 0$$

we see by subtraction that $c_4 \sin\lambda L = 0$. This equation yields the eigenvalues $\lambda = n\pi L$ for $n = 1, 2, 3, \ldots$. The corresponding eigenfunctions are

$$X = c_4 \sin\frac{n\pi}{L}x.$$

Thus

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n^2\pi^2}{L^2}at + B_n \sin\frac{n^2\pi^2}{L^2}at\right)\sin\frac{n\pi}{L}x.$$

From

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x$$

we obtain

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\, dx.$$

From

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left(-A_n\frac{n^2\pi^2 a}{L^2}\sin\frac{n^2\pi^2}{L^2}at + B_n\frac{n^2\pi^2 a}{L^2}\cos\frac{n^2\pi^2}{L^2}at\right)\sin\frac{n\pi}{L}x$$

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and

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = \sum_{n=1}^{\infty} B_n \frac{n^2\pi^2 a}{L^2} \sin\frac{n\pi}{L}x$$

we obtain

$$B_n \frac{n^2\pi^2 a}{L^2} = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

and

$$B_n = \frac{2L}{n^2\pi^2 a}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx.$$

12. (a) Using $$X = c_1\cosh\lambda x + c_2\sinh\lambda x + c_3\cos\lambda x + c_4\sin\lambda x$$

and $X(0) = 0$, $X'(0) = 0$ we find, in turn, $c_3 = -c_1$ and $c_4 = -c_2$. The conditions $X(L) = 0$ and $X'(L) = 0$ then yield the system of equations for c_1 and c_2:

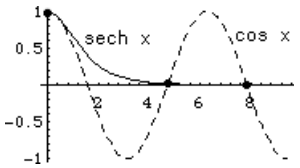
$$c_1(\cosh\lambda L - \cos\lambda L) + c_2(\sinh\lambda L - \sin\lambda L) = 0$$
$$c_1(\lambda\sinh\lambda L + \lambda\sin\lambda L) + c_2(\lambda\cosh\lambda L - \lambda\cos\lambda L) = 0.$$

In order that this system have nontrivial solutions the determinant of the coefficients must be zero:

$$\lambda(\cosh\lambda L - \cos\lambda L)^2 - \lambda(\sinh^2\lambda L - \sin^2\lambda L) = 0.$$

$\lambda = 0$ is not an eigenvalue since this leads to $X = 0$. Thus the last equation simplifies to $\cosh\lambda L\cos\lambda L = 1$ or $\cosh x\cos x = 1$, where $x = \lambda L$.

(b) The equation $\cosh x\cos x = 1$ is the same as $\cos x = \operatorname{sech} x$. The figure indicates that the equation has an infinite number of roots.



(c) Using a CAS we find the first four positive roots to be $x_1 = 4.7300$, $x_2 = 7.8532$, $x_3 = 10.9956$, and $x_4 = 14.1372$. Thus the first four eigenvalues are $\lambda_1 = x_1/L = 4.7300/L$, $\lambda_2 = x_2/L = 7.8532/L$, $\lambda_3 = x_3/L = 10.9956/L$, and $\lambda_4 = 14.1372/L$.

13. From (5) in the text we have

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{L}t + B_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x.$$

Since $u_t(x,0) = g(x) = 0$ we have $B_n = 0$ and

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n\cos\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x \\ &= \sum_{n=1}^{\infty} A_n\frac{1}{2}\left[\sin\left(\frac{n\pi}{L}x + \frac{n\pi a}{L}t\right) + \sin\left(\frac{n\pi}{L}x - \frac{n\pi a}{L}t\right)\right] \\ &= \frac{1}{2}\sum_{n=1}^{\infty} A_n\left[\sin\frac{n\pi}{L}(x+at) + \sin\frac{n\pi}{L}(x-at)\right]. \end{aligned}$$

From

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi}{L}x$$

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we identify

$$f(x+at)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{L}(x+at)$$

and

$$f(x-at)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{L}(x-at),$$

so that

$$u(x,t)=\frac{1}{2}[f(x+at)+f(x-at)].$$

14. **(a)** We note that $\xi_x=\eta_x=1$, $\xi_t=a$, and $\eta_t=-a$. Then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}=u_\xi+u_\eta$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial}{\partial x}(u_\xi+u_\eta)=\frac{\partial u_\xi}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u_\xi}{\partial \eta}\frac{\partial \eta}{\partial x}+\frac{\partial u_\eta}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u_\eta}{\partial \eta}\frac{\partial \eta}{\partial x}\\&=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}.\end{aligned}$$

Similarly

$$\frac{\partial^2 u}{\partial t^2}=a^2(u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta}).$$

Thus

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}\quad\text{becomes}\quad\frac{\partial^2 u}{\partial \xi\partial\eta}=0.$$

(b) Integrating

$$\frac{\partial^2 u}{\partial \xi\partial\eta}=\frac{\partial}{\partial\eta}u_\xi=0$$

we obtain

$$\int\frac{\partial}{\partial\eta}u_\xi\,d\eta=\int 0\,d\eta$$

$$u_\xi=f(\xi).$$

Integrating this result with respect to ξ we obtain

$$\int\frac{\partial u}{\partial\xi}\,d\xi=\int f(\xi)\,d\xi$$

$$u=F(\xi)+G(\eta).$$

Since $\xi=x+at$ and $\eta=x-at$, we then have

$$u=F(\xi)+G(\eta)=F(x+at)+G(x-at).$$

Next, we have

$$\begin{aligned}u(x,t)&=F(x+at)+G(x-at)\\u(x,0)&=F(x)+G(x)=f(x)\\u_t(x,0)&=aF'(x)-aG'(x)=g(x)\end{aligned}$$

Integrating the last equation with respect to x gives

$$F(x)-G(x)=\frac{1}{a}\int_{x_0}^{x}g(s)\,ds+c_1.$$

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Substituting $G(x) = f(x) - F(x)$ we obtain

$$F(x) = \frac{1}{2} f(x) + \frac{1}{2a} \int_{x_0}^{x} g(s)\, ds + C$$

where $c = c_1/2$. Thus

$$G(x) = \frac{1}{2} f(x) - \frac{1}{2a} \int_{x_0}^{x} g(s)\, ds - c.$$

(c) From the expressions for F and G,

$$F(x+at) = \frac{1}{2} f(x+at) + \frac{1}{2a} \int_{x_0}^{x+at} g(s)\, ds + c$$

$$G(x-at) = \frac{1}{2} f(x-at) - \frac{1}{2a} \int_{x_0}^{x-at} g(s)\, ds - c.$$

Thus,

$$u(x,t) = F(x+at) + G(x-at) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a} \int_{x-at}^{x+at} g(s)\, ds.$$

Here we have used $-\displaystyle\int_{x_0}^{x-at} g(s)\, ds = \int_{x-at}^{x_0} g(s)\, ds.$

15. $u(x,t) = \dfrac{1}{2}[\sin(x+at) + \sin(x-at)] + \dfrac{1}{2a} \displaystyle\int_{x-at}^{x+at} ds$

$= \dfrac{1}{2}[\sin x \cos at + \cos x \sin at + \sin x \cos at - \cos x \sin at] + \dfrac{1}{2a}\, s \Big|_{x-at}^{x+at} = \sin x \cos at + t$

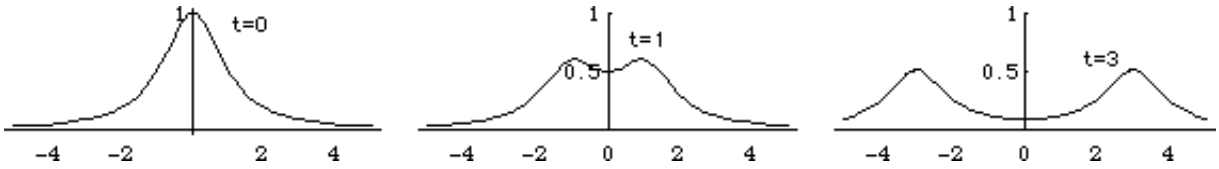
16. $u(x,t) = \dfrac{1}{2} \sin(x+at) + \sin(x-at)] + \dfrac{1}{2a} \displaystyle\int_{x-at}^{x+at} \cos s\, ds$

$= \sin x \cos at + \dfrac{1}{2a}[\sin(x+at) - \sin(x-at)] = \sin x \cos at + \dfrac{1}{a} \cos x \sin at$

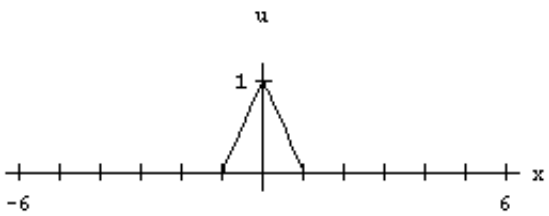
17. $u(x,t) = 0 + \dfrac{1}{2a} \displaystyle\int_{x-at}^{x+at} \sin 2s\, ds = \frac{1}{2a} \left[\frac{-\cos(2x+2at) + \cos(2x-2at)}{2}\right]$

$= \dfrac{1}{4a}[-\cos 2x \cos 2at + \sin 2x \sin 2at + \cos 2x \cos 2at + \sin 2x \sin 2at] = \dfrac{1}{2a} \sin 2x \sin 2at$

18.



19. (a)



(b) Since $g(x) = 0$, d'Alembert's solution with $a = 1$ is

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)].$$

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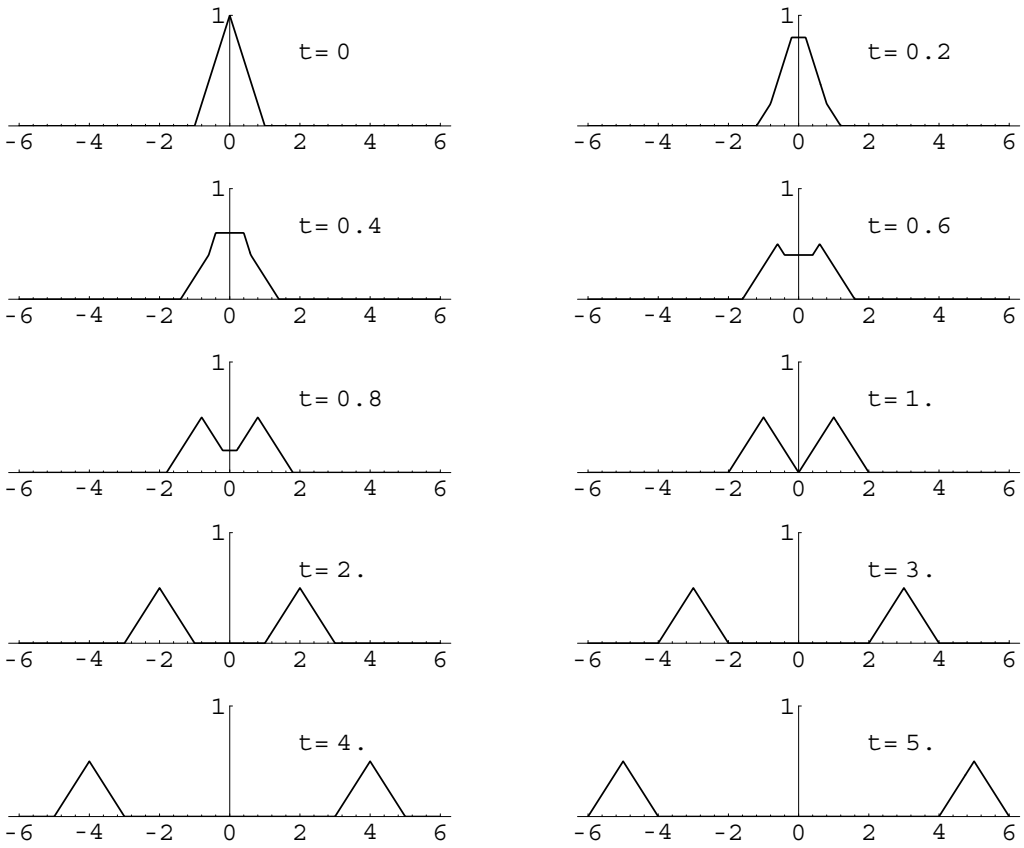
Sample plots are shown below.

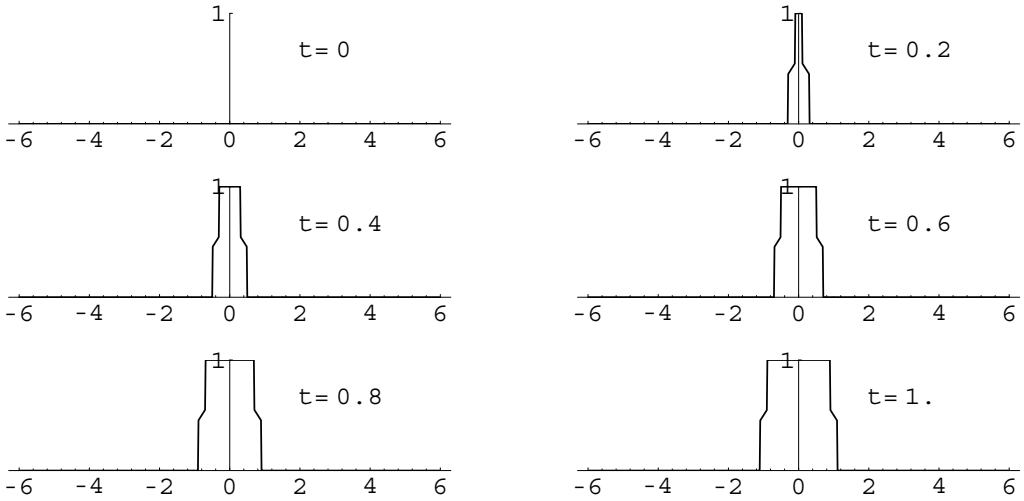


(c) The single peaked wave disolves into two peaks moving outward.

20. (a) With $a = 1$, d'Alembert's solution is

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \qquad \text{where} \qquad g(s) = \begin{cases} 1, & |s| \le 0.1 \\ 0, & |s| > 0.1 \end{cases}.$$

Sample plots are shown below.



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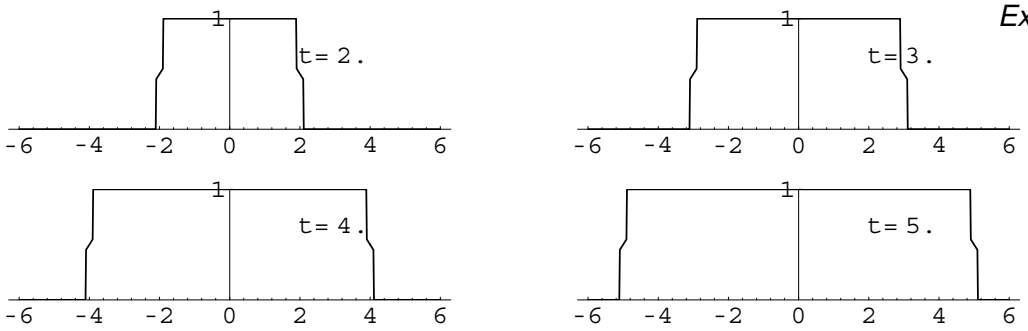


(b) Some frames of the movie are shown in part (a), The string has a roughly rectangular shape with the base on the x-axis increasing in length.

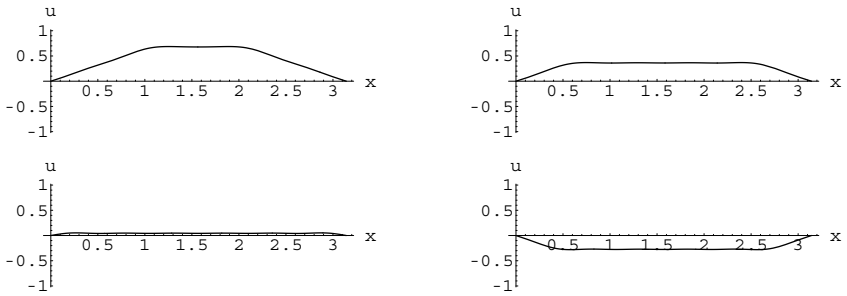
21. (a) and **(b)** With the given parameters, the solution is

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos nt\sin nx.$$

For n even, $\sin(n\pi/2) = 0$, so the first six nonzero terms correspond to $n = 1, 3, 5, 7, 9, 11$. In this case $\sin(n\pi/2) = \sin(2p-1)/2 = (-1)^{p+1}$ for $p = 1, 2, 3, 4, 5, 6$, and

$$u(x,t) = \frac{8}{\pi^2}\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{(2p-1)^2}\cos(2p-1)t\sin(2p-1)x.$$

Frames of the movie corresponding to $t = 0.5$, 1, 1.5, and 2 are shown.



Exercises 13.5

1. Using $u = XY$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(a) = 0,$$

and

$$Y'' - \lambda^2 Y = 0,$$
$$Y(0) = 0.$$

Then

$$X = c_1\sin\frac{n\pi}{a}x \quad \text{and} \quad Y = c_2\sinh\frac{n\pi}{a}y$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}x\,\sinh\frac{n\pi}{a}y.$$

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Imposing

$$u(x,b) = f(x) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi}{a} x$$

gives

$$A_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx$$

so that

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} y$$

where

$$A_n = \frac{2}{a} \operatorname{csch} \frac{n\pi b}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx.$$

2. Using $u = XY$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(a) = 0,$$

and

$$Y'' - \lambda^2 Y = 0,$$
$$Y'(0) = 0.$$

Then

$$X = c_1 \sin \frac{n\pi}{a} x \quad \text{and} \quad Y = c_2 \cosh \frac{n\pi}{a} y$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \cosh \frac{n\pi}{a} y.$$

Imposing

$$u(x,b) = f(x) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi b}{a} \sin \frac{n\pi}{a} x$$

gives

$$A_n \cosh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx$$

so that

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \cosh \frac{n\pi}{a} y$$

where

$$A_n = \frac{2}{a} \operatorname{sech} \frac{n\pi b}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx.$$

3. Using $u = XY$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(a) = 0,$$

and

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$$Y'' - \lambda^2 Y = 0,$$
$$Y(b) = 0.$$

Then

$$X = c_1 \sin \frac{n\pi}{a}x \quad \text{and} \quad Y = c_2 \cosh \frac{n\pi}{a}y - c_2 \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi}{a}y$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a}x \left(\cosh \frac{n\pi}{a}y - \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi}{a}y \right).$$

Imposing

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a}x$$

gives

$$A_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a}x \, dx$$

so that

$$u(x, y) = \frac{2}{a} \sum_{n=1}^{\infty} \left(\int_0^a f(x) \sin \frac{n\pi}{a}x \, dx \right) \sin \frac{n\pi}{a}x \left(\cosh \frac{n\pi}{a}y - \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi}{a}y \right).$$

4. Using $u = XY$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X'(0) = 0,$$
$$X'(a) = 0,$$

and

$$Y'' - \lambda^2 Y = 0,$$
$$Y(b) = 0.$$

Then

$$X = c_1 \cos \frac{n\pi}{a}x$$

for $n = 0, 1, 2, \ldots$ and

$$Y = c_2(y - b) \quad \text{or} \quad Y = c_2 \cosh \frac{n\pi}{a}y - c_2 \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi}{a}y$$

so that

$$u = A_0(y - b) + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{a}x \left(\cosh \frac{n\pi}{a}y - \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi}{a}y \right).$$

Imposing

$$u(x, 0) = x = -A_0 b + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{a}x$$

gives

$$-A_0 b = \frac{1}{a} \int_0^a x \, dx = \frac{1}{2}a$$

and

$$A_n = \frac{2}{a} \int_0^a x \cos \frac{n\pi}{a}x \, dx = \frac{2a}{n^2\pi^2}[(-1)^n - 1]$$

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so that

$$u(x,y) = \frac{a}{2b}(b-y) + \frac{2a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\frac{n\pi}{a}x\left(\cosh\frac{n\pi}{a}y - \frac{\cosh\frac{n\pi b}{a}}{\sinh\frac{n\pi b}{a}}\sinh\frac{n\pi}{a}y\right).$$

5. Using $u = XY$ and λ^2 as a separation constant leads to

$$X'' + -\lambda^2 X = 0,$$
$$X(0) = 0,$$

and

$$Y'' + \lambda^2 Y = 0,$$
$$Y'(0) = 0,$$
$$Y'(1) = 0.$$

Then

$$Y = c_1 \cos n\pi y$$

for $n = 0, 1, 2, \ldots$ and

$$X = c_2 x \quad \text{or} \quad X = c_2 \sinh n\pi x$$

for $n = 1, 2, 3, \ldots$ so that

$$u = A_0 x + \sum_{n=1}^{\infty} A_n \sinh n\pi x \cos n\pi y.$$

Imposing

$$u(1,y) = 1 - y = A_0 + \sum_{n=1}^{\infty} A_n \sinh n\pi x \cos n\pi y$$

gives

$$A_0 = \int_0^1 (1-y)\,dy$$

and

$$A_n \sinh n\pi = 2\int_0^1 (1-y)\cos n\pi y = \frac{2[1-(-1)^n]}{n^2\pi^2 \sinh n\pi}$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x,y) = \frac{1}{2}x + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2 \sinh n\pi}\sinh n\pi x \cos n\pi y.$$

6. Using $u = XY$ and λ^2 as a separation constant leads to

$$X'' - \lambda^2 X = 0,$$
$$X'(1) = 0,$$

and

$$Y'' + \lambda^2 Y = 0,$$
$$Y'(0) = 0,$$
$$Y'(\pi) = 0.$$

Then

$$Y = c_1 \cos ny$$

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for $n = 0, 1, 2, \ldots$ and

$$X = c_2 \cosh nx - c_2 \frac{\sinh n}{\cosh n} \sinh nx$$

for $n = 0, 1, 2, \ldots$ so that

$$u = A_0 + \sum_{n=1}^{\infty} A_n \left(\cosh nx - \frac{\sinh n}{\cosh n} \sinh nx \right) \cos ny.$$

Imposing

$$u(0, y) = g(y) = A_0 + \sum_{n=1}^{\infty} A_n \cos ny$$

gives

$$A_0 = \frac{1}{\pi} \int_0^{\pi} g(y)\, dy \quad \text{and} \quad A_n = \frac{2}{\pi} \int_0^{\pi} g(y) \cos ny\, dy$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x, y) = \frac{1}{\pi} \int_0^{\pi} g(y)\, dy + \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \int_0^{\pi} g(y) \cos ny\, dy \right) \left(\cosh nx - \frac{\sinh n}{\cosh n} \sinh nx \right) \cos ny.$$

7. Using $u = XY$ and λ^2 as a separation constant leads to

$$X'' - \lambda^2 X = 0,$$
$$X'(0) = X(0),$$

and

$$Y'' + \lambda^2 Y = 0,$$
$$Y(0) = 0,$$
$$Y(\pi) = 0.$$

Then

$$Y = c_1 \sin ny \quad \text{and} \quad X = c_2(n \cosh nx + \sinh nx)$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n (n \cosh nx + \sinh nx) \sin ny.$$

Imposing

$$u(\pi, y) = 1 = \sum_{n=1}^{\infty} A_n (n \cosh n\pi + \sinh n\pi) \sin ny$$

gives

$$A_n (n \cosh n\pi + \sinh n\pi) = \frac{2}{\pi} \int_0^{\pi} \sin ny\, dy = \frac{2[1 - (-1)^n]}{n\pi}$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \frac{n \cosh nx + \sinh nx}{n \cosh n\pi + \sinh n\pi} \sin ny.$$

8. Using $u = XY$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(1) = 0$$

and

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$$Y'' - \lambda^2 Y = 0,$$
$$Y'(0) = Y(0).$$

Then

$$X = c_1 \sin n\pi x \quad \text{and} \quad Y = c_2(n\cosh n\pi y + \sinh n\pi y)$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n(n\cosh n\pi y + \sinh n\pi y)\sin n\pi x.$$

Imposing

$$u(x, 1) = f(x) = \sum_{n=1}^{\infty} A_n(n\cosh n\pi + \sinh n\pi)\sin n\pi x$$

gives

$$A_n(n\cosh n\pi + \sinh n\pi) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin n\pi x\,dx$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x, y) = \sum_{n=1}^{\infty} A_n(n\cosh n\pi y + \sinh n\pi y)\sin n\pi x$$

where

$$A_n = \frac{2}{n\pi\cosh n\pi + \pi\sinh n\pi}\int_0^1 f(x)\sin n\pi x\,dx.$$

9. This boundary-value problem has the form of Problem 1 in this section, with $a = b = 1$, $f(x) = 100$, and $g(x) = 200$. The solution, then, is

$$u(x, y) = \sum_{n=1}^{\infty}(A_n\cosh n\pi y + B_n\sinh n\pi y)\sin n\pi x,$$

where

$$A_n = 2\int_0^1 100\sin n\pi x\,dx = 200\left(\frac{1-(-1)^n}{n\pi}\right)$$

and

$$\begin{aligned} B_n &= \frac{1}{\sinh n\pi}\left[2\int_0^1 200\sin n\pi x\,dx - A_n\cosh n\pi\right] \\ &= \frac{1}{\sinh n\pi}\left[400\left(\frac{1-(-1)^n}{n\pi}\right) - 200\left(\frac{1-(-1)^n}{n\pi}\right)\cosh n\pi\right] \\ &= 200\left[\frac{1-(-1)^n}{n\pi}\right][2\operatorname{csch} n\pi - \coth n\pi]. \end{aligned}$$

10. This boundary-value problem has the form of Problem 2 in this section, with $a = 1$ and $b = 1$. Thus, the solution has the form

$$u(x, y) = \sum_{n=1}^{\infty}(A_n\cosh n\pi x + B_n\sinh n\pi x)\sin n\pi y.$$

The boundary condition $u(0, y) = 10y$ implies

$$10y = \sum_{n=1}^{\infty} A_n\sin n\pi y$$

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and

$$A_n = \frac{2}{1}\int_0^1 10y \sin n\pi y \, dy = \frac{20}{n\pi}(-1)^{n+1}.$$

The boundary condition $u_x(1, y) = -1$ implies

$$-1 = \sum_{n=1}^{\infty} (n\pi A_n \sinh n\pi + n\pi B_n \cosh n\pi) \sin n\pi y$$

and

$$n\pi A_n \sinh n\pi + n\pi B_n \cosh n\pi = \frac{2}{1}\int_0^1 (-\sin n\pi y) dy$$

$$A_n \sinh n\pi + B_n \cos n\pi = -\frac{2}{n\pi}\left[1 - (-1)^n\right]$$

$$B_n = \frac{2}{n\pi}\left[(-1)^n - 1\right] \operatorname{sech} n\pi - \frac{20}{n\pi}(-1)^{n+1} \tanh n\pi.$$

11. Using $u = XY$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(\pi) = 0,$$

and

$$Y'' - \lambda^2 Y = 0.$$

Then the boundedness of u as $y \to \infty$ gives $Y = c_1 e^{-ny}$ and $X = c_2 \sin nx$ for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n e^{-ny} \sin nx.$$

Imposing

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin nx$$

gives

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

so that

$$u(x, y) = \sum_{n=1}^{\infty} \left(\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx\right) e^{-ny} \sin nx.$$

12. Using $u = XY$ and $-\lambda^2$ as a separation constant leads to

$$X'' + \lambda^2 X = 0,$$
$$X'(0) = 0,$$
$$X'(\pi) = 0,$$

and

$$Y'' - \lambda^2 Y = 0.$$

By the boundedness of u as $y \to \infty$ we obtain $Y = c_1 e^{-ny}$ for $n = 1, 2, 3, \ldots$ or $Y = c_1$ and $X = c_2 \cos nx$ for $n = 0, 1, 2, \ldots$ so that

$$u = A_0 + \sum_{n=1}^{\infty} A_n e^{-ny} \cos nx.$$

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Imposing

$$u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx$$

gives

$$A_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \quad \text{and} \quad A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

so that

$$u(x,y) = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx\right)e^{-ny}\cos nx.$$

13. Since the boundary conditions at $y = 0$ and $y = b$ are functions of x we choose to separate Laplace's equation as

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

so that

$$X'' + \lambda^2 X = 0$$
$$Y'' - \lambda^2 Y = 0$$

and

$$X(x) = c_1\cos\lambda x + c_2\sin\lambda x$$
$$Y(y) = c_3\cosh\lambda y + c_2\sinh\lambda y.$$

Now $X(0) = 0$ gives $c_1 = 0$ and $X(a) = 0$ implies $\sin\lambda a = 0$ or $\lambda = n\pi/a$ for $n = 1, 2, 3, \ldots$. Thus

$$u_n(x,y) = XY = \left(A_n\cosh\frac{n\pi}{a}y + B_n\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

and

$$u(x,y) = \sum_{n=1}^{\infty}\left(A_n\cosh\frac{n\pi}{a}y + B_n\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x. \tag{1}$$

At $y = 0$ we then have

$$f(x) = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi}{a}x$$

and consequently

$$A_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx. \tag{2}$$

At $y = b$,

$$g(y) = \sum_{n=1}^{\infty}\left(A_n\cosh\frac{n\pi}{a}b + B_n\sinh\frac{n\pi}{b}a\right)\sin\frac{n\pi}{a}x$$

indicates that the entire expression in the parentheses is given by

$$A_n\cosh\frac{n\pi}{a}b + B_n\sinh\frac{n\pi}{a}b = \frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx.$$

We can now solve for B_n:

$$B_n\sinh\frac{n\pi}{a}b = \frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx - A_n\cosh\frac{n\pi}{a}b$$

$$B_n = \frac{1}{\sinh\frac{n\pi}{a}b}\left(\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx - A_n\cosh\frac{n\pi}{a}b\right). \tag{3}$$

A solution to the given boundary-value problem consists of the series (1) with coefficients A_n and B_n given in (2) and (3), respectively.

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14. Since the boundary conditions at $x = 0$ and $x = a$ are functions of y we choose to separate Laplace's equation as

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2$$

so that

$$X'' - \lambda^2 X = 0$$
$$Y'' + \lambda^2 Y = 0$$

and

$$X(x) = c_1 \cosh \lambda x + c_2 \sinh \lambda x$$
$$Y(y) = c_3 \cos \lambda y + c_2 \sin \lambda y.$$

Now $Y(0) = 0$ gives $c_3 = 0$ and $Y(b) = 0$ implies $\sin \lambda b = 0$ or $\lambda = n\pi/b$ for $n = 1, 2, 3, \ldots$. Thus

$$u_n(x, y) = XY = \left(A_n \cosh \frac{n\pi}{b}x + B_n \sinh \frac{n\pi}{b}x\right) \sin \frac{n\pi}{b}y$$

and

$$u(x, y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi}{b}x + B_n \sinh \frac{n\pi}{b}x\right) \sin \frac{n\pi}{b}y. \tag{4}$$

At $x = 0$ we then have

$$F(y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{b}y$$

and consequently

$$A_n = \frac{2}{b}\int_0^b F(y) \sin \frac{n\pi}{b}y \, dy. \tag{5}$$

At $x = a$,

$$G(y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi}{b}a + B_n \sinh \frac{n\pi}{b}a\right) \sin \frac{n\pi}{b}y$$

indicates that the entire expression in the parentheses is given by

$$A_n \cosh \frac{n\pi}{b}a + B_n \sinh \frac{n\pi}{b}a = \frac{2}{b}\int_0^b G(y) \sin \frac{n\pi}{b}y \, dy.$$

We can now solve for B_n:

$$B_n \sinh \frac{n\pi}{b}a = \frac{2}{b}\int_0^b G(y) \sin \frac{n\pi}{b}y \, dy - A_n \cosh \frac{n\pi}{b}a$$

$$B_n = \frac{1}{\sinh \frac{n\pi}{b}a}\left(\frac{2}{b}\int_0^b G(y) \sin \frac{n\pi}{b}y \, dy - A_n \cosh \frac{n\pi}{b}a\right). \tag{6}$$

A solution to the given boundary-value problem consists of the series (4) with coefficients A_n and B_n given in (5) and (6), respectively.

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In Problems 15 and 16 we refer to the discussion in the text under the heading **Superposition Principle**.

15. We identify $a = b = \pi$, $f(x) = 0$, $g(x) = 1$, $F(y) = 1$, and $G(y) = 1$. Then $A_n = 0$ and

$$u_1(x, y) = \sum_{n=1}^{\infty} B_n \sinh ny \sin nx$$

where

$$B_n = \frac{2}{\pi \sinh n\pi} \int_0^{\pi} \sin nx\, dx = \frac{2[1 - (-1)^n]}{n\pi \sinh n\pi}.$$

Next

$$u_2(x, y) = \sum_{n=1}^{\infty} (A_n \cosh nx + B_n \sinh nx) \sin ny$$

where

$$A_n = \frac{2}{\pi} \int_0^{\pi} \sin ny\, dy = \frac{2[1 - (-1)^n]}{n\pi}$$

and

$$\begin{aligned} B_n &= \frac{1}{\sinh n\pi} \left(\frac{2}{\pi} \int_0^{\pi} \sin ny\, dy - A_n \cosh n\pi \right) \\ &= \frac{1}{\sinh n\pi} \left(\frac{2[1 - (-1)^n]}{n\pi} - \frac{2[1 - (-1)^n]}{n\pi} \cosh n\pi \right) \\ &= \frac{2[1 - (-1)^n]}{n\pi \sinh n\pi} (1 - \cosh n\pi). \end{aligned}$$

Now

$$\begin{aligned} A_n \cosh nx + B_n \sinh nx &= \frac{2[1 - (-1)^n]}{n\pi} \left[\cosh nx + \frac{\sinh nx}{\sinh n\pi} (1 - \cosh n\pi) \right] \\ &= \frac{2[1 - (-1)^n]}{n\pi \sinh n\pi} [\cosh nx \sinh n\pi + \sinh nx - \sinh nx \cosh n\pi] \\ &= \frac{2[1 - (-1)^n]}{n\pi \sinh n\pi} [\sinh nx + \sinh n(\pi - x)] \end{aligned}$$

and

$$\begin{aligned} u(x, y) = u_1 + u_2 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n \sinh n\pi} \sinh ny \sin nx \\ + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n][\sinh nx + \sinh n(\pi - x)]}{n \sinh n\pi} \sin ny. \end{aligned}$$

16. We identify $a = b = 2$, $f(x) = 0$, $g(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \end{cases}$, $F(y) = 0$, and $G(y) = y(2 - y)$. Then $A_n = 0$ and

$$u_1(x, y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{2} y \sin \frac{n\pi}{2} x$$

where

$$\begin{aligned} B_n &= \frac{1}{\sinh n\pi} \int_0^2 g(x) \sin \frac{n\pi}{2} x\, dx \\ &= \frac{1}{\sinh n\pi} \left(\int_0^1 x \sin \frac{n\pi}{2} x\, dx + \int_1^2 (2 - x) \sin \frac{n\pi}{2} x\, dx \right) \\ &= \frac{8 \sin \frac{n\pi}{2}}{n^2 \pi^2 \sinh n\pi}. \end{aligned}$$

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Next, since $A_n = 0$ in u_2, we have

$$u_2(x, y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{2} x \sin \frac{n\pi}{2}$$

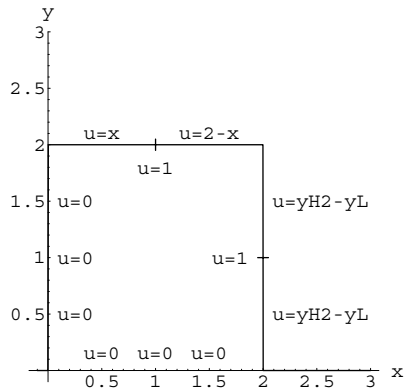
where

$$B_n = \frac{1}{\sinh n\pi} \int_0^b y(2 - y) \sin \frac{n\pi}{2} y \, dy = \frac{16[1 - (-1)^n]}{n^3 \pi^3 \sinh n\pi}.$$

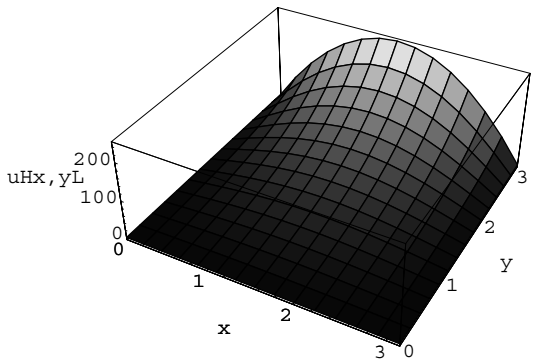
Thus

$$u(x, y) = u_1 + u_2 = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^2 \sinh n\pi} \sinh \frac{n\pi}{2} y \sin \frac{n\pi}{2} x + \frac{16}{\pi^3} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n^3 \sinh n\pi} \sinh \frac{n\pi}{2} x \sin \frac{n\pi}{2} y.$$

17. From the figure showing the boundary conditions we see that the maximum value of the temperature is 1 at $(1, 2)$ and $(2, 1)$.



18. **(a)**



(b) The maximum value occurs at $(\pi/2, \pi)$ and is $f(\pi/2) = 25\pi^2$.

(c) The coefficients are

$$A_n = \frac{2}{\pi} \operatorname{csch} n\pi \int_0^{\pi} 100x(\pi - x) \sin nx \, dx$$
$$= \frac{200 \operatorname{csch} n\pi}{\pi} \left[\frac{200}{n^3} \left(1 - (-1)^n \right) \right] = \frac{400}{n^3 \pi} \left[1 - (-1)^n \right] \operatorname{csch} n\pi.$$

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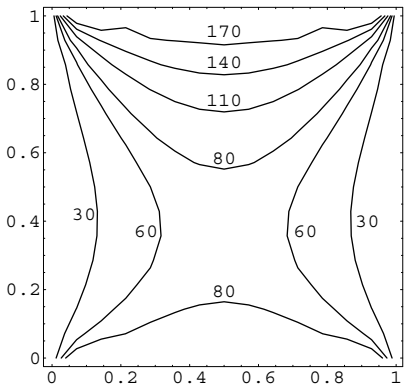
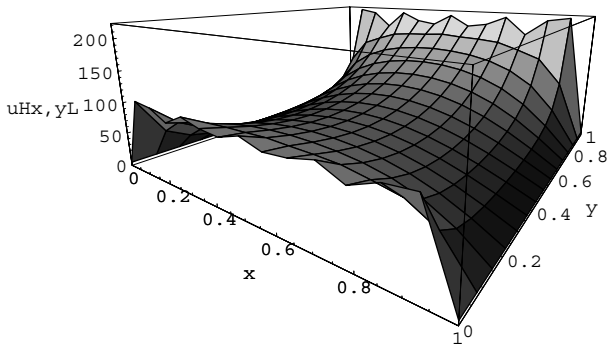
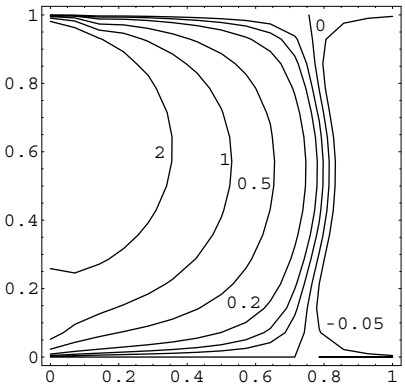
19. **(a)**



(b)



20.



Exercises 13.6

1. Using $v(x,t) = u(x,t) - 100$ we wish to solve $kv_{xx} = v_t$ subject to $v(0,t) = 0$, $v(1,t) = 0$, and $v(x,0) = -100$. Let $v = XT$ and use $-\lambda^2$ as a separation constant so that

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(1) = 0,$$

and

$$T' + \lambda^2 kT = 0.$$

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Then

$$X = c_1 \sin n\pi x \quad \text{and} \quad T = c_2 e^{-kn^2\pi^2 t}$$

for $n = 1, 2, 3, \ldots$ so that

$$v = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x.$$

Imposing

$$v(x, 0) = -100 = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

gives

$$A_n = 2\int_0^1 (-100) \sin n\pi x \, dx = \frac{-200}{n\pi}[1 - (-1)^n]$$

so that

$$u(x, t) = v(x, t) + 100 = 100 + \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n} e^{-kn^2\pi^2 t} \sin n\pi x.$$

2. Letting $u(x, t) = v(x, t) + \psi(x)$ and proceeding as in Example 1 in the text we find $\psi(x) = u_0 - u_0 x$. Then $v(x, t) = u(x, t) + u_0 x - u_0$ and we wish to solve $kv_{xx} = v_t$ subject to $v(0, t) = 0$, $v(1, t) = 0$, and $v(x, 0) = f(x) + u_0 x - u_0$. Let $v = XT$ and use $-\lambda^2$ as a separation constant so that

$$X'' + \lambda^2 X = 0,$$
$$X(0) = 0,$$
$$X(1) = 0,$$

and

$$T' + \lambda^2 kT = 0.$$

Then

$$X = c_1 \sin n\pi x \quad \text{and} \quad T = c_2 e^{-kn^2\pi^2 t}$$

for $n = 1, 2, 3, \ldots$ so that

$$v = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x.$$

Imposing

$$v(x, 0) = f(x) + u_0 x - u_0 = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

gives

$$A_n = 2\int_0^1 \big(f(x) + u_0 x - u_0\big) \sin n\pi x \, dx$$

so that

$$u(x, t) = v(x, t) + u_0 - u_0 x = u_0 - u_0 x + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x.$$

3. If we let $u(x, t) = v(x, t) + \psi(x)$, then we obtain as in Example 1 in the text

$$k\psi'' + r = 0$$

or

$$\psi(x) = -\frac{r}{2k} x^2 + c_1 x + c_2.$$

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The boundary conditions become

$$u(0,t) = v(0,t) + \psi(0) = u_0$$
$$u(1,t) = v(1,t) + \psi(1) = u_0.$$

Letting $\psi(0) = \psi(1) = u_0$ we obtain homogeneous boundary conditions in v:

$$v(0,t) = 0 \qquad \text{and} \qquad v(1,t) = 0.$$

Now $\psi(0) = \psi(1) = u_0$ implies $c_2 = u_0$ and $c_1 = r/2k$. Thus

$$\psi(x) = -\frac{r}{2k}x^2 + \frac{r}{2k}x + u_0 = u_0 - \frac{r}{2k}x(x-1).$$

To determine $v(x,t)$ we solve

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{dt}, \quad 0 < x < 1,\ t > 0$$
$$v(0,t) = 0, \quad v(1,t) = 0,$$
$$v(x,0) = \frac{r}{2k}x(x-1) - u_0.$$

Separating variables, we find

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x,$$

where

$$A_n = 2\int_0^1 \left[\frac{r}{2k}x(x-1) - u_0\right] \sin n\pi x\, dx = 2\left[\frac{u_0}{n\pi} + \frac{r}{kn^3\pi^3}\right][(-1)^n - 1]. \tag{1}$$

Hence, a solution of the original problem is

$$u(x,t) = \psi(x) + v(x,t)$$
$$= u_0 - \frac{r}{2k}x(x-1) + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x,$$

where A_n is defined in (1).

4. If we let $u(x,t) = v(x,t) + \psi(x)$, then we obtain as in Example 1 in the text

$$k\psi'' + r = 0$$

or

$$\psi(x) = -\frac{r}{2k}x^2 + c_1 x + c_2.$$

The boundary conditions become

$$u(0,t) = v(0,t) + \psi(0) = u_0$$
$$u(1,t) = v(1,t) + \psi(1) = u_1.$$

Letting $\psi(0) = u_0$ and $\psi(1) = u_1$ we obtain homogeneous boundary conditions in v:

$$v(0,t) = 0 \qquad \text{and} \qquad v(1,t) = 0.$$

Now $\psi(0) = u_0$ and $\psi(1) = u_1$ imply $c_2 = u_0$ and $c_1 = u_1 - u_0 + r/2k$. Thus

$$\psi(x) = -\frac{r}{2k}x^2 + \left(u_1 - u_0 + \frac{r}{2k}\right)x + u_0.$$

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To determine $v(x,t)$ we solve

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{dt}, \quad 0 < x < 1,\ t > 0$$
$$v(0,t) = 0, \quad v(1,t) = 0,$$
$$v(x,0) = f(x) - \psi(x).$$

Separating variables, we find

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x,$$

where

$$A_n = 2\int_0^1 [f(x) - \psi(x)] \sin n\pi x\, dx. \tag{2}$$

Hence, a solution of the original problem is

$$u(x,t) = \psi(x) + v(x,t)$$
$$= -\frac{r}{2k}x^2 + \left(u_1 - u_0 + \frac{r}{2k}\right)x + u_0 + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x,$$

where A_n is defined in (2).

5. Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation gives

$$k\,\frac{\partial^2 v}{\partial x^2} + k\psi'' + Ae^{-\beta x} = \frac{\partial v}{\partial t}\,.$$

This equation will be homogeneous provided ψ satisfies

$$k\psi'' + Ae^{-\beta x} = 0.$$

The general solution of this differential equation is

$$\psi(x) = -\frac{A}{\beta^2 k}e^{-\beta x} + c_1 x + c_2.$$

From $\psi(0) = 0$ and $\psi(1) = 0$ we find

$$c_1 = \frac{A}{\beta^2 k}(e^{-\beta} - 1) \quad \text{and} \quad c_2 = \frac{A}{\beta^2 k}\,.$$

Hence

$$\psi(x) = -\frac{A}{\beta^2 k}e^{-\beta x} + \frac{A}{\beta^2 k}(e^{-\beta} - 1)x + \frac{A}{\beta^2 k}$$
$$= \frac{A}{\beta^2 k}\left[1 - e^{-\beta x} + (e^{-\beta} - 1)x\right].$$

Now the new problem is

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \quad ,\ 0 < x < 1, \quad t > 0,$$
$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0,$$
$$v(x,0) = f(x) - \psi(x), \quad 0 < x < 1.$$

Identifying this as the heat equation solved in Section 13.3 in the text with $L = 1$ we obtain

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x$$

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where

$$A_n = 2\int_0^1 [f(x) - \psi(x)] \sin n\pi x \, dx.$$

Thus

$$u(x,t) = \frac{A}{\beta^2 k}\left[1 - e^{-\beta x} + (e^{-\beta} - 1)x\right] + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x.$$

6. Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation gives

$$k\frac{\partial^2 v}{\partial x^2} + k\psi'' - hv - h\psi = \frac{\partial v}{\partial t}.$$

This equation will be homogeneous provided ψ satisfies

$$k\psi'' - h\psi = 0.$$

Since k and h are positive, the general solution of this latter equation is

$$\psi(x) = c_1 \cosh\sqrt{\frac{h}{k}}\, x + c_2 \sinh\sqrt{\frac{h}{k}}\, x.$$

From $\psi(0) = 0$ and $\psi(\pi) = u_0$ we find $c_1 = 0$ and $c_2 = u_0/\sinh\sqrt{h/k}\,\pi$. Hence

$$\psi(x) = u_0 \frac{\sinh\sqrt{h/k}\, x}{\sinh\sqrt{h/k}\,\pi}.$$

Now the new problem is

$$k\frac{\partial^2 v}{\partial x^2} - hv = \frac{\partial v}{\partial t}, \quad 0 < x < \pi, \ t > 0$$

$$v(0,t) = 0, \quad v(\pi,t) = 0, \quad t > 0$$

$$v(x,0) = -\psi(x), \quad 0 < x < \pi.$$

If we let $v = XT$ then

$$\frac{X''}{X} = \frac{T' + hT}{kT} = -\lambda^2$$

gives the separated differential equations

$$X'' + \lambda^2 X = 0 \qquad \text{and} \qquad T' + \left(h + k\lambda^2\right) T = 0.$$

The respective solutions are

$$X(x) = c_3 \cos\lambda x + c_4 \sin\lambda x$$

$$T(t) = c_5 e^{-\left(h + k\lambda^2\right)t}.$$

From $X(0) = 0$ we get $c_3 = 0$ and from $X(\pi) = 0$ we find $\lambda = n$ for $n = 1, 2, 3, \ldots$. Consequently, it follows that

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(h + kn^2\right)t} \sin nx$$

where

$$A_n = -\frac{2}{\pi}\int_0^{\pi} \psi(x) \sin nx \, dx.$$

Hence a solution of the original problem is

$$u(x,t) = u_0 \frac{\sinh\sqrt{h/k}\, x}{\sinh\sqrt{h/k}\,\pi} + e^{-ht} \sum_{n=1}^{\infty} A_n e^{-kn^2 t} \sin nx$$

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where

$$A_n = -\frac{2}{\pi}\int_0^{\pi} u_0 \frac{\sinh\sqrt{h/k}\,x}{\sinh\sqrt{h/k}\,\pi}\sin nx\,dx.$$

Using the exponential definition of the hyperbolic sine and integration by parts we find

$$A_n = \frac{2u_0 nk(-1)^n}{\pi\,(h+kn^2)}\,.$$

7. Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation gives

$$k\frac{\partial^2 v}{\partial x^2} + k\psi'' - hv - h\psi + hu_0 = \frac{\partial v}{\partial t}\,.$$

This equation will be homogeneous provided ψ satisfies

$$k\psi'' - h\psi + hu_0 = 0 \quad \text{or} \quad k\psi'' - h\psi = -hu_0.$$

This second-order, linear, non-homogeneous differential equation has solution

$$\psi(x) = c_1\cosh\sqrt{\frac{h}{k}}\,x + c_2\sinh\sqrt{\frac{h}{k}}\,x + u_0,$$

where we assume $h > 0$ and $k > 0$. From $\psi(0) = u_0$ and $\psi(1) = 0$ we find $c_1 = 0$ and $c_2 = -u_0/\sinh\sqrt{h/k}$. Thus, the steady-state solution is

$$\psi(x) = -\frac{u_0}{\sinh\sqrt{\frac{h}{k}}}\sinh\sqrt{\frac{h}{k}}\,x + u_0 = u_0\left(1 - \frac{\sinh\sqrt{\frac{h}{k}}\,x}{\sinh\sqrt{\frac{h}{k}}}\right).$$

8. The partial differential equation is

$$k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}\,.$$

Substituting $u(x,t) = v(x,t) + \psi(x)$ gives

$$k\frac{\partial^2 v}{\partial x^2} + k\psi'' - hv - h\psi = \frac{\partial v}{\partial t}\,.$$

This equation will be homogeneous provided ψ satisfies

$$k\psi'' - h\psi = 0.$$

Assuming $h > 0$ and $k > 0$, we have

$$\psi = c_1 e^{\sqrt{h/k}\,x} + c_2 e^{-\sqrt{h/k}\,x},$$

where we have used the exponential form of the solution since the rod is infinite. Now, in order that the steady-state temperature $\psi(x)$ be bounded as $x \to \infty$, we require $c_1 = 0$. Then

$$\psi(x) = c_2 e^{-\sqrt{h/k}\,x}$$

and $\psi(0) = u_0$ implies $c_2 = u_0$. Thus

$$\psi(x) = u_0 e^{-\sqrt{h/k}\,x}.$$

9. Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation gives

$$a^2\frac{\partial^2 v}{\partial x^2} + a^2\psi'' + Ax = \frac{\partial^2 v}{\partial t^2}\,.$$

This equation will be homogeneous provided ψ satisfies

$$a^2\psi'' + Ax = 0.$$

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The general solution of this differential equation is

$$\psi(x) = -\frac{A}{6a^2}x^3 + c_1 x + c_2.$$

From $\psi(0) = 0$ we obtain $c_2 = 0$, and from $\psi(1) = 0$ we obtain $c_1 = A/6a^2$. Hence

$$\psi(x) = \frac{A}{6a^2}(x - x^3).$$

Now the new problem is

$$a^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2}$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0,$$

$$v(x,0) = -\psi(x), \quad v_t(x,0) = 0, \quad 0 < x < 1.$$

Identifying this as the wave equation solved in Section 13.4 in the text with $L = 1$, $f(x) = -\psi(x)$, and $g(x) = 0$ we obtain

$$v(x,t) = \sum_{n=1}^{\infty} A_n \cos n\pi a t \sin n\pi x$$

where

$$A_n = 2\int_0^1 [-\psi(x)] \sin n\pi x \, dx = \frac{A}{3a^2}\int_0^1 (x^3 - x) \sin n\pi x \, dx = \frac{2A(-1)^n}{a^2\pi^3 n^3}.$$

Thus

$$u(x,t) = \frac{A}{6a^2}(x - x^3) + \frac{2A}{a^2\pi^3}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \cos n\pi a t \sin n\pi x.$$

10. We solve

$$a^2 \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \ t > 0$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0$$

$$u(x,0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < 1.$$

The partial differential equation is nonhomogeneous. The substitution $u(x,t) = v(x,t) + \psi(x)$ yields a homogeneous partial differential equation provided ψ satisfies

$$a^2\psi'' - g = 0.$$

By integrating twice we find

$$\psi(x) = \frac{g}{2a^2}x^2 + c_1 x + c_2.$$

The imposed conditions $\psi(0) = 0$ and $\psi(1) = 0$ then lead to $c_2 = 0$ and $c_1 = -g/2a^2$. Hence

$$\psi(x) = \frac{g}{2a^2}\left(x^2 - x\right).$$

The new problem is now

$$a^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2}, \quad 0 < x < 1, \ t > 0$$

$$v(0,t) = 0, \quad v(1,t) = 0$$

$$v(x,0) = \frac{g}{2a^2}\left(x - x^2\right), \quad \left.\frac{\partial v}{\partial t}\right|_{t=0} = 0.$$

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Substituting $v = XT$ we find in the usual manner

$$X'' + \lambda^2 X = 0$$
$$T'' + a^2\lambda^2 T = 0$$

with solutions

$$X(x) = c_3 \cos \lambda x + c_4 \sin \lambda x$$
$$T(t) = c_5 \cos a\lambda t + c_6 \sin a\lambda t.$$

The conditions $X(0) = 0$ and $X(1) = 0$ imply in turn that $c_3 = 0$ and $\lambda = n\pi$ for $n = 1, 2, 3, \ldots$. The condition $T'(0) = 0$ implies $c_6 = 0$. Hence, by the superposition principle

$$v(x,t) = \sum_{n=1}^{\infty} A_n \cos(an\pi t) \sin(n\pi x).$$

At $t = 0$,

$$\frac{g}{2a^2}\left(x - x^2\right) = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

and so

$$A_n = \frac{g}{a^2}\int_0^1 \left(x - x^2\right) \sin n\pi x \, dx = \frac{2g}{a^2 n^3 \pi^3}\left[1 - (-1)^n\right].$$

Thus the solution to the original problem is

$$u(x,t) = \psi(x) + v(x,t) = \frac{g}{2a^2}\left(x^2 - x\right) + \frac{2g}{a^2\pi^3}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \cos(an\pi t) \sin(n\pi x).$$

11. Substituting $u(x,y) = v(x,y) + \psi(y)$ into Laplace's equation we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \psi''(y) = 0.$$

This equation will be homogeneous provided ψ satisfies $\psi(y) = c_1 y + c_2$. Considering

$$u(x,0) = v(x,0) + \psi(0) = u_1$$
$$u(x,1) = v(x,1) + \psi(1) = u_0$$
$$u(0,y) = v(0,y) + \psi(y) = 0$$

we require that $\psi(0) = u_1$, $\psi_1 = u_0$ and $v(0,y) = -\psi(y)$. Then $c_1 = u_0 - u_1$ and $c_2 = u_1$. The new boundary-value problem is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$
$$v(x,0) = 0, \quad v(x,1) = 0,$$
$$v(0,y) = -\psi(y), \quad 0 < y < 1,$$

where $v(x,y)$ is bounded at $x \to \infty$. This problem is similar to Problem 11 in Section 13.5. The solution is

$$\begin{aligned} v(x,y) &= \sum_{n=1}^{\infty}\left(2\int_0^1 [-\psi(y)\sin n\pi y]\, dy\right) e^{-n\pi x} \sin n\pi y \\ &= 2\sum_{n=1}^{\infty}\left[(u_1 - u_0)\int_0^1 y \sin n\pi y \, dy - u_1 \int_0^1 \sin n\pi y \, dy\right] e^{-n\pi x} \sin n\pi y \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{u_0(-1)^n - u_1}{n} e^{-n\pi x} \sin n\pi y. \end{aligned}$$

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Thus

$$\begin{aligned} u(x,y) &= v(x,y) + \psi(y) \\ &= (u_0 - u_1)y + u_1 + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{u_0(-1)^n - u_1}{n}\, e^{-n\pi x} \sin n\pi y. \end{aligned}$$

12. Substituting $u(x,y) = v(x,y) + \psi(x)$ into Poisson's equation we obtain

$$\frac{\partial^2 v}{\partial x^2} + \psi''(x) + h + \frac{\partial^2 v}{\partial y^2} = 0.$$

The equation will be homogeneous provided ψ satisfies $\psi''(x)+h = 0$ or $\psi(x) = -\frac{h}{2}x^2 + c_1 x + c_2$. From $\psi(0) = 0$ we obtain $c_2 = 0$. From $\psi(\pi) = 1$ we obtain

$$c_1 = \frac{1}{\pi} + \frac{h\pi}{2}.$$

Then

$$\psi(x) = \left(\frac{1}{\pi} + \frac{h\pi}{2}\right) x - \frac{h}{2}x^2.$$

The new boundary-value problem is

$$\begin{gathered} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \\ v(0,y) = 0, \quad v(\pi,y) = 0, \\ v(x,0) = -\psi(x), \quad 0 < x < \pi. \end{gathered}$$

This is Problem 11 in Section 13.5. The solution is

$$v(x,y) = \sum_{n=1}^{\infty} A_n e^{-ny} \sin nx$$

where

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^{\pi} [-\psi(x)\sin nx]\, dx \\ &= \frac{2(-1)^n}{m}\left(\frac{1}{\pi} + \frac{h\pi}{2}\right) - h(-1)^n\left(\frac{\pi}{n} + \frac{2}{n^2}\right). \end{aligned}$$

Thus

$$u(x,y) = v(x,y) + \psi(x) = \left(\frac{1}{\pi} + \frac{h\pi}{2}\right) x - \frac{h}{2}x^2 + \sum_{n=1}^{\infty} A_n e^{-ny} \sin nx.$$

Exercises 13.7

1. Referring to Example 1 in the text we have

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

and

$$T(t) = c_3 e^{-k\lambda^2 t}.$$

From $X'(0) = 0$ (since the left end of the rod is insulated), we find $c_2 = 0$. Then $X(x) = c_1 \cos \lambda x$ and the other boundary condition $X'(1) = -hX(1)$ implies

$$-\lambda \sin \lambda + h \cos \lambda = 0 \quad \text{or} \quad \cot \lambda = \frac{\lambda}{h}.$$

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Denoting the consecutive positive roots of this latter equation by λ_n for $n = 1, 2, 3, \ldots$, we have

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda_n^2 t} \cos \lambda_n x.$$

From the initial condition $u(x,0) = 1$ we obtain

$$1 = \sum_{n=1}^{\infty} A_n \cos \lambda_n x$$

and

$$\begin{aligned} A_n &= \frac{\int_0^1 \cos \lambda_n x \, dx}{\int_0^1 \cos^2 \lambda_n x \, dx} = \frac{\sin \lambda_n / \lambda_n}{\frac{1}{2}\left[1 + \frac{1}{2\lambda_n} \sin 2\lambda_n\right]} \\ &= \frac{2 \sin \lambda_n}{\lambda_n \left[1 + \frac{1}{\lambda_n} \sin \lambda_n \cos \lambda_n\right]} = \frac{2 \sin \lambda_n}{\lambda_n \left[1 + \frac{1}{h\lambda_n} \sin \lambda_n (\lambda_n \sin \lambda_n)\right]} \\ &= \frac{2h \sin \lambda_n}{\lambda_n [h + \sin^2 \lambda_n]}. \end{aligned}$$

The solution is

$$u(x,t) = 2h \sum_{n=1}^{\infty} \frac{\sin \lambda_n}{\lambda_n (h + \sin^2 \lambda_n)} e^{-k\lambda_n^2 t} \cos \lambda_n x.$$

2. Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation gives

$$k \frac{\partial^2 v}{\partial x^2} + k\psi'' = \frac{\partial v}{\partial t}.$$

This equation will be homogeneous if $\psi''(x) = 0$ or $\psi(x) = c_1 x + c_2$. The boundary condition $u(0,t) = 0$ implies $\psi(0) = 0$ which implies $c_2 = 0$. Thus $\psi(x) = c_1 x$. Using the second boundary condition we obtain

$$-\left(\frac{\partial v}{\partial x} + \psi'\right)\Big|_{x=1} = -h[v(1,t) + \psi(1) - u_0],$$

which will be homogeneous when

$$-\psi'(1) = -h\psi(1) + hu_0.$$

Since $\psi(1) = \psi'(1) = c_1$ we have $-c_1 = -hc_1 + hu_0$ and $c_1 = hu_0/(h-1)$. Thus

$$\psi(x) = \frac{hu_0}{h-1} x.$$

The new boundary-value problem is

$$k \frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

$$v(0,t) = 0, \quad \frac{\partial v}{\partial x}\Big|_{x=1} = -hv(1,t), \quad h > 0, \quad t > 0$$

$$v(x,0) = f(x) - \frac{hu_0}{h-1} x, \quad 0 < x < 1.$$

Referring to Example 1 in the text we see that

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda_n^2 t} \sin \lambda_n x$$

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and

$$u(x,t) = v(x,t) + \psi(x) = \frac{hu_0}{h-1}x + \sum_{n=1}^{\infty} A_n e^{-k\lambda_n^2 t} \sin \lambda_n x$$

where

$$f(x) - \frac{hu_0}{h-1}x = \sum_{n=1}^{\infty} A_n \sin \lambda_n x$$

and λ_n is a solution of $\lambda_n \cos \lambda_n = -h \sin \lambda_n$. The coefficients are

$$\begin{aligned} A_n &= \frac{\int_0^1 [f(x) - hu_0 x/(h-1)] \sin \lambda_n x \, dx}{\int_0^1 \sin^2 \lambda_n x \, dx} \\ &= \frac{\int_0^1 [f(x) - hu_0 x/(h-1)] \sin \lambda_n x \, dx}{\frac{1}{2}\left[1 - \frac{1}{2\lambda_n} \sin 2\lambda_n\right]} \\ &= \frac{2\int_0^1 [f(x) - hu_0 x/(h-1)] \sin \lambda_n x \, dx}{1 - \frac{1}{\lambda_n} \sin \lambda_n \cos \lambda_n} \\ &= \frac{2\int_0^1 [f(x) - hu_0 x/(h-1)] \sin \lambda_n x \, dx}{1 - \frac{1}{h\lambda_n}(h \sin \lambda_n) \cos \lambda_n} \\ &= \frac{2\int_0^1 [f(x) - hu_0 x/(h-1)] \sin \lambda_n x \, dx}{1 - \frac{1}{h\lambda_n}(-\lambda_n \cos \lambda_n) \cos \lambda_n} \\ &= \frac{2h}{h + \cos^2 \lambda_n} \int_0^1 \left[f(x) - \frac{hu_0}{h-1}x\right] \sin \lambda_n x \, dx. \end{aligned}$$

3. Separating variables in Laplace's equation gives

$$X'' + \lambda^2 X = 0$$
$$Y'' - \lambda^2 Y = 0$$

and

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$
$$Y(y) = c_3 \cosh \lambda y + c_4 \sinh \lambda y.$$

From $u(0,y) = 0$ we obtain $X(0) = 0$ and $c_1 = 0$. From $u_x(a,y) = -hu(a,y)$ we obtain $X'(a) = -hX(a)$ and

$$\lambda \cos \lambda a = -h \sin \lambda a \quad \text{or} \quad \tan \lambda a = -\frac{\lambda}{h}.$$

Let λ_n, where $n = 1, 2, 3, \ldots$, be the consecutive positive roots of this equation. From $u(x,0) = 0$ we obtain $Y(0) = 0$ and $c_3 = 0$. Thus

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh \lambda_n y \sin \lambda_n x.$$

Now

$$f(x) = \sum_{n=1}^{\infty} A_n \sinh \lambda_n b \sin \lambda_n x$$

and

$$A_n \sinh \lambda_n b = \frac{\int_0^a f(x) \sin \lambda_n x \, dx}{\int_0^a \sin^2 \lambda_n x \, dx}.$$

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Since

$$\int_0^a \sin^2 \lambda_n x\, dx = \frac{1}{2}\left[a - \frac{1}{2\lambda_n}\sin 2\lambda_n a\right] = \frac{1}{2}\left[a - \frac{1}{\lambda_n}\sin\lambda_n a\cos\lambda_n a\right]$$
$$= \frac{1}{2}\left[a - \frac{1}{h\lambda_n}(h\sin\lambda_n a)\cos\lambda_n a\right]$$
$$= \frac{1}{2}\left[a - \frac{1}{h\lambda_n}(-\lambda_n\cos\lambda_n a)\cos\lambda_n a\right] = \frac{1}{2h}\left[ah + \cos^2\lambda_n a\right],$$

we have

$$A_n = \frac{2h}{\sinh\lambda_n b[ah + \cos^2\lambda_n a]}\int_0^a f(x)\sin\lambda_n x\, dx.$$

4. Letting $u(x, y) = X(x)Y(y)$ and separating variables gives

$$X''Y + XY'' = 0.$$

The boundary conditions

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 \qquad \text{and} \qquad \left.\frac{\partial u}{\partial y}\right|_{y=1} = -hu(x, 1)$$

correspond to

$$X(x)Y'(0) = 0 \qquad \text{and} \qquad X(x)Y'(1) = -hX(x)Y(1)$$

or

$$Y'(0) = 0 \qquad \text{and} \qquad Y'(1) = -hY(1).$$

Since these homogeneous boundary conditions are in terms of Y, we separate the differential equation as

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2.$$

Then

$$Y'' + \lambda^2 Y = 0$$

and

$$X'' - \lambda^2 X = 0$$

have solutions

$$Y(y) = c_1\cos\lambda y + c_2\sin\lambda y$$

and

$$X(x) = c_3 e^{-\lambda x} + c_4 e^{\lambda x}.$$

We use exponential functions in the solution of $X(x)$ since $\cosh\lambda x$ and $\sinh\lambda x$ are both unbounded as $x \to \infty$. Now, $Y'(0) = 0$ implies $c_2 = 0$, so $Y(y) = c_1\cos\lambda y$. Since $Y'(y) = -c_1\lambda\sin\lambda y$, the boundary condition $Y'(1) = -hY(1)$ implies

$$-c_1\lambda\sin\lambda = -hc_1\cos\lambda \qquad \text{or} \qquad \cot\lambda = \frac{\lambda}{h}.$$

Consideration of the graphs of $f(\lambda) = \cot\lambda$ and $g(\lambda) = \lambda/h$ show that $\cos\lambda = \lambda h$ has an infinite number of roots. The consecutive positive roots λ_n for $n = 1, 2, 3, \ldots$, are the eigenvalues of the problem. The corresponding eigenfunctions are $Y_n(y) = c_1\cos\lambda_n y$. The condition $\lim_{x\to\infty} u(x, y) = 0$ is equivalent to $\lim_{x\to\infty} X(x) = 0$. Thus $c_4 = 0$ and $X(x) = c_3 e^{-\lambda x}$. Therefore

$$u_n(x, y) = X_n(x)Y_n(x) = A_n e^{-\lambda_n x}\cos\lambda_n y$$

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and by the superposition principle

$$u(x, y) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n x} \cos \lambda_n y.$$

[It is easily shown that there are no eigenvalues corresponding to $\lambda = 0$.] Finally, the condition $u(0, y) = u_0$ implies

$$u_0 = \sum_{n=1}^{\infty} A_n \cos \lambda_n y.$$

This is not a Fourier cosine series since the coefficients λ_n of y are not integer multiples of π/p, where $p = 1$ in this problem. The functions $\cos \lambda_n y$ are however orthogonal since they are eigenfunctions of the Sturm-Lionville problem

$$Y'' + \lambda^2 Y = 0,$$

$$Y'(0) = 0$$

$$Y'(1) + hY(1) = 0,$$

with weight function $p(x) = 1$. Thus we find

$$A_n = \frac{\int_0^1 u_0 \cos \lambda_n y \, dy}{\int_0^1 \cos^2 \lambda_n y \, dy}.$$

Now

$$\int_0^1 u_0 \cos \lambda_n y \, dy = \frac{u_0}{\lambda_n} \sin \lambda_n y \Big|_0^1 = \frac{u_0}{\lambda_n} \sin \lambda_n$$

and

$$\int_0^1 \cos^2 \lambda_n y \, dy = \frac{1}{2} \int_0^1 (1 + \cos 2\lambda_n y) \, dy = \frac{1}{2} \left[y + \frac{1}{2\lambda_n} \sin 2\lambda_n y \right]_0^1$$

$$= \frac{1}{2} \left[1 + \frac{1}{2\lambda_n} \sin 2\lambda_n \right] = \frac{1}{2} \left[1 + \frac{1}{\lambda_n} \sin \lambda_n \cos \lambda_n \right].$$

Since $\cot \lambda = \lambda/h$,

$$\frac{\cos \lambda}{\lambda} = \frac{\sin \lambda}{h}$$

and

$$\int_0^1 \cos^2 \lambda_n y \, dy = \frac{1}{2} \left[1 + \frac{\sin^2 \lambda_n}{h} \right].$$

Then

$$A_n = \frac{\frac{u_0}{\lambda_n} \sin \lambda_n}{\frac{1}{2} \left[1 + \frac{1}{h} \sin^2 \lambda_n \right]} = \frac{2hu_0 \sin \lambda_n}{\lambda_n \left(h + \sin^2 \lambda_n \right)}$$

and

$$u(x, y) = 2hu_0 \sum_{n=1}^{\infty} \frac{\sin \lambda_n}{\lambda_n \left(1 + \sin^2 \lambda_n \right)} e^{-\lambda_n x} \cos \lambda_n y$$

where λ_n for $n = 1, 2, 3, \ldots$ are the consecutive positive roots of $\cot \lambda = \lambda/h$.

5. The boundary-value problem is

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0,$$

$$u(0, t) = 0, \quad \frac{\partial u}{\partial x} \Big|_{x=L} = 0, \quad t > 0,$$

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$$u(x,0) = f(x), \quad 0 < x < L.$$

Separation of variables leads to

$$X'' + \lambda^2 X = 0$$
$$T' + k\lambda^2 T = 0$$

and

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$
$$T(t) = c_3 e^{-k\lambda^2 t}.$$

From $X(0) = 0$ we find $c_1 = 0$. From $X'(L) = 0$ we obtain $\cos \lambda L = 0$ and

$$\lambda = \frac{\pi(2n-1)}{2L}, \ n = 1, 2, 3, \ldots \ .$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k(2n-1)^2\pi^2 t/4L^2} \sin\left(\frac{2n-1}{2L}\right)\pi x$$

where

$$A_n = \frac{\int_0^L f(x) \sin\left(\frac{2n-1}{2L}\right)\pi x \, dx}{\int_0^L \sin^2\left(\frac{2n-1}{2L}\right)\pi x \, dx} = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{2n-1}{2L}\right)\pi x \, dx.$$

6. Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation gives

$$a^2 \frac{\partial^2 v}{\partial x^2} + \psi''(x) = \frac{\partial^2 v}{\partial t^2}.$$

This equation will be homogeneous if $\psi''(x) = 0$ or $\psi(x) = c_1 x + c_2$. The boundary condition $u(0,t) = 0$ implies $\psi(0) = 0$ which implies $c_2 = 0$. Thus $\psi(x) = c_1 x$. Using the second boundary condition, we obtain

$$E\left(\frac{\partial v}{\partial x} + \psi'\right)\Big|_{x=L} = F_0,$$

which will be homogeneous when

$$E\psi'(L) = F_0.$$

Since $\psi'(x) = c_1$ we conclude that $c_1 = F_0/E$ and

$$\psi(x) = \frac{F_0}{E}x.$$

The new boundary-value problem is

$$a^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2}, \quad 0 < x < L, \quad t > 0$$
$$v(0,t) = 0, \quad \frac{\partial v}{\partial x}\Big|_{x=L} = 0, \quad t > 0,$$
$$v(x,0) = -\frac{F_0}{E}x, \quad \frac{\partial v}{\partial t}\Big|_{t=0} = 0, \quad 0 < x < L.$$

Referring to Example 2 in the text we see that

$$v(x,t) = \sum_{n=1}^{\infty} A_n \cos a\left(\frac{2n-1}{2L}\right)\pi t \, \sin\left(\frac{2n-1}{2L}\right)\pi x$$

where

$$-\frac{F_0}{E}x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2L}\right)\pi x$$

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and

$$A_n = \frac{-F_0 \int_0^L x \sin\left(\frac{2n-1}{2L}\right)\pi x\,dx}{E\int_0^L \sin^2\left(\frac{2n-1}{2L}\right)\pi x\,dx} = \frac{8F_0L(-1)^n}{E\pi^2(2n-1)^2}\,.$$

Thus

$$\begin{aligned} u(x,t) &= v(x,t) + \psi(x) \\ &= \frac{F_0}{E}x + \frac{8F_0L}{E\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\cos a\left(\frac{2n-1}{2L}\right)\pi t\,\sin\left(\frac{2n-1}{2L}\right)\pi x. \end{aligned}$$

7. Separation of variables leads to

$$Y'' + \lambda^2 Y = 0$$
$$X'' - \lambda^2 X = 0$$

and

$$Y(y) = c_1\cos\lambda y + c_2\sin\lambda y$$
$$X(x) = c_3\cosh\lambda x + c_4\sinh\lambda x.$$

From $Y(0) = 0$ we find $c_1 = 0$. From $Y'(1) = 0$ we obtain $\cos\lambda = 0$ and

$$\lambda = \frac{\pi(2n-1)}{2}\,,\ n = 1, 2, 3, \ldots\ .$$

Thus

$$Y(y) = c_2\sin\left(\frac{2n-1}{2}\right)\pi y.$$

From $X'(0) = 0$ we find $c_4 = 0$. Then

$$u(x,y) = \sum_{n=1}^{\infty} A_n\cosh\left(\frac{2n-1}{2}\right)\pi x\,\sin\left(\frac{2n-1}{2}\right)\pi y$$

where

$$u_0 = u(1,y) = \sum_{n=1}^{\infty} A_n\cosh\left(\frac{2n-1}{2}\right)\pi\,\sin\left(\frac{2n-1}{2}\right)\pi y$$

and

$$A_n\cosh\left(\frac{2n-1}{2}\right)\pi = \frac{\int_0^1 u_0\sin\left(\frac{2n-1}{2}\right)\pi y\,dy}{\int_0^1 \sin^2\left(\frac{2n-1}{2}\right)\pi y\,dy} = \frac{4u_0}{(2n-1)\pi}\,.$$

Thus

$$u(x,y) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\cosh\left(\frac{2n-1}{2}\right)\pi}\cosh\left(\frac{2n-1}{2}\right)\pi x\,\sin\left(\frac{2n-1}{2}\right)\pi y.$$

8. The boundary-value problem is

$$k\,\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\,,\quad 0 < x < 1,\quad t > 0$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = hu(0,t),\quad \left.\frac{\partial u}{\partial x}\right|_{x=1} = -hu(1,t),\quad h > 0,\quad t > 0,$$
$$u(x,0) = f(x),\quad 0 < x < 1.$$

Referring to Example 1 in the text we have

$$X(x) = c_1\cos\lambda x + c_2\sin\lambda x$$

and

$$T(t) = c_3 e^{-k\lambda^2 t}.$$

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Applying the boundary conditions, we obtain

$$X'(0) = hX(0)$$
$$X'(1) = -hX(1)$$

or

$$\lambda c_2 = hc_1$$
$$-\lambda c_1 \sin\lambda + \lambda c_2 \cos\lambda = -hc_1 \cos\lambda - hc_2 \sin\lambda.$$

Choosing $c_1 = \lambda$ and $c_2 = h$ (to satisfy the first equation above) we obtain

$$-\lambda^2 \sin\lambda + h\lambda\cos\lambda = -h\lambda\cos\lambda - h^2\sin\lambda$$
$$2h\lambda\cos\lambda = (\lambda^2 - h^2)\sin\lambda.$$

The eigenvalues λ_n are the consecutive positive roots of

$$\tan\lambda = \frac{2h\lambda}{\lambda^2 - h^2}\,.$$

Then

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda_n^2 t}(\lambda_n \cos\lambda_n x + h\sin\lambda_n x)$$

where

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n(\lambda_n\cos\lambda_n x + h\sin\lambda_n x)$$

and

$$A_n = \frac{\int_0^1 f(x)(\lambda_n\cos\lambda_n x + h\sin\lambda_n x)dx}{\int_0^1(\lambda_n\cos\lambda_n x + h\sin\lambda_n x)^2 dx}$$
$$= \frac{2}{\lambda_n^2 + 2h + h^2}\int_0^1 f(x)(\lambda_n\cos\lambda_n x + h\sin\lambda_n x)dx.$$

[Note: the evaluation and simplification of the integral in the denominator requires the use of the relationship $(\lambda^2 - h^2)\sin\lambda = 2h\lambda\cos\lambda$.]

9. (a) Using $u = XT$ and separation constant λ^4 we find

$$X^{(4)} - \lambda^4 X = 0$$

and

$$X(x) = c_1\cos\lambda x + c_2\sin\lambda x + c_3\cosh\lambda x + c_4\sinh\lambda x.$$

Since $u = XT$ the boundary conditions become

$$X(0) = 0, \quad X'(0) = 0, \quad X''(1) = 0, \quad X'''(1) = 0.$$

Now $X(0) = 0$ implies $c_1 + c_3 = 0$, while $X'(0) = 0$ implies $c_2 + c_4 = 0$. Thus

$$X(x) = c_1\cos\lambda x + c_2\sin\lambda x - c_1\cosh\lambda x - c_2\sinh\lambda x.$$

The boundary condition $X''(1) = 0$ implies

$$-c_1\cos\lambda - c_2\sin\lambda - c_1\cosh\lambda - c_2\sinh\lambda = 0$$

while the boundary condition $X'''(1) = 0$ implies

$$c_1\sin\lambda - c_2\cos\lambda - c_1\sinh\lambda - c_2\cosh\lambda = 0.$$

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We then have the system of two equations in two unknowns

$$(\cos\lambda + \cosh\lambda)c_1 + (\sin\lambda + \sinh\lambda)c_2 = 0$$
$$(\sin\lambda - \sinh\lambda)c_1 - (\cos\lambda + \cosh\lambda)c_2 = 0.$$

This homogeneous system will have nontrivial solutions for c_1 and c_2 provided

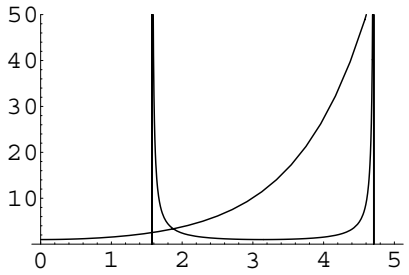
$$\begin{vmatrix} \cos\lambda + \cosh\lambda & \sin\lambda + \sinh\lambda \\ \sin\lambda - \sinh\lambda & -\cos\lambda - \cosh\lambda \end{vmatrix} = 0$$

or

$$-2 - 2\cos\lambda\cosh\lambda = 0.$$

Thus, the eigenvalues are determined by the equation $\cos\lambda\cosh\lambda = -1$.

(b) Using a computer to graph $\cosh\lambda$ and $-1/\cos\lambda = -\sec\lambda$ we see that the first two positive eigenvalues occur near 1.9 and 4.7. Applying Newton's method with these initial values we find that the eigenvalues are $\lambda_1 = 1.8751$ and $\lambda_2 = 4.6941$.



10. (a) In this case the boundary conditions are

$$u(0,t) = 0, \qquad \frac{\partial u}{\partial x}\bigg|_{x=0} = 0$$
$$u(1,t) = 0, \qquad \frac{\partial u}{\partial x}\bigg|_{x=1} = 0.$$

Separating variables leads to

$$X(x) = c_1\cos\lambda x + c_2\sin\lambda x + c_3\cosh\lambda x + c_4\sinh\lambda x$$

subject to

$$X(0) = 0, \quad X'(0) = 0, \quad X(1) = 0, \quad \text{and} \quad X'(1) = 0.$$

Now $X(0) = 0$ implies $c_1 + c_3 = 0$ while $X'(0) = 0$ implies $c_2 + c_4 = 0$. Thus

$$X(x) = c_1\cos\lambda x + c_2\sin\lambda x - c_1\cosh\lambda x - c_2\sinh\lambda x.$$

The boundary condition $X(1) = 0$ implies

$$c_1\cos\lambda + c_2\sin\lambda - c_1\cosh\lambda - c_2\sinh\lambda = 0$$

while the boundary condition $X'(1) = 0$ implies

$$-c_1\sin\lambda + c_2\cos\lambda - c_1\sinh\lambda - c_2\cosh\lambda = 0.$$

We then have the system of two equations in two unknowns

$$(\cos\lambda - \cosh\lambda)c_1 + (\sin\lambda - \sinh\lambda)c_2 = 0$$
$$-(\sin\lambda + \sinh\lambda)c_1 + (\cos\lambda - \cosh\lambda)c_2 = 0.$$

This homogeneous system will have nontrivial solutions for c_1 and c_2 provided

$$\begin{vmatrix} \cos\lambda - \cosh\lambda & \sin\lambda - \sinh\lambda \\ -\sin\lambda - \sinh\lambda & \cos\lambda - \cosh\lambda \end{vmatrix} = 0$$

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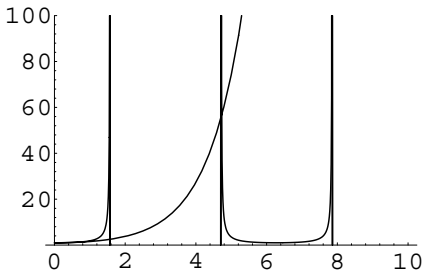
or

$$2 - 2\cos\lambda\cosh\lambda = 0.$$

Thus, the eigenvalues are determined by the equation $\cos\lambda\cosh\lambda = 1$.

(b) Using a computer to graph $\cosh\lambda$ and $1/\cos\lambda = \sec\lambda$ we see that the first two positive eigenvalues occur near the vertical asymptotes of $\sec\lambda$, at $3\pi/2$ and $5\pi/2$. Applying Newton's method with these initial values we find that the eigenvalues are $\lambda_1 = 4.7300$ and $\lambda_2 = 7.8532$.



Exercises 13.8

1. This boundary-value problem was solved in Example 1 in the text. Identifying $b = c = \pi$ and $f(x, y) = u_0$ we have

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} e^{-k(m^2+n^2)t} \sin mx \sin ny$$

where

$$\begin{aligned} A_{mn} &= \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} u_0 \sin mx \sin ny\, dx\, dy \\ &= \frac{4u_0}{\pi^2}\int_0^{\pi} \sin mx\, dx \int_0^{\pi} \sin ny\, dy \\ &= \frac{4u_0}{mn\pi^2}[1 - (-1)^m][1 - (-1)^n]. \end{aligned}$$

2. This boundary-value problem was solved in Example 1 in the text. Identifying $b = c = 1$ and $f(x, y) = xy(x - 1)(y - 1)$ we have

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} e^{-k\pi(m^2+n^2)t} \sin m\pi x \sin n\pi y$$

where

$$\begin{aligned} A_{mn} &= 4\int_0^1\int_0^1 xy(x-1)(y-1)\sin m\pi x \sin n\pi y\, dx\, dy \\ &= 4\left(\int_0^1 x(x-1)\sin m\pi x\, dx\right)\left(\int_0^1 y(y-1)\sin n\pi y\, dy\right) \\ &= 4\left(\frac{2}{m^3\pi^3}\left[(-1)^m - 1\right]\right)\left(\frac{2}{n^3\pi^3}\left[(-1)^n - 1\right]\right) \\ &= \frac{16}{m^3n^3\pi^6}\left[(-1)^m - 1\right]\left[(-1)^n - 1\right]. \end{aligned}$$

In Problems 3 and 4 we need to solve the partial differential equation

$$a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^2 u}{\partial t^2}.$$

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To separate this equation we try $u(x, y, t) = X(x)Y(y)T(t)$:

$$a^2(X''YT + XY''T) = XYT''$$

$$\frac{X''}{X} = -\frac{Y''}{Y} + \frac{T''}{a^2T} = -\lambda^2.$$

Then

$$X'' + \lambda^2 X = 0 \tag{1}$$

$$\frac{Y''}{Y} = \frac{T''}{a^2T} + \lambda^2 = -\mu^2$$

$$Y'' + \mu^2 Y = 0 \tag{2}$$

$$T'' + a^2\left(\lambda^2 + \mu^2\right)T = 0. \tag{3}$$

The general solutions of equations (1), (2), and (3) are, respectively,

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

$$Y(y) = c_3 \cos \mu y + c_4 \sin \mu y$$

$$T(t) = c_5 \cos a\sqrt{\lambda^2 + \mu^2}\, t + c_6 \sin a\sqrt{\lambda^2 + \mu^2}\, t.$$

3. The conditions $X(0) = 0$ and $Y(0) = 0$ give $c_1 = 0$ and $c_3 = 0$. The conditions $X(\pi) = 0$ and $Y(\pi) = 0$ yield two sets of eigenvalues:

$$\lambda = m,\ m = 1, 2, 3, \ldots \quad \text{and} \quad \mu = n,\ n = 1, 2, 3, \ldots .$$

A product solution of the partial differential equation that satisfies the boundary conditions is

$$u_{mn}(x, y, t) = \left(A_{mn} \cos a\sqrt{m^2 + n^2}\, t + B_{mn} \sin a\sqrt{m^2 + n^2}\, t\right) \sin mx \sin ny.$$

To satisfy the initial conditions we use the superposition principle:

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(A_{mn} \cos a\sqrt{m^2 + n^2}\, t + B_{mn} \sin a\sqrt{m^2 + n^2}\, t\right) \sin mx \sin ny.$$

The initial condition $u_t(x, y, 0) = 0$ implies $B_{mn} = 0$ and

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos a\sqrt{m^2 + n^2}\, t \sin mx \sin ny.$$

At $t = 0$ we have

$$xy(x - \pi)(y - \pi) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny.$$

Using (11) and (12) in the text, it follows that

$$\begin{aligned} A_{mn} &= \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} xy(x - \pi)(y - \pi) \sin mx \sin ny\, dx\, dy \\ &= \frac{4}{\pi^2} \int_0^{\pi} x(x - \pi) \sin mx\, dx \int_0^{\pi} y(y - \pi) \sin ny\, dy \\ &= \frac{16}{m^3 n^3 \pi^2}[(-1)^m - 1][(-1)^n - 1]. \end{aligned}$$

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4. The conditions $X(0) = 0$ and $Y(0) = 0$ give $c_1 = 0$ and $c_3 = 0$. The conditions $X(b) = 0$ and $Y(c) = 0$ yield two sets of eigenvalues

$$\lambda = m\pi/b,\ m = 1,\ 2,\ 3,\ \ldots \qquad \text{and} \qquad \mu = n\pi/c,\ n = 1,\ 2,\ 3,\ \ldots .$$

A product solution of the partial differential that satisfies the boundary conditions is

$$u_{mn}(x, y, t) = (A_{mn} \cos a\omega_{mn} t + B_{mn} \sin a\omega_{mn} t) \sin\left(\frac{m\pi}{b} x\right) \sin\left(\frac{n\pi}{c} y\right),$$

where $\omega_{mn} = \sqrt{(m\pi/b)^2 + (n\pi/c)^2}$. To satisfy the initial conditions we use the superposition principle:

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn} \cos a\omega_{mn} t + B_{mn} \sin a\omega_{mn} t) \sin\left(\frac{m\pi}{b} x\right) \sin\left(\frac{n\pi}{c} y\right).$$

At $t = 0$ we have

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi}{b} x\right) \sin\left(\frac{n\pi}{c} y\right)$$

and

$$g(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} a\omega_{mn} \sin\left(\frac{m\pi}{b} x\right) \sin\left(\frac{n\pi}{c} y\right).$$

Using (11) and (12) in the text, it follows that

$$A_{mn} = \frac{4}{bc} \int_0^c \int_0^b f(x, y) \sin\left(\frac{m\pi}{b} x\right) \sin\left(\frac{n\pi}{c} y\right) dx\, dy$$

$$B_{mn} = \frac{4}{abc\omega_{mn}} \int_0^c \int_0^b g(x, y) \sin\left(\frac{m\pi}{b} x\right) \sin\left(\frac{n\pi}{c} y\right) dx\, dy.$$

To separate Laplace's equation in three dimensions we try $u(x, y, z) = X(x)Y(y)Z(z)$:

$$X''YZ + XY''Z + XYZ'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda^2.$$

Then

$$X'' + \lambda^2 X = 0 \tag{4}$$

$$\frac{Y''}{Y} = -\frac{Z''}{Z} + \lambda^2 = -\mu^2$$

$$Y'' + \mu^2 Y = 0 \tag{5}$$

$$Z'' - (\lambda^2 + \mu^2) Z = 0. \tag{6}$$

The general solutions of equations (4), (5), and (6) are, respectively

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

$$Y(y) = c_3 \cos \mu y + c_4 \sin \mu y$$

$$Z(z) = c_5 \cosh \sqrt{\lambda^2 + \mu^2}\, z + c_6 \sinh \sqrt{\lambda^2 + \mu^2}\, z.$$

5. The boundary and initial conditions are

$$u(0, y, z) = 0, \quad u(a, y, z) = 0$$

$$u(x, 0, z) = 0, \quad u(x, b, z) = 0$$

$$u(x, y, 0) = 0, \quad u(x, y, c) = f(x, y).$$

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The conditions $X(0) = Y(0) = Z(0) = 0$ give $c_1 = c_3 = c_5 = 0$. The conditions $X(a) = 0$ and $Y(b) = 0$ yield two sets of eigenvalues:

$$\lambda = \frac{m\pi}{a}\,,\ m = 1, 2, 3, \ldots \quad \text{and} \quad \mu = \frac{n\pi}{b}\,,\ n = 1, 2, 3, \ldots\,.$$

By the superposition principle

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sinh \omega_{mn} z \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$$

where

$$\omega_{mn}^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}$$

and

$$A_{mn} = \frac{4}{ab \sinh \omega_{mn} c} \int_0^b \int_0^a f(x, y) \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \, dx \, dy.$$

6. The boundary and initial conditions are

$$\begin{aligned} u(0, y, z) &= 0, & u(a, y, z) &= 0, \\ u(x, 0, z) &= 0, & u(x, b, z) &= 0, \\ u(x, y, 0) &= f(x, y), & u(x, y, c) &= 0. \end{aligned}$$

The conditions $X(0) = Y(0) = 0$ give $c_1 = c_3 = 0$. The conditions $X(a) = Y(b) = 0$ yield two sets of eigenvalues:

$$\lambda = \frac{m\pi}{a}\,,\ m = 1, 2, 3, \ldots \quad \text{and} \quad \mu = \frac{n\pi}{b}\,,\ n = 1, 2, 3, \ldots\,.$$

Let

$$\omega_{mn}^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\,.$$

Then the boundary condition $Z(c) = 0$ gives

$$c_5 \cosh c\omega_{mn} + c_6 \sinh c\omega_{mn} = 0$$

from which we obtain

$$\begin{aligned} Z(z) &= c_5 \left(\cosh w_{mn} z - \frac{\cosh c\omega_{mn}}{\sinh c\omega_{mn}} \sinh \omega z \right) \\ &= \frac{c_5}{\sinh c\omega_{mn}} (\sinh c\omega_{mn} \cosh \omega_{mn} z - \cosh c\omega_{mn} \sinh \omega_{mn} z) \\ &= c_{mn} \sinh \omega_{mn} (c - z). \end{aligned}$$

By the superposition principle

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sinh \omega_{mn}(c - z) \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$$

where

$$A_{mn} = \frac{4}{ab \sinh c\omega_{mn}} \int_0^b \int_0^a f(x, y) \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \, dx \, dy.$$

7. The boundary and initial conditions are

$$\begin{aligned} u(0, y, z) &= 0, & u(1, y, z) &= 0, \\ u(x, 0, z) &= 0, & u(x, 1, z) &= 0, \\ u(x, y, 0) &= -u_0, & u(x, y, 1) &= u_0. \end{aligned}$$

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Applying the superposition principle to the solutions in Problems 5 and 6, with $a = b = c = 1$ and $f(x, y) = u_0$ in Problem 5 and $f(x, y) = -u_0$ in Problem 6, we get

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}[\sinh \omega_{mn} z - \sinh \omega_m (1 - z)] \sin m\pi x \sin n\pi y$$

where

$$\omega_{mn}^2 = (m^2 + n^2)\pi^2$$

and

$$A_{mn} = \frac{4}{\sinh \omega_{mn}} \int_0^1 \int_0^1 u_0 \sin m\pi x \sin n\pi y \, dx \, dy = \frac{4u_0}{\sinh \omega_{mn}} \left(\int_0^1 \sin m\pi x \, dx \right) \left(\int_0^1 \sin n\pi y \, dy \right)$$

$$= \frac{4u_0}{\sinh \omega_{mn}} \left(\frac{1}{m\pi} [1 - (-1)^n] \right) \left(\frac{1}{n\pi} [1 - (-1)^n] \right) = \frac{4u_0}{mn\pi^2 \sinh \omega_{mn}} [1 - (-1)^m][1 - (-1)^n].$$

Chapter 13 Review Exercises

1. Let $u = XY$ so that

$$u_{yx} = X'Y',$$
$$X'Y' = XY,$$

and

$$\frac{X'}{X} = \frac{Y}{Y'} = \pm\lambda^2.$$

Then

$$X' \mp \lambda^2 X = 0 \quad \text{and} \quad Y \mp \lambda^2 Y' = 0.$$

For $\lambda^2 > 0$ we obtain

$$X = c_1 e^{\lambda^2 x} \quad \text{and} \quad Y = c_2 e^{y/\lambda^2}.$$

For $-\lambda^2 < 0$ we obtain

$$X = c_1 e^{-\lambda^2 x} \quad \text{and} \quad Y = c_2 e^{-y/\lambda^2}.$$

For $\lambda^2 = 0$ we obtain $X = c_1$ and $Y = 0$. The general form is

$$u = XY = A_1 e^{A_2 x + y/A_2}.$$

2. If $u = XY$ then

$$u_x = X'Y,$$
$$u_y = XY',$$
$$u_{xx} = X''Y,$$
$$u_{yy} = XY'',$$

and

$$X''Y + XY'' + 2X'Y + 2XY' = 0$$

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so that

$$(X'' + 2X')Y + X(Y'' + 2Y') = 0,$$
$$\frac{X'' + 2X'}{-X} = \frac{Y'' + 2Y'}{Y} = \pm\lambda^2,$$
$$X'' + 2X' \pm \lambda^2 X = 0,$$

and

$$Y'' + 2Y' \mp \lambda^2 Y = 0.$$

Using λ^2 as a separation constant we obtain

$$Y = c_1 e^{(-1+\sqrt{1+\lambda^2})y} + c_2 e^{(-1-\sqrt{1+\lambda^2})y}.$$

If $1 - \lambda^2 < 0$ then

$$X = e^{-x}\left(c_3 \cos\sqrt{\lambda^2 - 1}\, x + c_4 \sin\sqrt{\lambda^2 - 1}\, x\right).$$

If $1 - \lambda^2 > 0$ then

$$X = c_3 e^{(-1+\sqrt{1-\lambda^2})x} + c_4 e^{(-1-\sqrt{1-\lambda^2})x}.$$

If $1 - \lambda^2 = 0$ then

$$X = c_3 e^{-x} + c_4 x e^{-x}$$

so that

$$\begin{aligned} u = XY &= \left(A_1 e^{(-1+\sqrt{1+A_5})y} + A_2 e^{(-1-\sqrt{1+A_5})y}\right) e^{-x}\left(A_3 \cos\sqrt{A_5 - 1}\, x + A_4 \sin\sqrt{A_5 - 1}\, x\right) \\ &= \left(A_1 e^{(-1+\sqrt{1+A_5})y} + A_2 e^{(-1-\sqrt{1+A_5})y}\right)\left(A_3 e^{(-1+\sqrt{1-A_5})x} + A_4 e^{(-1-\sqrt{1-A_5})x}\right) \\ &= \left(A_1 e^{(-1+\sqrt{2})y} + A_2 e^{(-1-\sqrt{2})y}\right)\left(A_3 e^{-x} + A_4 x e^{-x}\right). \end{aligned}$$

Using $-\lambda^2$ we obtain the same three solutions except that x and y are interchanged. Using $\lambda^2 = 0$ we obtain

$$u = XY = \left(A_1 + A_2 e^{-2x}\right)\left(A_3 + A_4 e^{-2y}\right).$$

3. Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation we obtain

$$k\frac{\partial^2 v}{\partial x^2} + k\psi''(x) = \frac{\partial v}{\partial t}.$$

This equation will be homogeneous provided ψ satisfies

$$k\psi'' = 0 \quad \text{or} \quad \psi = c_1 x + c_2.$$

Considering

$$u(0,t) = v(0,t) + \psi(0) = u_0$$

we set $\psi(0) = u_0$ so that $\psi(x) = c_1 x + u_0$. Now

$$-\frac{\partial u}{\partial x}\bigg|_{x=\pi} = -\frac{\partial v}{\partial x}\bigg|_{x=\pi} - \psi'(x) = v(\pi,t) + \psi(\pi) - u_1$$

is equivalent to

$$\frac{\partial v}{\partial x}\bigg|_{x=\pi} + v(\pi,t) = u_1 - \psi'(x) - \psi(\pi) = u_1 - c_1 - (c_1\pi + u_0),$$

which will be homogeneous when

$$u_1 - c_1 - c_1\pi - u_0 = 0 \quad \text{or} \quad c_1 = \frac{u_1 - u_0}{1 + \pi}.$$

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The steady-state solution is

$$\psi(x) = \left(\frac{u_1 - u_0}{1 + \pi}\right) x + u_0.$$

4. The solution of the problem represents the heat of a thin rod of length π. The left boundary $x = 0$ is kept at constant temperature u_0 for $t > 0$. Heat is lost from the right end of the rod by being in contact with a medium that is held at constant temperature u_1.

5. The boundary-value problem is

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \quad t > 0,$$
$$u(0, t) = 0, \quad u = (1, t) = 0, \quad t > 0,$$
$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < 1.$$

From Section 13.4 in the text we see that $A_n = 0$,

$$B_n = \frac{2}{n\pi a}\int_0^1 g(x) \sin n\pi x \, dx = \frac{2}{n\pi a}\int_{1/4}^{3/4} h \sin n\pi x \, dx$$
$$= \frac{2h}{n\pi a}\left(-\frac{1}{n\pi}\cos n\pi x\right)\Bigg|_{1/4}^{3/4} = \frac{2h}{n^2\pi^2 a}\left(\cos\frac{n\pi}{4} - \cos\frac{3n\pi}{4}\right)$$

and

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin n\pi a t \sin n\pi x.$$

6. The boundary-value problem is

$$\frac{\partial^2 u}{\partial x^2} + x^2 = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \quad t > 0,$$
$$u(0, t) = 1, \quad u(1, t) = 0, \quad t > 0,$$
$$u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad 0 < x < 1.$$

Substituting $u(x, t) = v(x, t) + \psi(x)$ into the partial differential equation gives

$$\frac{\partial^2 v}{\partial x^2} + \psi''(x) + x^2 = \frac{\partial^2 v}{\partial t^2}.$$

This equation will be homogeneous provided $\psi''(x) + x^2 = 0$ or

$$\psi(x) = -\frac{1}{12}x^4 + c_1 x + c_2.$$

From $\psi(0) = 1$ and $\psi(1) = 0$ we obtain $c_1 = -11/12$ and $c_2 = 1$. The new problem is

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2}, \quad 0 < x < 1, \quad t > 0,$$
$$v(0, t) = 0, \quad v(1, t) = 0, \quad t > 0,$$
$$v(x, 0) = f(x) - \psi(x), \quad v_t(x, 0) = 0, \quad 0 < x < 1.$$

From Section 13.4 in the text we see that $B_n = 0$,

$$A_n = 2\int_0^1 [f(x) - \psi(x)] \sin n\pi x \, dx = 2\int_0^1 \left[f(x) + \frac{1}{12}x^4 + \frac{11}{12}x - 1\right] \sin n\pi x \, dx,$$

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and

$$v(x,t) = \sum_{n=1}^{\infty} A_n \cos n\pi t \sin n\pi x.$$

Thus

$$u(x,t) = v(x,t) + \psi(x) = -\frac{1}{12}x^4 - \frac{11}{12}x + 1 + \sum_{n=1}^{\infty} A_n \cos n\pi t \sin n\pi x.$$

7. Using $u = XY$ and λ^2 as a separation constant leads to

$$X'' - \lambda^2 X = 0,$$
$$X(0) = 0,$$

and

$$Y'' + \lambda^2 Y = 0,$$
$$Y(0) = 0,$$
$$Y(\pi) = 0.$$

Then

$$Y = c_1 \sin ny \quad \text{and} \quad X = c_2 \sinh nx$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n \sinh nx \, \sin ny.$$

Imposing

$$u(\pi, y) = 50 = \sum_{n=1}^{\infty} A_n \sinh n\pi \, \sin ny$$

gives

$$A_n \sinh n\pi = \frac{100}{n\pi} \frac{1 - (-1)^n}{\sinh n\pi}$$

so that

$$u(x,y) = \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n \sinh n\pi} \sinh nx \, \sin ny.$$

8. Using $u = XY$ and λ^2 as a separation constant leads to

$$X'' - \lambda^2 X = 0,$$

and

$$Y'' + \lambda^2 Y = 0,$$
$$Y'(0) = 0,$$
$$Y'(\pi) = 0.$$

Then

$$Y = c_1 \cos ny$$

for $n = 0, 1, 2, \ldots$ and

$$X = c_2 \quad \text{or} \quad X = c_2 e^{-nx}$$

for $n = 1, 2, 3, \ldots$ (since u must be bounded as $x \to \infty$) so that

$$u = A_0 + \sum_{n=1}^{\infty} A_n e^{-nx} \cos ny.$$

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Imposing

$$u(0, y) = 50 = A_0 + \sum_{n=1}^{\infty} A_n \cos ny$$

gives

$$A_0 = \frac{1}{\pi} \int_0^{\pi} 50\, dy = 50$$

and

$$A_n = \frac{2}{\pi} \int_0^{\pi} 50 \cos ny\, dy = 0$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x, y) = 50.$$

9. Using $u = XY$ and λ^2 as a separation constant leads to

$$X'' - \lambda^2 X = 0,$$

and

$$Y'' + \lambda^2 Y = 0,$$
$$Y(0) = 0,$$
$$Y(\pi) = 0.$$

Then

$$Y = c_1 \sin ny \quad \text{and} \quad X = c_2 e^{-nx}$$

for $n = 1, 2, 3, \ldots$(since u must be bounded as $x \to \infty$) so that

$$u = \sum_{n=1}^{\infty} A_n e^{-nx} \sin ny.$$

Imposing

$$u(0, y) = 50 = \sum_{n=1}^{\infty} A_n \sin ny$$

gives

$$A_n = \frac{2}{\pi} \int_0^{\pi} 50 \sin ny\, dy = \frac{100}{n\pi}[1 - (-1)^n]$$

so that

$$u(x, y) = \sum_{n=1}^{\infty} \frac{100}{n\pi}[1 - (-1)^n] e^{-nx} \sin ny.$$

10. The boundary-value problem is

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad -L < x < L, \quad t > 0,$$
$$u(-L, t) = 0, \quad u(L, t) = 0, \quad t > 0,$$
$$u(x, 0) = u_0, \quad -L < x < L.$$

Referring to Section 13.3 in the text we have

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

and

$$T(t) = c_3 e^{-k\lambda^2 t}.$$

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Using the boundary conditions $u(-L,0) = X(-L)T(0) = 0$ and $u(L,0) = X(L)T(0) = 0$ we obtain

$$c_1 \cos(-\lambda L) + c_2 \sin(-\lambda L) = 0$$
$$c_1 \cos \lambda L + c_2 \sin \lambda L = 0$$

or

$$c_1 \cos \lambda L - c_2 \sin \lambda L = 0$$
$$c_1 \cos \lambda L + c_2 \sin \lambda L = 0.$$

Adding, we find $\cos \lambda L = 0$ which gives the eigenvalues

$$\lambda = \frac{2n-1}{2L}\pi, \quad n = 1, 2, 3, \ldots .$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{2n-1}{2L}\pi\right)^2 kt} \cos\left(\frac{2n-1}{2L}\right)\pi x.$$

From

$$u(x,0) = u_0 = \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n-1}{2L}\right)\pi x$$

we find

$$A_n = \frac{2\int_0^L u_0 \cos\left(\frac{2n-1}{2L}\right)\pi x\, dx}{2\int_0^L \cos^2\left(\frac{2n-1}{2L}\right)\pi x\, dx} = \frac{u_0(-1)^{n+1}2L/\pi(2n-1)}{L/2} = \frac{4u_0(-1)^{n+1}}{\pi(2n-1)}.$$

11. The coefficients of the series

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx$$

are

$$B_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}[\cos(1-n)x - \cos(1+n)x]\, dx$$
$$= \frac{1}{\pi}\left[\frac{\sin(1-n)x}{1-n}\,\Big|_0^{\pi} - \frac{\sin(1+n)x}{1+n}\,\Big|_0^{\pi}\right] = 0 \text{ for } n \neq 1.$$

For $n = 1$,

$$B_1 = \frac{2}{\pi}\int_0^{\pi} \sin^2 x\, dx = \frac{1}{\pi}\int_0^{\pi} (1 - \cos 2x)\, dx = 1.$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin nx$$

reduces to $u(x,t) = e^{-t}\sin x$ for $n = 1$.

12. Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation gives

$$k\frac{\partial^2 v}{\partial x^2} + k\psi'' + \sin 2\pi x = \frac{\partial v}{\partial t}.$$

This equation will be homogeneous provided ψ satisfies

$$k\psi'' + \sin 2\pi x = 0.$$

The general solution of this equation is

$$\psi(x) = \frac{1}{4k\pi^2}\sin 2\pi x + c_1 x + c_2.$$

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From $\psi(0) = \psi(1) = 0$ we find that $c_1 = c_2 = 0$ and

$$\psi(x) = \frac{1}{4k\pi^2} \sin 2\pi x.$$

Now the new problem is

$$k\,\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}\,, \quad 0 < x < 1, \quad t > 0$$
$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0$$
$$v(x,0) = \sin \pi x - \psi(x), \quad 0 < x < 1.$$

If we let $v = XT$ then

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda^2$$

gives the separated differential equations

$$X'' + \lambda^2 X = 0 \quad \text{and} \quad T' + k\lambda^2 T = 0.$$

The respective solutions are

$$X(x) = c_3 \cos \lambda x + c_4 \sin \lambda x$$
$$T(t) = c_5 e^{-k\lambda^2 t}.$$

From $X(0) = 0$ we get $c_3 = 0$ and from $X(1) = 0$ we find $\lambda = n\pi$ for $n = 1, 2, 3, \ldots$. Consequently, it follows that

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x$$

where

$$v(x,0) = \sin \pi x - \frac{1}{4k\pi^2} \sin 2\pi x = 0$$

implies

$$A_n = 2\int_0^1 \left(\sin \pi x - \frac{1}{4k\pi^2} \sin 2\pi x\right) \sin n\pi x\, dx.$$

By orthogonality $A_n = 0$ for $n = 3, 4, 5, \ldots$, and only A_1 and A_2 can be nonzero. We have

$$A_1 = 2\left[\int_0^1 \sin^2 \pi x\, dx - \frac{1}{4k\pi^2}\int_0^1 \sin 2\pi x \sin \pi x\, dx\right] = 2\int_0^1 \frac{1}{2}(1 - \cos 2\pi x)\, dx = 1$$

and

$$A_2 = 2\left[\int_0^1 \sin \pi x \sin 2\pi x\, dx - \frac{1}{4k\pi^2}\int_0^1 \sin^2 2\pi x\, dx\right]$$
$$= -\frac{1}{2k\pi^2}\int_0^1 \frac{1}{2}(1 - \cos 4\pi x)\, dx = -\frac{1}{4k\pi^2}\,.$$

Therefore

$$v(x,t) = A_1 e^{-k\pi^2 t} \sin \pi x + A_2 e^{-k4\pi^2 t} \sin 2\pi x$$
$$= e^{-k\pi^2 t} \sin \pi x - \frac{1}{4k\pi^2}\, e^{-4k\pi^2 t} \sin 2\pi x$$

and

$$u(x,t) = v(x,t) + \psi(x) = e^{-k\pi^2 t} \sin \pi x + \frac{1}{4k\pi^2}(1 - e^{-4k\pi^2 t}) \sin 2\pi x.$$

13. Using $u = XT$ and $-\lambda^2$ as a separation constant we find

$$X'' + 2X' + \lambda^2 X = 0 \quad \text{and} \quad T'' + 2T' + (1 + \lambda^2)T = 0.$$

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Thus for $\lambda > 1$

$$X = c_1 e^{-x} \cos \sqrt{\lambda^2 - 1}\, x + c_2 e^{-x} \sin \sqrt{\lambda^2 - 1}\, x$$
$$T = c_3 e^{-t} \cos \lambda t + c_4 e^{-t} \sin \lambda t.$$

For $0 \le \lambda \le 1$ we only obtain $X = 0$. Now the boundary conditions $X(0) = 0$ and $X(\pi) = 0$ give, in turn, $c_1 = 0$ and $\sqrt{\lambda^2 - 1}\,\pi = n\pi$ or $\lambda^2 = n^2 + 1$, $n = 1, 2, 3, \ldots$. The corresponding solutions are $X = c_2 e^{-x} \sin nx$. The initial condition $T'(0) = 0$ implies $c_3 = \lambda c_4$ and so

$$T = c_4 e^{-t} \left[\sqrt{n^2 + 1} \cos \sqrt{n^2 + 1}\, t + \sin \sqrt{n^2 + 1}\, t \right].$$

Using $u = XT$ and the superposition principle, a formal series solution is

$$u(x,t) = e^{-(x+t)} \sum_{n=1}^{\infty} A_n \left[\sqrt{n^2 + 1} \cos \sqrt{n^2 + 1}\, t + \sin \sqrt{n^2 + 1}\, t \right] \sin nx.$$

14. Letting $c = XT$ and separating variables we obtain

$$\frac{kX'' - hX'}{X} = \frac{T'}{T} \quad \text{or} \quad \frac{X'' - \alpha X'}{X} = \frac{T'}{kT} = -\lambda^2$$

where $\alpha = h/k$. This leads to the separated differential equations

$$X'' - \alpha X' + \lambda^2 X = 0 \quad \text{and} \quad T' + k\lambda^2 T = 0.$$

The solution of the second equation is

$$T(t) = c_3 e^{-k\lambda^2 t}.$$

For the first equation we have $m = \frac{1}{2}(\alpha \pm \sqrt{\alpha^2 - 4\lambda^2}\,)$, and we consider three cases using the boundary conditions $X(0) = X(1) = 0$:

$\boxed{\alpha^2 > 4\lambda^2}$ The solution is $X = c_1 e^{m_1 x} + c_2 e^{m_2 x}$, where the boundary conditions imply $c_1 = c_2 = 0$, so $X = 0$. (Note in this case that if $\lambda = 0$, the solution is $X = c_1 + c_2 e^{\alpha x}$ and the boundary conditions again imply $c_1 = c_2 = 0$, so $X = 0$.)

$\boxed{\alpha^2 = 4\lambda^2}$ The solution is $X = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$, where the boundary conditions imply $c_1 = c_2 = 0$, so $X = 0$.

$\boxed{\alpha^2 < 4\lambda^2}$ The solution is

$$X(x) = c_1 e^{\alpha x/2} \cos \frac{\sqrt{4\lambda^2 - \alpha^2}}{2}\, x + c_2 e^{\alpha x/2} \sin \frac{\sqrt{4\lambda^2 - \alpha^2}}{2}\, x.$$

From $X(0) = 0$ we see that $c_1 = 0$. From $X(1) = 0$ we find

$$\frac{1}{2}\sqrt{4\lambda^2 - \alpha^2} = n\pi \quad \text{or} \quad \lambda^2 = \frac{1}{4}(4n^2\pi^2 + \alpha^2).$$

Thus

$$X(x) = c_2 e^{\alpha x/2} \sin n\pi x,$$

and

$$c(x,t) = \sum_{n=1}^{\infty} A_n e^{\alpha x/2} e^{-k(4n^2\pi^2 + \alpha^2)t/4} \sin 2\pi x.$$

The initial condition $c(x,0) = c_0$ implies

$$c_0 = \sum_{n=1}^{\infty} A_n e^{\alpha/2} \sin n\pi x. \tag{1}$$

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From the self-adjoint form

$$\frac{d}{dx}\left[e^{-\alpha x}X'\right] + \lambda^2 e^{-\alpha x}X = 0$$

the eigenfunctions are orthogonal on $[0, 1]$ with weight function $e^{-\alpha x}$. That is

$$\int_0^1 e^{-\alpha x}(e^{\alpha x/2}\sin n\pi x)(e^{\alpha x/2}\sin m\pi x)\,dx = 0, \quad n \neq m.$$

Multiplying (1) by $e^{-\alpha x}e^{\alpha x/2}\sin m\pi x$ and integrating we obtain

$$\int_0^1 c_0 e^{-\alpha x}e^{\alpha x/2}\sin m\pi x\,dx = \sum_{n=1}^{\infty} A_n e^{-\alpha x}e^{\alpha x/2}(\sin m\pi x)e^{\alpha x/2}\sin n\pi x\,dx$$

$$c_0\int_0^1 e^{-\alpha x/2}\sin n\pi x\,dx = A_n\int_0^1 \sin^2 n\pi x\,dx = \frac{1}{2}A_n$$

and

$$A_n = 2c_0\int_0^1 e^{-\alpha x/2}\sin n\pi x\,dx = \frac{4c_0[2e^{\alpha/2}n\pi - 2n\pi(-1)^n]}{e^{\alpha/2}(\alpha^2 + 4n^2\pi^2)} = \frac{8n\pi c_0[e^{\alpha/2} - (-1)^n]}{e^{\alpha/2}(\alpha^2 + 4n^2\pi^2)}.$$

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14 Boundary-Value Problems in Other Coordinate Systems

Exercises 14.1

1. We have

$$A_0 = \frac{1}{2\pi}\int_0^\pi u_0\,d\theta = \frac{u_0}{2}$$

$$A_n = \frac{1}{\pi}\int_0^\pi u_0\cos n\theta\,d\theta = 0$$

$$B_n = \frac{1}{\pi}\int_0^\pi u_0\sin n\theta\,d\theta = \frac{u_0}{n\pi}[1-(-1)^n]$$

and so

$$u(r,\theta) = \frac{u_0}{2} + \frac{u_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\,r^n\sin n\theta.$$

2. We have

$$A_0 = \frac{1}{2\pi}\int_0^\pi \theta\,d\theta + \frac{1}{2\pi}\int_\pi^{2\pi}(\pi-\theta)\,d\theta = 0$$

$$A_n = \frac{1}{\pi}\int_0^\pi \theta\cos n\theta\,d\theta + \frac{1}{\pi}\int_\pi^{2\pi}(\pi-\theta)\cos n\theta\,d\theta = \frac{2}{n^2\pi}[(-1)^n-1]$$

$$B_n = \frac{1}{\pi}\int_0^\pi \theta\sin n\theta\,d\theta + \frac{1}{\pi}\int_\pi^{2\pi}(\pi-\theta)\sin n\theta\,d\theta = \frac{1}{n}[1-(-1)^n]$$

and so

$$u(r,\theta) = \sum_{n=1}^{\infty} r^n\left[\frac{(-1)^n-1}{n^2\pi}\cos n\theta + \frac{1-(-1)^n}{n}\sin n\theta\right].$$

3. We have

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi}(2\pi\theta-\theta^2)\,d\theta = \frac{2\pi^2}{3}$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi}(2\pi\theta-\theta^2)\cos n\theta\,d\theta = -\frac{4}{n^2}$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi}(2\pi\theta-\theta^2)\sin n\theta\,d\theta = 0$$

and so

$$u(r,\theta) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{r^n}{n^2}\cos n\theta.$$

4. We have

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi}\theta\,d\theta = \pi$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi}\theta\cos n\theta\,d\theta = 0$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi}\theta\sin n\theta\,d\theta = -\frac{2}{n}$$

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and so

$$u(r,\theta) = \pi - 2\sum_{n=1}^{\infty} \frac{r^n}{n} \sin n\theta.$$

5. As in Example 1 in the text we have $R(r) = c_3 r^n + c_4 r^{-n}$. In order that the solution be bounded as $r \to \infty$ we must define $c_3 = 0$. Hence

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta)$$

where

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta$$

$$A_n = \frac{c^n}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta$$

$$B_n = \frac{c^n}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta.$$

6. We solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < \theta < \frac{\pi}{2}, \quad 0 < r < c,$$

$$u(c,\theta) = f(\theta), \quad 0 < \theta < \frac{\pi}{2},$$

$$u(r,0) = 0, \quad u(r,\pi/2) = 0, \quad 0 < r < c.$$

Proceeding as in Example 1 in the text we obtain the separated differential equations

$$r^2 R'' + rR' - \lambda^2 R = 0$$

$$\Theta'' + \lambda^2 \Theta = 0$$

with solutions

$$\Theta(\theta) = c_1 \cos\lambda\theta + c_2 \sin\lambda\theta$$

$$R(r) = c_3 r^{\lambda} + c_4 r^{-\lambda}.$$

Since we want $R(r)$ to be bounded as $r \to 0$ we require $c_4 = 0$. Applying the boundary conditions $\Theta(0) = 0$ and $\Theta(\pi/2) = 0$ we find that $c_1 = 0$ and $\lambda = 2n$ for $n = 1, 2, 3, \ldots$. Therefore

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n} \sin 2n\theta.$$

From

$$u(c,\theta) = f(\theta) = \sum_{n=1}^{\infty} A_n c^n \sin 2n\theta$$

we find

$$A_n = \frac{4}{\pi c^{2n}}\int_0^{\pi/2} f(\theta)\sin 2n\theta\, d\theta.$$

7. Referring to the solution of Problem 6 above we have

$$\Theta(\theta) = c_1 \cos\lambda\theta + c_2 \sin\lambda\theta$$

$$R(r) = c_3 r^n.$$

Applying the boundary conditions $\Theta'(0) = 0$ and $\Theta'(\pi/2) = 0$ we find that $c_2 = 0$ and $\lambda = 2n$ for

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$n = 0, 1, 2, \ldots$. Therefore

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^{2n} \cos 2n\theta.$$

From

$$u(c,\theta) = \begin{cases} 1, & 0 < \theta < \pi/4 \\ 0, & \pi/4 < \theta < \pi/2 \end{cases} = A_0 + \sum_{n=1}^{\infty} A_n c^{2n} \cos 2n\theta$$

we find

$$A_0 = \frac{1}{\pi/2} \int_0^{\pi/4} d\theta = \frac{1}{2}$$

and

$$c^{2n} A_n = \frac{2}{\pi/2} \int_0^{\pi/4} \cos 2n\theta \, d\theta = \frac{2}{n\pi} \sin \frac{n\pi}{2}.$$

Thus

$$u(r,\theta) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \left(\frac{r}{c}\right)^{2n} \cos 2n\theta.$$

8. We solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < \theta < \pi/4, \quad r > 0$$

$$u(r,0) = 0, \quad r > 0$$

$$u(r,\pi/4) = 30, \quad r > 0.$$

Proceeding as in Example 1 in the text we find the separated ordinary differential equations to be

$$r^2 R'' + rR' - \lambda^2 R = 0$$

$$\Theta'' + \lambda^2 \Theta = 0.$$

The corresponding general solutions are

$$R(r) = c_1 r^{\lambda} + c_2 r^{-\lambda}$$

$$\Theta(\theta) = c_3 \cos \lambda\theta + c_4 \sin \lambda\theta.$$

The condition $\Theta(0) = 0$ implies $c_3 = 0$ so that $\Theta = c_4 \sin \lambda\theta$. Now, in order that the temperature be bounded as $r \to \infty$ we define $c_1 = 0$. Similarly, in order that the temperature be bounded as $r \to 0$ we are forced to define $c_2 = 0$. Thus $R(r) = 0$ and so no nontrivial solution exists for $\lambda > 0$. For $\lambda = 0$ the separated differential equations are

$$r^2 R'' + rR' = 0 \quad \text{and} \quad \Theta'' = 0.$$

Solutions of these latter equations are

$$R(r) = c_1 + c_2 \ln r \quad \text{and} \quad \Theta(\theta) = c_2\theta + c_3.$$

$\Theta(0) = 0$ still implies $c_3 = 0$, whereas boundedness as $r \to 0$ demands $c_2 = 0$. Thus, a product solution is

$$u = c_1 c_2 \theta = A\theta.$$

From $u(r,\pi/4) = 0$ we obtain $A = 120/\pi$. Thus, a solution to the problem is

$$u(r,\theta) = \frac{120}{\pi}\theta.$$

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9. Proceeding as in Example 1 in the text and again using the periodicity of $u(r,\theta)$, we have

$$\Theta(\theta) = c_1 \cos\lambda\theta + c_2 \sin\lambda\theta$$

where $\lambda = n$ for $n = 0, 1, 2, \ldots$. Then

$$R(r) = c_3 r^n + c_4 r^{-n}.$$

[We do not have $c_4 = 0$ in this case since $0 < a \le r$.] Since $u(b,\theta) = 0$ we have

$$u(r,\theta) = A_0 \ln\frac{r}{b} + \sum_{n=1}^{\infty}\left[\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right][A_n \cos n\theta + B_n \sin n\theta].$$

From

$$u(a,\theta) = f(\theta) = A_0 \ln\frac{a}{b} + \sum_{n=1}^{\infty}\left[\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right][A_n \cos n\theta + B_n \sin n\theta]$$

we find

$$A_0 \ln\frac{a}{b} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta,$$

$$\left[\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right] A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta,$$

and

$$\left[\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right] B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta.$$

10. Substituting $u(r,\theta) = v(r,\theta) + \psi(r)$ into the partial differential equation we obtain

$$\frac{\partial^2 v}{\partial r^2} + \psi''(r) + \frac{1}{r}\left[\frac{\partial v}{\partial r} + \psi'(r)\right] + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} = 0.$$

This equation will be homogeneous provided

$$\psi''(r) + \frac{1}{r}\psi'(r) = 0 \quad \text{or} \quad r^2\psi''(r) + r\psi'(r) = 0.$$

The general solution of this Cauchy-Euler differential equation is

$$\psi(r) = c_1 + c_2 \ln r.$$

From

$$u_0 = u(a,\theta) = v(a,\theta) + \psi(a) \quad \text{and} \quad u_1 = u(b,\theta) = v(b,\theta) + \psi(b)$$

we see that in order for the boundary values $v(a,\theta)$ and $v(b,\theta)$ to be 0 we need $\psi(a) = u_0$ and $\psi(b) = u_1$. From this we have

$$\psi(a) = c_1 + c_2 \ln a = u_0$$
$$\psi(b) = c_1 + c_2 \ln b = u_1.$$

Solving for c_1 and c_2 we obtain

$$c_1 = \frac{u_1 \ln a - u_0 \ln b}{\ln a/b} \quad \text{and} \quad c_2 = \frac{u_0 - u_1}{\ln a/b}.$$

Then

$$\psi(r) = \frac{u_1 \ln a - u_0 \ln b}{\ln a/b} + \frac{u_0 - u_1}{\ln a/b}\ln r = \frac{u_0 \ln r/b - u_1 \ln r/a}{\ln a/b}.$$

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From Problem 9 with $f(\theta) = 0$ we see that the solution of

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0, \quad 0 < \theta < 2\pi, \quad a < r < b,$$
$$v(a, \theta) = 0, \quad v(b, \theta) = 0, \quad 0 < \theta < 2\pi$$

is $v(r, \theta) = 0$. Thus the steady-state temperature of the ring is

$$u(r, \theta) = v(r, \theta) + \psi(r) = \frac{u_0 \ln r/b - u_1 \ln r/a}{\ln a/b}.$$

11. We solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < \theta < \pi, \quad a < r < b,$$
$$u(a, \theta) = \theta(\pi - \theta), \quad u(b, \theta) = 0, \quad 0 < \theta < \pi,$$
$$u(r, 0) = 0, \quad u(r, \pi) = 0, \quad a < r < b.$$

Proceeding as in Example 1 in the text we obtain the separated differential equations

$$r^2 R'' + rR' - \lambda^2 R = 0$$
$$\Theta'' + \lambda^2 \Theta = 0$$

with solutions

$$\Theta(\theta) = c_1 \cos \lambda\theta + c_2 \sin \lambda\theta$$
$$R(r) = c_3 r^{\lambda} + c_4 r^{-\lambda}.$$

Applying the boundary conditions $\Theta(0) = 0$ and $\Theta(\pi) = 0$ we find that $c_1 = 0$ and $\lambda = n$ for $n = 1, 2, 3, \ldots$. The boundary condition $R(b) = 0$ gives

$$c_3 b^n + c_4 b^{-n} = 0 \quad \text{and} \quad c_4 = -c_3 b^{2n}.$$

Then

$$R(r) = c_3 \left(r^n - \frac{b^{2n}}{r^n} \right) = c_3 \left(\frac{r^{2n} - b^{2n}}{r^n} \right)$$

and

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n \left(\frac{r^{2n} - b^{2n}}{r^n} \right) \sin n\theta.$$

From

$$u(a, \theta) = \theta(\pi - \theta) = \sum_{n=1}^{\infty} A_n \left(\frac{a^{2n} - b^{2n}}{a^n} \right) \sin n\theta$$

we find

$$A_n \left(\frac{a^{2n} - b^{2n}}{a^n} \right) = \frac{2}{\pi} \int_0^{\pi} (\theta\pi - \theta^2) \sin n\theta \, d\theta = \frac{4}{n^3 \pi}[1 - (-1)^n].$$

Thus

$$u(r, \theta) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \frac{r^{2n} - b^{2n}}{a^{2n} - b^{2n}} \left(\frac{a}{r} \right)^n \sin n\theta.$$

12. Letting $u(r, \theta) = v(r, \theta) + \psi(\theta)$ we obtain $\psi''(\theta) = 0$ and so $\psi(\theta) = c_1\theta + c_2$. From $\psi(0) = 0$ and $\psi(\pi) = u_0$ we find, in turn, $c_2 = 0$ and $c_1 = u_0/\pi$. Therefore $\psi(\theta) = \frac{u_0}{\pi}\theta$. Now $u(1, \theta) = v(1, \theta) + \psi(\theta)$ so that $v(1, \theta) = u_0 - \frac{u_0}{\pi}\theta$. From

$$v(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta \quad \text{and} \quad v(1, \theta) = \sum_{n=1}^{\infty} A_n \sin n\theta$$

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we obtain

$$A_n = \frac{2}{\pi}\int_0^{\pi}\left(u_0 - \frac{u_0}{\pi}\,\theta\right)\sin n\theta\,d\theta = \frac{2u_0}{\pi n}\,.$$

Thus

$$u(r,\theta) = \frac{u_0}{\pi}\,\theta + \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{r^n}{n}\sin n\theta.$$

13. We solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < \theta < \pi\,, \quad 0 < r < 2,$$

$$u(2,\theta) = \begin{cases} u_0, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta < \pi \end{cases}$$

$$\left.\frac{\partial u}{\partial \theta}\right|_{\theta=0} = 0, \quad \left.\frac{\partial u}{\partial \theta}\right|_{\theta=\pi} = 0, \quad 0 < r < 2.$$

Proceeding as in Example 1 in the text we obtain the separated differential equations

$$r^2R'' + rR' - \lambda^2 R = 0$$

$$\Theta'' + \lambda^2\Theta = 0$$

with solutions

$$\Theta(\theta) = c_1\cos\lambda\theta + c_2\sin\lambda\theta$$

$$R(r) = c_3 r^{\lambda} + c_4 r^{-\lambda}.$$

Applying the boundary conditions $\Theta'(0) = 0$ and $\Theta'(\pi) = 0$ we find that $c_2 = 0$ and $\lambda = n$ for $n = 0,\ 1,\ 2,\ \ldots$. Since we want $R(r)$ to be bounded as $r \to 0$ we require $c_4 = 0$. Thus

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\theta.$$

From

$$u(2,\theta) = \begin{cases} u_0, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta < \pi \end{cases} = A_0 + \sum_{n=1}^{\infty} A_n 2^n \cos n\theta$$

we find

$$A_0 = \frac{1}{2}\,\frac{2}{\pi}\int_0^{\pi/2} u_0\,d\theta = \frac{u_0}{2}$$

and

$$2^n A_n = \frac{2u_0}{\pi}\int_0^{\pi/2}\cos n\theta\,d\theta = \frac{2u_0}{\pi}\,\frac{\sin n\pi/2}{n}\,.$$

Therefore

$$u(r,\theta) = \frac{u_0}{2} + \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi}{2}\right)\left(\frac{r}{2}\right)^n\cos n\theta.$$

14. Let u_1 be the solution of the boundary-value problem

$$\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\,\frac{\partial u_1}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 u_1}{\partial \theta^2} = 0, \quad 0 < \theta < \pi, \quad a < r < b$$

$$u_1(a,\theta) = f(\theta), \quad 0 < \theta < \pi$$

$$u_1(b,\theta) = 0, \quad 0 < \theta < \pi,$$

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and let u^2 be the solution of the boundary-value problem

$$\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_1}{\partial \theta^2} = 0, \quad 0 < \theta < \pi, \quad a < r < b$$

$$u_2(a, \theta) = 0, \quad 0 < \theta < \pi$$

$$u_2(b, \theta) = g(\theta), \quad 0 < \theta < \pi.$$

Each of these problems can be solved using the methods shown in Problem 9 of this section. Now if $u(r, \theta) = u_1(t, \theta) + u_2(r, \theta)$, then

$$u(a, \theta) = u_1(a, \theta) + u_2(a, \theta) = f(\theta)$$

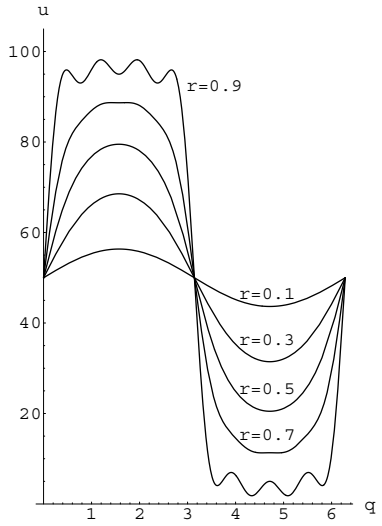
$$u(b, \theta) = u_1(b, \theta) + u_2(b, \theta) = g(\theta)$$

and $u(r, \theta)$ will be the steady-state temperature of the circular ring with boundary conditions $u(a, \theta) = f(\theta)$ and $u(b, \theta) = g(\theta)$.

15. **(a)** From Problem 1 in this section, with $u_0 = 100$,

$$u(r, \theta) = 50 + \frac{100}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} r^n \sin n\theta.$$

(b)



(c) Using a partial sum including the term with r^{99} we find

$u(0.9, 1.3) \approx 96.5268$	$u(0.9, 2\pi - 1.3) \approx 3.4731$
$u(0.7, 2) \approx 87.871$	$u(0.7, 2\pi - 2) \approx 12.129$
$u(0.5, 3.5) \approx 36.0744$	$u(0.5, 2\pi - 3.5) \approx 63.9256$
$u(0.3, 4) \approx 35.2674$	$u(0.3, 2\pi - 4) \approx 64.7326$
$u(0.1, 5.5) \approx 45.4934$	$u(0.1, 2\pi - 5.5) \approx 54.5066$

(d) At the center of the plate $u(0, 0) = 50$. From the graphs in part (b) we observe that the solution curves are symmetric about the point $(\pi, 50)$. In part (c) we observe that the horizontal pairs add up to 100, and hence average 50. This is consistent with the observation about part (b), so it is appropriate to say the average temperature in the plate is 50°.

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Exercises 14.2

1. Referring to the solution of Example 1 in the text we have

$$R(r) = c_1 J_0(\lambda_n r) \quad \text{and} \quad T(t) = c_3 \cos a\lambda_n t + c_4 \sin a\lambda_n t$$

where the eigenvalues λ_n are the positive roots of $J_0(\lambda c) = 0$. Now, the initial condition $u(r, 0) = R(r)T(0) = 0$ implies $c_3 = 0$. Thus

$$u(r,t) = \sum_{n=1}^{\infty} A_n \sin a\lambda_n t J_0(\lambda_n r) \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} a\lambda_n A_n \cos a\lambda_n t J_0(\lambda_n r).$$

From

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 1 = \sum_{n=1}^{\infty} a\lambda_n A_n J_0 \lambda_n r$$

we find

$$\begin{aligned} a\lambda_n A_n &= \frac{2}{c^2 J_1^2(\lambda_n c)} \int_0^c r J_0(\lambda_n r)\, dr \qquad \boxed{x = \lambda_n r,\ dx = \lambda_n\, dr} \\ &= \frac{2}{c^2 J_1^2(\lambda_n c)} \int_0^{\lambda_n c} \frac{1}{\lambda_n^2} x J_0(x)\, dx \\ &= \frac{2}{c^2 J_1^2(\lambda_n c)} \int_0^{\lambda_n c} \frac{1}{\lambda_n^2} \frac{d}{dx}[x J_1(x)]\, dx \qquad \boxed{\text{see (4) of Section 11.5 in text}} \\ &= \frac{2}{c^2 \lambda_n^2 J_1^2(\lambda_n c)} \big(x J_1(x)\big) \Big|_0^{\lambda_n c} = \frac{2}{c\lambda_n J_1(\lambda_n c)}. \end{aligned}$$

Then

$$A_n = \frac{2}{ac\lambda_n^2 J_1(\lambda_n c)}$$

and

$$u(r,t) = \frac{2}{ac} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n^2 J_1(\lambda_n c)} \sin a\lambda_n t.$$

2. From Example 1 in the text we have $B_n = 0$ and

$$A_n = \frac{2}{J_1^2(\lambda_n)} \int_0^1 r(1 - r^2) J_0(\lambda_n r)\, dr.$$

From Problem 10, Exercises 12.6 we obtained $A_n = \dfrac{4J_2(\lambda_n)}{\lambda_n^2 J_1^2(\lambda_n)}$. Thus

$$u(r,t) = 4 \sum_{n=1}^{\infty} \frac{J_2(\lambda_n)}{J_1^2(\lambda_n)} \cos a\lambda_n t J_0(\lambda_n r).$$

3. Referring to Example 2 in the text we have

$$\begin{aligned} R(r) &= c_1 J_0(\lambda r) + c_2 Y_0(\lambda r) \\ Z(z) &= c_3 \cosh \lambda z + c_4 \sinh \lambda z \end{aligned}$$

where $c_2 = 0$ and $J_0(2\lambda) = 0$ defines the positive eigenvalues λ_n. From $Z(4) = 0$ we obtain

$$c_3 \cosh 4\lambda_n + c_4 \sinh 4\lambda_n = 0 \quad \text{or} \quad c_4 = -c_3 \frac{\cosh 4\lambda_n}{\sinh 4\lambda_n}.$$

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Then

$$Z(z) = c_3\left[\cosh\lambda_n z - \frac{\cosh 4\lambda_n}{\sinh 4\lambda_n}\sinh\lambda_n z\right] = c_3\,\frac{\sinh 4\lambda_n\cosh\lambda_n z - \cosh 4\lambda_n\sinh\lambda_n z}{\sinh 4\lambda_n}$$

$$= c_3\,\frac{\sinh\lambda_n(4-z)}{\sinh 4\lambda_n}$$

and

$$u(r,z) = \sum_{n=1}^{\infty} A_n\,\frac{\sinh\lambda_n(4-z)}{\sinh 4\lambda_n}J_0(\lambda_n r).$$

From

$$u(r,0) = u_0 = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$$

we obtain

$$A_n = \frac{2u_0}{4J_1^2(2\lambda_n)}\int_0^2 rJ_0(\lambda_n r)\,dr = \frac{u_0}{\lambda_n J_1(2\lambda_n)}\,.$$

Thus the temperature in the cylinder is

$$u(r,z) = u_0\sum_{n=1}^{\infty}\frac{\sinh\lambda_n(4-z)J_0(\lambda_n r)}{\lambda_n\sinh 4\lambda_n J_1(2\lambda_n)}\,.$$

4. **(a)** The boundary condition $u_r(2,z) = 0$ implies $R'(2) = 0$ or $J_0'(2\lambda) = 0$. Thus $\lambda = 0$ is also an eigenvalue and the separated equations are in this case $rR'' + R' = 0$ and $z'' = 0$. The solutions of these equations are then

$$R(r) = c_1 + c_2\ln r, \quad Z(z) = c_3 z + c_4.$$

Now $Z(0) = 0$ yields $c_4 = 0$ and the implicit condition that the temperature is bounded as $r \to 0$ demands that we define $c_2 = 0$. Using the superposition principle then gives

$$u(r,z) = A_1 z + \sum_{n=2}^{\infty} A_n\sinh\lambda_n z J_0(\lambda_n r). \tag{1}$$

At $z = 4$ we obtain

$$f(r) = 4A_1 + \sum_{n=2}^{\infty} A_n\sinh 4\lambda_n J_0(\lambda_n r).$$

Thus from (19) and (20) of Section 12.6 in the text we can write with $b = 2$,

$$A_1 = \frac{1}{8}\int_0^2 rf(r)\,dr \tag{2}$$

$$A_n = \frac{1}{2\sinh 4\lambda_n J_0^2(2\lambda_n)}\int_0^2 rf(r)J_0(\lambda_n r)\,dr. \tag{3}$$

A solution of the problem consists of the series (1) with coefficients A_1 and A_n defined in (2) and (3), respectively.

(b) When $f(r) = u_0$ we get $A_1 = u_0/4$ and

$$A_n = \frac{u_0 J_1(2\lambda_n)}{\lambda_n\sinh 4\lambda_n J_0^2(2\lambda_n)} = 0$$

since $J_0'(2\lambda) = 0$ is equivalent to $J_1(2\lambda) = 0$. A solution of the problem is then $u(r,z) = \dfrac{u_0}{4}z$.

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5. Letting $u(r,t) = R(r)T(t)$ and separating variables we obtain

$$\frac{R'' + \frac{1}{r}R'}{R} = \frac{T'}{kT} = \mu \quad \text{and} \quad R'' + \frac{1}{r}R' - \mu R = 0, \quad T' - \mu kT = 0.$$

From the second equation we find $T(t) = e^{\mu kt}$. If $\mu > 0$, $T(t)$ increases without bound as $t \to \infty$. Thus we assume $\mu = -\lambda^2 \leq 0$. Now

$$R'' + \frac{1}{r}R' + \lambda^2 R = 0$$

is a parametric Bessel equation with solution

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r).$$

Since Y_0 is unbounded as $r \to 0$ we take $c_2 = 0$. Then $R(r) = c_1 J_0(\lambda r)$ and the boundary condition $u(c,t) = R(c)T(t) = 0$ implies $J_0(\lambda c) = 0$. This latter equation defines the positive eigenvalues λ_n. Thus

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) e^{-\lambda_n^2 kt}.$$

From

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$$

we find

$$A_n = \frac{2}{c^2 J_1^2(\lambda_n c)} \int_0^c r J_0(\lambda_n r) f(r)\, dr, \;\; n = 1, 2, 3, \ldots .$$

6. If the edge $r = c$ is insulated we have the boundary condition $u_r(c,t) = 0$. Referring to the solution of Problem 5 above we have

$$R'(c) = \lambda c_1 J_0'(\lambda c) = 0$$

which defines an eigenvalue $\lambda = 0$ and positive eigenvalues λ_n. Thus

$$u(r,t) = A_0 + \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) e^{-\lambda_n^2 kt}.$$

From

$$u(r,0) = f(r) = A_0 + \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$$

we find

$$A_0 = \frac{2}{c^2} \int_0^c r f(r)\, dr$$

$$A_n = \frac{2}{c^2 J_0^2(\lambda_n c)} \int_0^c r J_0(\lambda_n r) f(r)\, dr.$$

7. Referring to Problem 5 above we have $T(t) = e^{-\lambda^2 kt}$ and $R(r) = c_1 J_0(\lambda r)$. The boundary condition $hu(1,t) + u_r(1,t) = 0$ implies $hJ_0(\lambda) + \lambda J_0'(\lambda) = 0$ which defines positive eigenvalues λ_n. Now

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) e^{-\lambda_n^2 kt}$$

where

$$A_n = \frac{2\lambda_n^2}{(\lambda_n^2 + h^2) J_0^2(\lambda_n)} \int_0^1 r J_0(\lambda_n r) f(r)\, dr.$$

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8. We solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < 1, \quad z > 0$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = -hu(1, z), \quad z > 0$$

$$u(r, 0) = u_0, \quad 0 < r < 1.$$

assuming $u = RZ$ we get

$$\frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = -\lambda^2$$

and so

$$rR'' + R' + \lambda^2 rR = 0 \quad \text{and} \quad Z'' - \lambda^2 Z = 0.$$

Therefore

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r) \quad \text{and} \quad Z(z) = c_3 e^{-\lambda z} + c_4 e^{\lambda z}.$$

We use the exponential form of the solution of $Z'' - \lambda^2 Z = 0$ since the domain of the variable z is a semi-infinite interval. As usual we define $c_2 = 0$ since the temperature is surely bounded as $r \to 0$. Hence $R(r) = c_1 J_0(\lambda r)$. Now the boundary-condition $u_r(1, z) + hu(1, z) = 0$ is equivalent to

$$\lambda J_0'(\lambda) + hJ_0(\lambda) = 0. \tag{4}$$

The eigenvalues λ_n are the positive roots of (4). Finally, we must now define $c_4 = 0$ since the temperature is also expected to be bounded as $z \to \infty$. A product solution of the partial differential equation that satisfies the first boundary condition is given by

$$u_n(r, z) = A_n e^{-\lambda_n z} J_0(\lambda_n r).$$

Therefore

$$u(r, z) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n z} J_0(\lambda_n r)$$

is another formal solution. At $z = 0$ we have $u_0 = A_n J_0(\lambda_n r)$. In view of (4) we use (17) and (18) of Section 12.6 in the text with the identification $b = 1$:

$$A_n = \frac{2\lambda_n^2}{(\lambda_n^2 + h^2)\, J_0^2(\lambda_n)} \int_0^1 r J_0(\lambda_n r) u_0 \, dr = \frac{2\lambda_n^2 u_0}{(\lambda_n^2 + h^2)\, J_0^2(\lambda_n)\lambda_n^2} t J_1(t)\Big|_0^{\lambda_n} = \frac{2\lambda_n u_0 J_1(\lambda_n)}{(\lambda_n^2 + h^2)\, J_0^2(\lambda_n)}. \tag{5}$$

Since $J_0' = -J_1$ [see (5) of Section 12.6] it follows from (4) that $\lambda_n J_1(\lambda_n) = hJ_0(\lambda_n)$. Thus (5) simplifies to

$$A_n = \frac{2u_0 h}{(\lambda_n^2 + h^2)\, J_0^2(\lambda_n)}.$$

A solution to the boundary-value problem is then

$$u(r, z) = 2u_0 h \sum_{n=1}^{\infty} \frac{e^{-\lambda_n z}}{(\lambda_n^2 + h^2)\, J_0(\lambda_n)} J_0(\lambda_n r).$$

9. Substituting $u(r, t) = v(r, t) + \psi(r)$ into the partial differential equation gives

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \psi'' + \frac{1}{r}\psi' = \frac{\partial v}{\partial t}.$$

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This equation will be homogeneous provided $\psi'' + \frac{1}{r}\psi' = 0$ or $\psi(r) = c_1 \ln r + c_2$. Since $\ln r$ is unbounded as $r \to 0$ we take $c_1 = 0$. Then $\psi(r) = c_2$ and using $u(2,t) = v(2,t) + \psi(2) = 100$ we set $c_2 = \psi(r) = 100$. Referring to Problem 5 above, the solution of the boundary-value problem

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{\partial v}{\partial t}, \quad 0 < r < 2, \quad t > 0,$$
$$v(2,t) = 0, \quad t > 0,$$
$$v(r,0) = u(r,0) - \psi(r)$$

is

$$v(r,t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) e^{-\lambda_n^2 t}$$

where

$$\begin{aligned} A_n &= \frac{2}{2^2 J_1^2(2\lambda_n)} \int_0^2 r J_0(\lambda_n r)[u(r,0) - \psi(r)]\, dr \\ &= \frac{1}{2J_1^2(2\lambda_n)} \left[\int_0^1 r J_0(\lambda_n r)[200 - 100]\, dr + \int_1^2 r J_0(\lambda_n r)[100 - 100]\, dr \right] \\ &= \frac{50}{J_1^2(2\lambda_n)} \int_0^1 r J_0(\lambda_n r)\, dr \qquad \boxed{x = \lambda_n r,\ dx = \lambda_n\, dr} \\ &= \frac{50}{J_1^2(2\lambda_n)} \int_0^{\lambda_n} \frac{1}{\lambda_n^2} x J_0(x)\, dx \\ &= \frac{50}{\lambda_n^2 J_1^2(2\lambda_n)} \int_0^{\lambda_n} \frac{d}{dx}[x J_1(x)]\, dx \qquad \boxed{\text{see (4) of Section 12.6 in text}} \\ &= \frac{50}{\lambda_n^2 J_1^2(2\lambda_n)} \big(x J_1(x)\big) \Big|_0^{\lambda_n} = \frac{50 J_1(\lambda_n)}{\lambda_n J_1^2(2\lambda_n)}. \end{aligned}$$

Thus

$$u(r,t) = v(r,t) + \psi(r) = 100 + 50 \sum_{n=1}^{\infty} \frac{J_1(\lambda_n) J_0(\lambda_n r)}{\lambda_n J_1^2(2\lambda_n)} e^{-\lambda^2 t}.$$

10. Letting $u(r,t) = u(r,t) + \psi(r)$ we obtain $r\psi'' + \psi' = -\beta r$. The general solution of this nonhomogeneous equation is found with the aid of variation of parameters: $\psi = c_1 + c_2 \ln r - \beta \dfrac{r^2}{4}$. In order that this solution be bounded as $r \to 0$ we define $c_2 = 0$. Using $\psi(1) = 0$ then gives $c_1 = \dfrac{\beta}{4}$ and so $\psi(r) = \dfrac{\beta}{4}(1 - r^2)$. Using $v = RT$ we find that a solution of

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{\partial v}{\partial t}, \quad 0 < r < 1, \quad t > 0$$
$$v(1,t) = 0, \quad t > 0$$
$$v(r,0) = -\psi(r), \quad 0 < r < 1$$

is

$$v(r,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 t} J_0(\lambda_n r)$$

where

$$A_n = -\frac{\beta}{4} \frac{2}{J_1^2(\lambda_n)} \int_0^1 r(1 - r^2) J_0(\lambda_n r)\, dr$$

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and the eigenvalues are defined by $J_0(\lambda) = 0$. From the result of Problem 10, Exercises 12.6 (see also Problem 2 of this exercise set) we get

$$A_n = -\frac{\beta J_2(\lambda_n)}{\lambda_n^2 J_1^2(\lambda_n)}\,.$$

Thus from $u = v + \psi(r)$ it follows that

$$u(r,t) = \frac{\beta}{4}(1-r^2) - \beta\sum_{n=1}^{\infty}\frac{J_2(\lambda_n)}{\lambda_n^2 J_1^2(\lambda_n)}\,e^{-\lambda_n^2 t}J_0(\lambda_n r).$$

11. (a) Writing the partial differential equation in the form

$$g\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial t^2}$$

and separating variables we obtain

$$\frac{xX'' + X'}{X} = \frac{T''}{gT} = -\lambda^2.$$

Then

$$xX'' + X' + \lambda^2 X = 0 \quad \text{and} \quad T'' + g\lambda^2 T = 0.$$

Letting $x = \tau^2/4$ in the first equation we obtain $dx/d\tau = \tau/2$ or $d\tau/dx = 2\tau$. Then

$$\frac{dX}{dx} = \frac{dX}{d\tau}\,\frac{d\tau}{dx} = \frac{2}{\tau}\,\frac{dX}{d\tau}$$

and

$$\begin{aligned}\frac{d^2X}{dx^2} &= \frac{d}{dx}\left(\frac{2}{\tau}\,\frac{dX}{d\tau}\right) = \frac{2}{\tau}\,\frac{d}{dx}\left(\frac{dX}{d\tau}\right) + \frac{dX}{d\tau}\,\frac{d}{dx}\left(\frac{2}{\tau}\right)\\ &= \frac{2}{\tau}\,\frac{d}{d\tau}\left(\frac{dX}{d\tau}\right)\frac{d\tau}{dx} + \frac{dX}{d\tau}\,\frac{d}{d\tau}\left(\frac{2}{\tau}\right)\frac{d\tau}{dx} = \frac{4}{\tau^2}\,\frac{d^2X}{d\tau^2} - \frac{4}{\tau^3}\,\frac{dX}{d\tau}\,.\end{aligned}$$

Thus

$$xX'' + X' + \lambda^2 X = \frac{\tau^2}{4}\left(\frac{4}{\tau^2}\,\frac{d^2X}{d\tau^2} - \frac{4}{\tau^3}\,\frac{dX}{d\tau}\right) + \frac{2}{\tau}\,\frac{dX}{d\tau} + \lambda^2 X = \frac{d^2X}{d\tau^2} + \frac{1}{\tau}\,\frac{dX}{d\tau} + \lambda^2 X = 0.$$

This is a parametric Bessel equation with solution

$$X(\tau) = c_1 J_0(\lambda\tau) + c_2 Y_0(\lambda\tau).$$

(b) To insure a finite solution at $x = 0$ (and thus $\tau = 0$) we set $c_2 = 0$. The condition $u(L,t) = X(L)T(t) = 0$ implies $X\,\big|_{x=L} = X\,\big|_{\tau=2\sqrt{L}} = c_1 J_0(2\lambda\sqrt{L}\,) = 0$, which defines positive eigenvalues λ_n. The solution of $T'' + g\lambda^2 T = 0$ is

$$T(t) = c_3\cos\lambda_n\sqrt{g}\,t + c_4\sin\lambda_n\sqrt{g}\,t.$$

The boundary condition $u_t(x,0) = X(x)T'(0) = 0$ implies $c_4 = 0$. Thus

$$u(\tau,t) = \sum_{n=1}^{\infty} A_n\cos\lambda_n\sqrt{g}\,t J_0(\lambda_n\tau).$$

From

$$u(\tau,0) = f(\tau^2/4) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n\tau)$$

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we find

$$A_n = \frac{2}{(2\sqrt{L})^2 J_1^2(2\lambda_n\sqrt{L})}\int_0^{2\sqrt{L}} \tau J_0(\lambda_n\tau) f(\tau^2/4)\,d\tau \qquad \boxed{v = \tau/2,\ dv = d\tau/2}$$

$$= \frac{1}{2LJ_1^2(2\lambda_n\sqrt{L})}\int_0^{\sqrt{L}} 2vJ_0(2\lambda_n v) f(v^2) 2\,dv$$

$$= \frac{2}{LJ_1^2(2\lambda_n\sqrt{L})}\int_0^{\sqrt{L}} vJ_0(2\lambda_n v) f(v^2)\,dv.$$

12. (a) We need to solve $J_0(10\lambda) = 0$. From the Table 5.2 in Section 5.3 in the text we see that $\lambda_1 = 0.2405$, $\lambda_2 = 0.5520$, and $\lambda_3 = 0.8654$. Using a CAS to compute A_1 we find:

$$A_1 = \frac{2}{10^2 J_1^2(10\lambda_1)}\int_0^{10} rJ_0(\lambda_1 r)(1 - r/10)\,dr = 0.7845.$$

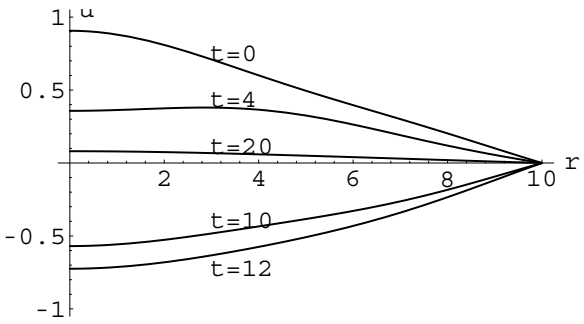
In a similar fashion we compute $A_2 = 0.0687$ and $A_3 = 0.0531$. Then, from Example 1 in the text with $a = 1$ and $g(r) = 0$ [so that $B_n = 0$] we have

$$u(r,t) \approx \sum_{n=1}^{3} A_n(\cos\lambda_n t) J_0(\lambda_n r)$$

$$= 0.7845\cos(0.2405t)J_0(0.2405r) + 0.0687\cos(0.5520t)J_0(0.5520r)$$

$$+ 0.0531\cos(0.8654t)J_0(0.8654r).$$

(b)



13. Because of the nonhomogeneous boundary condition $u(c,t) = 200$ we use the substitution $u(r,t) = v(r,t) + \psi(r)$. This gives

$$k\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \psi'' + \frac{1}{r}\psi'\right) = \frac{\partial v}{\partial t}.$$

This equation will be homogeneous provided $\psi'' + (1/r)\psi' = 0$ or $\psi(r) = c_1 \ln r + c_2$. Since $\ln r$ is unbounded as $r \to 0$ we take $c_1 = 0$. Then $\psi(r) = c_2$ and using $u(c,t) = v(c,t) + c_2 = 200$ we set $c_2 = 200$, giving $v(c,t) = 0$. Referring to Problem 5 in this section, the solution of the boundary-value problem

$$k\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) = \frac{\partial v}{\partial t}, \quad 0 < r < c,\ t > 0$$

$$v(c,t) = 0, \quad t > 0$$

$$v(r,0) = -200, \quad 0 < r < c$$

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is

$$v(r,t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) e^{-\lambda_n^2 kt},$$

where the eigenvalues λ_n are defined by $J_0(\lambda c) = 0$, and

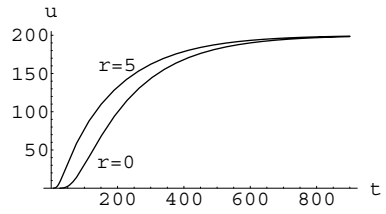
$$A_n = \frac{2}{c^2 J_1^2(\lambda_n c)} \int_0^c r J_0(\lambda_n r)(-200)dr = -\frac{400}{c_2 J_1^2(\lambda_n c)} \int_0^c r J_0(\lambda_n r)dr.$$

Taking $c = 10$ and $k = 0.1$ we have

$$u(r,t) = v(r,t) + 200 = 200 + \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) e^{-0.1\lambda n^2 t},$$

where the eigenvalues λ_n are defined by $J_0(10\lambda) = 0$ and

$$A_n = -\frac{4}{J_1^2(10\lambda_n)} \int_0^{10} r J_0(\lambda_n r)dr.$$



Using a CAS we find that the first five eigenvalues are $\lambda_1 = 0.240483$, $\lambda_2 = 0.552008$, $\lambda_3 = 0.865373$, $\lambda_4 = 1.17915$, and $\lambda_5 = 1.49309$. From the graph we see that $u(5,t) \approx 100$ when $t = 132$, $u(0,t) \approx 100$ when $t = 198$, $u(5,t) \approx 200$ when $t = 808$, and $u(0,t) \approx 200$ when $t = 885$.

14. (a) The boundary-value problem is

$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < 1,\ t > 0$$

$$u(1,t) = 0, \quad t > 0$$

$$u(r,0) = 0, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \begin{cases} -v_0, & 0 \le r < b \\ 0, & b \le r < 1 \end{cases}, \quad 0 < r < 1,$$

and the solution is

$$u(r,t) = \sum_{n=1}^{\infty} (A_n \cos a\lambda_n t + B_n \sin a\lambda_n t) J_0(\lambda_n r),$$

where the eigenvalues λ_n are defined by $J_0(\lambda) = 0$ and $A_n = 0$ since $f(r) = 0$. The coefficients B_n are given by

$$\begin{aligned} B_n &= \frac{2}{a\lambda_n J_1^2(\lambda_n)} \int_0^c r J_0(\lambda_n r) g(r)\, dr = -\frac{2v_0}{a\lambda_n J_1^2(\lambda_n)} \int_0^b r J_0(\lambda_n r) dr \\ &\qquad \boxed{\text{let } x = \lambda_n r} \\ &= -\frac{2v_0}{a\lambda_n J_1^2(\lambda_n)} \int_0^{\lambda_n b} \frac{x}{\lambda_n} J_0(x) \frac{1}{\lambda_n}\, dx = -\frac{2v_0}{a\lambda_n^3 J_1^2(\lambda_n)} \int_0^{\lambda_n b} x J_0(x)\, dx \\ &= -\frac{2v_0}{a\lambda_n^3 J_1^2(\lambda_n)} \big(x J_1(x)\big)\Big|_0^{\lambda_n b} = -\frac{2v_0}{a\lambda_n^3 J_1(\lambda_n)} \big(\lambda_n b J_1(\lambda_n b)\big) \\ &= -\frac{2v_0 b}{a\lambda_n^2} \frac{J_1(\lambda_n b)}{J_1^2(\lambda_n)}. \end{aligned}$$

Thus,

$$u(r,t) = \frac{-2v_0 b}{a} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \frac{J_1(\lambda_n b)}{J_1^2(\lambda_n)} \sin(a\lambda_n t) J_0(\lambda_n r).$$

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(b) The standing wave $u_n(r,t)$ is given by $u_n(r,t) = B_n \sin(a\lambda_n t) J_0(\lambda_n r)$, which has frequency $f_n = a\lambda_n/2\pi$, where λ_n is the nth positive zero of $J_0(x)$. The fundamental frequency is $f_1 = a\lambda_1/2\pi$. The next two frequencies are

$$f_2 = \frac{a\lambda_2}{2\pi} = \frac{\lambda_2}{\lambda_1}\left(\frac{a\lambda_1}{2\pi}\right) = \frac{5.520}{2.405} f_1 = 2.295 f_1$$

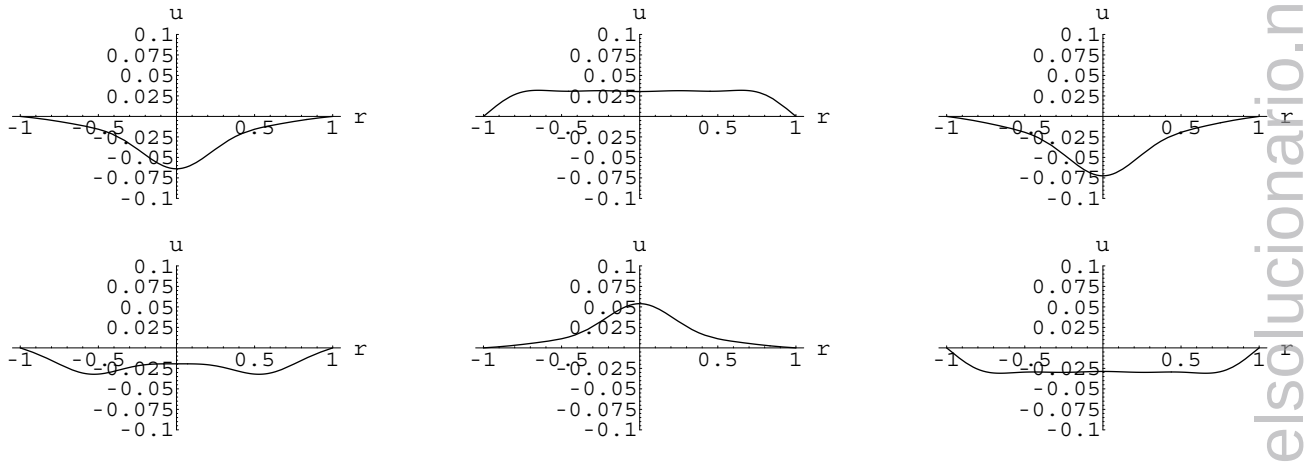
and

$$f_3 = \frac{a\lambda_3}{2\pi} = \frac{\lambda_3}{\lambda_1}\left(\frac{a\lambda_1}{2\pi}\right) = \frac{8.654}{2.405} f_1 = 3.598 f_1.$$

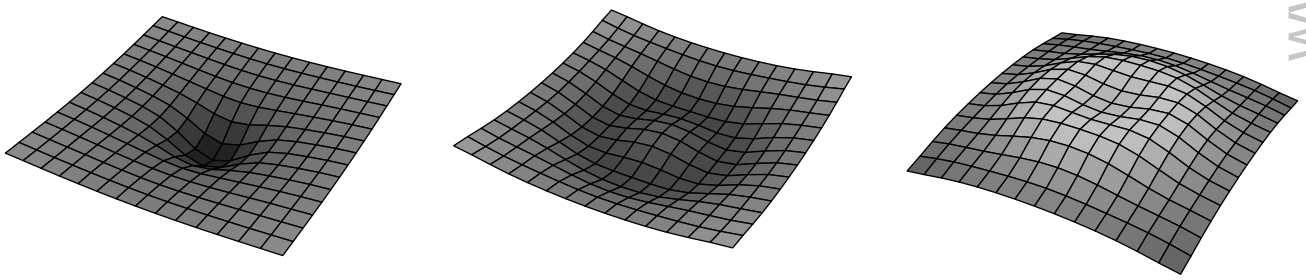
(c) With $a = 1$, $b = \frac{1}{4}$, and $v_0 = 1$, the solution becomes

$$u(r,t) = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \frac{J_1(\lambda_n/4)}{J_1(\lambda_n)} \sin(\lambda_n t) J_0(\lambda_n r).$$

The graphs of $S_5(r,t)$ for $t = 1, 2, 3, 4, 5, 6$ are shown below.



(d) Three frames from the movie are shown.



15. (a) First we see that

$$\frac{R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''}{R\Theta} = \frac{T''}{a^2 T} = -\lambda^2.$$

This gives $T'' + a^2\lambda^2 T = 0$. Then from

$$\frac{R'' + \frac{1}{r}R' + \lambda^2 R}{-R/r^2} = \frac{\Theta''}{\Theta} = -\nu^2$$

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we get $\Theta'' + \nu^2\Theta = 0$ and $r^2R'' + rR' + (\lambda^2 r^2 - \nu^2)R = 0$.

(b) The general solutions of the differential equations in part (a) are

$$T = c_1 \cos a\lambda t + c_2 \sin a\lambda t$$
$$\Theta = c_3 \cos \nu\theta + c_4 \cos \nu\theta$$
$$R = c_5 J_\nu(\lambda r) + c_6 Y_\nu(\lambda r).$$

(c) Implicitly we expect $u(r, \theta, t) = u(r, \theta + 2\pi, t)$ and so Θ must be 2π-periodic. Therefore $\nu = n$, $n = 0, 1, 2, \ldots$. The corresponding eigenfunctions are $1, \cos\theta, \cos 2\theta, \ldots, \sin\theta, \sin 2\theta, \ldots$. Arguing that $u(r, \theta, t)$ is bounded as $r \to 0$ we then define $c_6 = 0$ and so $R = c_3 J_n(\lambda r)$. But $R(c) = 0$ gives $J_n(\lambda c) = 0$; this equation defines the eigenvalues λ_n. For each n, $\lambda_{ni} = x_{ni}/c$, $i = 1, 2, 3, \ldots$.

(d) $$u(r, \theta, t) = \sum_{i=1}^{n} (A_{0i} \cos a\lambda_{0i} t + B_{0i} \sin a\lambda_{0i} t) J_0(\lambda_{0i} r)$$
$$+ \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \Big[(A_{ni} \cos a\lambda_{ni} t + B_{ni} \sin a\lambda_{ni} t) \cos n\theta$$
$$+ (C_{ni} \cos a\lambda_{ni} t + D_{ni} \sin a\lambda_{ni} t) \sin n\theta \Big] J_n(\lambda_{ni} r)$$

Exercises 14.3

1. To compute

$$A_n = \frac{2n+1}{2c^n} \int_0^\pi f(\theta) P_n(\cos\theta) \sin\theta \, d\theta$$

we substitute $x = \cos\theta$ and $dx = -\sin\theta \, d\theta$. Then

$$A_n = \frac{2n+1}{2c^n} \int_1^{-1} F(x) P_n(x)(-dx) = \frac{2n+1}{2c^n} \int_{-1}^{1} F(x) P_n(x) \, dx$$

where

$$F(x) = \begin{cases} 0, & -1 < x < 0 \\ 50, & 0 < x < 1 \end{cases} = 50 \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}.$$

The coefficients A_n are computed in Example 3 of Section 12.6. Thus

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$
$$= 50 \left[\frac{1}{2} P_0(\cos\theta) + \frac{3}{4} \left(\frac{r}{c}\right) P_1(\cos\theta) - \frac{7}{16} \left(\frac{r}{c}\right)^3 P_3(\cos\theta) + \frac{11}{32} \left(\frac{r}{c}\right)^5 P_5(\cos\theta) + \cdots \right].$$

2. In the solution of the Cauchy-Euler equation,

$$R(r) = c_1 r^n + c_2 r^{-(n+1)},$$

we define $c_1 = 0$ since we expect the potential u to be bounded as $r \to \infty$. Hence

$$u_n(r, \theta) = A_n r^{-(n+1)} P_n(\cos\theta)$$

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^{-(n+1)} P_n(\cos\theta).$$

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When $r = c$ we have

$$f(\theta) = \sum_{n=0}^{\infty} A_n c^{-(n+1)} P_n(\cos\theta)$$

so that

$$A_n = c^{n+1}\frac{(2n+1)}{2}\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta\, d\theta.$$

The solution of the problem is then

$$u(r,\theta) = \sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta d\theta\right)\left(\frac{c}{r}\right)^{n+1} P_n(\cos\theta).$$

3. The coefficients are given by

$$A_n = \frac{2n+1}{2c^n}\int_0^{\pi}\cos\theta P_n(\cos\theta)\sin\theta\, d\theta = \frac{2n+1}{2c^n}\int_0^{\pi} P_1(\cos\theta)P_n(\cos\theta)\sin\theta\, d\theta$$

$$\boxed{x = \cos\theta,\ dx = -\sin\theta\, d\theta}$$

$$= \frac{2n+1}{2c^n}\int_{-1}^{1} P_1(x)P_n(x)\, dx.$$

Since $P_n(x)$ and $P_m(x)$ are orthogonal for $m \neq n$, $A_n = 0$ for $n \neq 1$ and

$$A_1 = \frac{2(1)+1}{2c^1}\int_{-1}^{1} P_1(x)P_1(x)\, dx = \frac{3}{2c}\int_{-1}^{1} x^2 dx = \frac{1}{c}.$$

Thus

$$u(r,\theta) = \frac{r}{c}P_1(\cos\theta) = \frac{r}{c}\cos\theta.$$

4. The coefficients are given by

$$A_n = \frac{2n+1}{2c^n}\int_0^{\pi}(1-\cos 2\theta)P_n(\cos\theta)\sin\theta\, d\theta.$$

These were computed in Problem 18 of Section 12.6. Thus

$$u(r,\theta) = \frac{4}{3}P_0(\cos\theta) - \frac{4}{3}\left(\frac{r}{c}\right)^2 P_2(\cos\theta).$$

5. Referring to Example 1 in the text we have

$$\Theta = P_n(\cos\theta) \quad \text{and} \quad R = c_1 r^n + c_2 r^{-(n+1)}.$$

Since $u(b,\theta) = R(b)\Theta(\theta) = 0$,

$$c_1 b^n + c_2 b^{-(n+1)} = 0 \quad \text{or} \quad c_1 = -c_2 b^{-2n-1},$$

and

$$R(r) = -c_2 b^{-2n-1} r^n + c_2 r^{-(n+1)} = c_2\left(\frac{b^{2n+1} - r^{2n+1}}{b^{2n+1}r^{n+1}}\right).$$

Then

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n \frac{b^{2n+1} - r^{2n+1}}{b^{2n+1}r^{n+1}} P_n(\cos\theta)$$

where

$$\frac{b^{2n+1} - a^{2n+1}}{b^{2n+1}a^{n+1}} A_n = \frac{2n+1}{2}\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta\, d\theta.$$

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6. Referring to Example 1 in the text we have

$$R(r) = c_1 r^n \quad \text{and} \quad \Theta(\theta) = P_n(\cos\theta).$$

Now $\Theta(\pi/2) = 0$ implies that n is odd, so

$$u(r,\theta) = \sum_{n=0}^{\infty} A_{2n+1} r^{2n+1} P_{2n+1}(\cos\theta).$$

From

$$u(c,\theta) = f(\theta) = \sum_{n=0}^{\infty} A_{2n+1} c^{2n+1} P_{2n+1}(\cos\theta)$$

we see that

$$A_{2n+1} c^{2n+1} = (4n+3) \int_0^{\pi/2} f(\theta) \sin\theta \, P_{2n+1}(\cos\theta)\, d\theta.$$

Thus

$$u(r,\theta) = \sum_{n=0}^{\infty} A_{2n+1} r^{2n+1} P_{2n+1}(\cos\theta)$$

where

$$A_{2n+1} = \frac{4n+3}{c^{2n+1}} \int_0^{\pi/2} f(\theta) \sin\theta \, P_{2n+1}(\cos\theta)\, d\theta.$$

7. Referring to Example 1 in the text we have

$$r^2 R'' + 2rR' - \lambda^2 R = 0$$
$$\sin\theta\, \Theta'' + \cos\theta\, \Theta' + \lambda \sin\theta\, \Theta = 0.$$

Substituting $x = \cos\theta$, $0 \le \theta \le \pi/2$, the latter equation becomes

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \lambda^2 \Theta = 0, \quad 0 \le x \le 1.$$

Taking the solutions of this equation to be the Legendre polynomials $P_n(x)$ corresponding to $\lambda^2 = n(n+1)$ for $n = 1, 2, 3, \ldots$, we have $\Theta = P_n(\cos\theta)$. Since

$$\left.\frac{\partial u}{\partial \theta}\right|_{\theta=\pi/2} = \Theta'(\pi/2) R(r) = 0$$

we have

$$\Theta'(\pi/2) = -(\sin\pi/2) P_n'(\cos\pi/2) = -P_n'(0) = 0.$$

As noted in the hint, $P_n'(0) = 0$ only if n is even. Thus $\Theta = P_n(\cos\theta)$, $n = 0, 2, 4, \ldots$. As in Example 1, $R(r) = c_1 r^n$. Hence

$$u(r,\theta) = \sum_{n=0}^{\infty} A_{2n} r^{2n} P_{2n}(\cos\theta).$$

At $r = c$,

$$f(\theta) = \sum_{n=0}^{\infty} A_{2n} c^{2n} P_{2n}(\cos\theta).$$

Using Problem 19 in Section 12.6, we obtain

$$c^{2n} A_{2n} = (4n+1) \int_{\pi/2}^{0} f(\theta) P_{2n}(\cos\theta)(-\sin\theta)\, d\theta$$

and

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$$A_{2n} = \frac{4n+1}{c^{2n}} \int_0^{\pi/2} f(\theta) \sin\theta \, P_{2n}(\cos\theta)\, d\theta.$$

8. Referring to Example 1 in the text we have

$$R(r) = c_1 r^n + c_2 r^{-(n-1)} \quad \text{and} \quad \Theta(\theta) = P_n(\cos\theta).$$

Since we expect $u(r,\theta)$ to be bounded as $r \to \infty$, we define $c_1 = 0$. Also $\Theta(\pi/2) = 0$ implies that n is odd, so

$$u(r,\theta) = \sum_{n=0}^{\infty} A_{2n+1} r^{-2(n+1)} P_{2n+1}(\cos\theta).$$

From

$$u(c,\theta) = f(\theta) = \sum_{n=0}^{\infty} A_{2n+1} c^{-2(n+1)} P_{2n+1}(\cos\theta)$$

we see that

$$A_{2n+1} c^{-2(n+1)} = (4n+3) \int_0^{\pi/2} f(\theta) \sin\theta \, P_{2n+1}(\cos\theta)\, d\theta.$$

Thus

$$u(r,\theta) = \sum_{n=0}^{\infty} A_{2n+1} r^{-2(n+1)} P_{2n+1}(\cos\theta)$$

where

$$A_{2n+1} = (4n+3) c^{2(n+1)} \int_0^{\pi/2} f(\theta) \sin\theta \, P_{2n+1}(\cos\theta)\, d\theta.$$

9. Checking the hint, we find

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r} + u\right] = \frac{1}{r}\left[r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} + \frac{\partial u}{\partial r}\right] = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}.$$

The partial differential equation then becomes

$$\frac{\partial^2}{\partial r^2}(ru) = r\frac{\partial u}{\partial t}.$$

Now, letting $ru(r,t) = v(r,t) + \psi(r)$, since the boundary condition is nonhomogeneous, we obtain

$$\frac{\partial^2}{\partial r^2}\left[v(r,t) + \psi(r)\right] = r\frac{\partial}{\partial t}\left[\frac{1}{r}v(r,t) + \psi(r)\right]$$

or

$$\frac{\partial^2 v}{\partial r^2} + \psi''(r) = \frac{\partial v}{\partial t}.$$

This differential equation will be homogeneous if $\psi''(r) = 0$ or $\psi(r) = c_1 r + c_2$. Now

$$u(r,t) = \frac{1}{r}v(r,t) + \frac{1}{r}\psi(r) \quad \text{and} \quad \frac{1}{r}\psi(r) = c_1 + \frac{c_2}{r}.$$

Since we want $u(r,t)$ to be bounded as r approaches 0, we require $c_2 = 0$. Then $\psi(r) = c_1 r$. When $r = 1$

$$u(1,t) = v(1,t) + \psi(1) = v(1,t) + c_1 = 100,$$

and we will have the homogeneous boundary condition $v(1,t) = 0$ when $c_1 = 100$. Consequently, $\psi(r) = 100r$. The initial condition

$$u(r,0) = \frac{1}{r}v(r,0) + \frac{1}{r}\psi(r) = \frac{1}{r}v(r,0) + 100 = 0$$

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implies $v(r,0) = -100r$. We are thus led to solve the new boundary-value problem

$$\frac{\partial^2 v}{\partial r^2} = \frac{\partial v}{\partial t}, \quad 0 < r < 1, \quad t > 0,$$

$$v(1,t) = 0, \quad \lim_{r\to 0} \frac{1}{r} v(r,t) < \infty,$$

$$v(r,0) = -100r.$$

Letting $v(r,t) = R(r)T(t)$ and separating variables leads to

$$R'' + \lambda^2 R = 0 \quad \text{and} \quad T' + \lambda^2 T = 0$$

with solutions

$$R(r) = c_3 \cos \lambda r + c_4 \sin \lambda r \quad \text{and} \quad T(t) = c_5 e^{-\lambda^2 t}.$$

The boundary conditions are equivalent to $R(1) = 0$ and $\lim_{r\to 0} \frac{1}{r} R(r) < \infty$. Since

$$\lim_{r\to 0} \frac{1}{r} R(r) = \lim_{r\to 0} \frac{c_3 \cos \lambda r}{r} + \lim_{r\to 0} \frac{c_4 \sin \lambda r}{r} = \lim_{r\to 0} \frac{c_3 \cos \lambda r}{r} + c_4 \lambda < \infty$$

we must have $c_3 = 0$. Then $R(r) = c_4 \sin \lambda r$, and $R(1) = 0$ implies $\lambda = n\pi$ for $n = 1, 2, 3, \ldots$. Thus

$$v_n(r,t) = A_n e^{-n^2\pi^2 t} \sin n\pi r$$

for $n = 1, 2, 3, \ldots$. Using the condition $\lim_{r\to 0} \frac{1}{r} R(r) < \infty$ it is easily shown that there are no eigenvalues for $\lambda = 0$, nor does setting the common constant to $+\lambda^2$ when separating variables lead to any solutions. Now, by the superposition principle,

$$v(r,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \sin n\pi r.$$

The initial condition $v(r,0) = -100r$ implies

$$-100r = \sum_{n=1}^{\infty} A_n \sin n\pi r.$$

This is a Fourier sine series and so

$$A_n = 2\int_0^1 (-100r \sin n\pi r)\, dr = -200\left[-\frac{r}{n\pi}\cos n\pi r\Big|_0^1 + \int_0^1 \frac{1}{n\pi}\cos n\pi r\, dr\right]$$

$$= -200\left[-\frac{\cos n\pi}{n\pi} + \frac{1}{n^2\pi^2}\sin n\pi r\Big|_0^1\right] = -200\left[-\frac{(-1)^n}{n\pi}\right] = \frac{(-1)^n 200}{n\pi}.$$

A solution of the problem is thus

$$u(r,t) = \frac{1}{r}v(r,t) + \frac{1}{r}\psi(r) = \frac{1}{r}\sum_{n=1}^{\infty}(-1)^n \frac{20}{n\pi} e^{-n^2\pi^2 t}\sin n\pi r + \frac{1}{r}(100r)$$

$$= \frac{200}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} e^{-n^2\pi^2 t}\sin n\pi r + 100.$$

10. Referring to Problem 9 we have

$$\frac{\partial^2 v}{\partial r^2} + \psi''(r) = \frac{\partial v}{\partial t}$$

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where $\psi(r) = c_1 r$. Since

$$u(r,t) = \frac{1}{r}v(r,t) + \frac{1}{r}\psi(r) = \frac{1}{r}v(r,t) + c_1$$

we have

$$\frac{\partial u}{\partial r} = \frac{1}{r}v_r(r,t) - \frac{1}{r^2}v(r,t).$$

When $r = 1$,

$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = v_r(1,t) - v(1,t)$$

and

$$\left.\frac{\partial u}{\partial r}\right|_{r=1} + hu(1,t) = v_r(1,t) - v(1,t) + h[v(1,t) + \psi(1)] = v_r(1,t) + (h-1)v(1,t) + hc_1.$$

Thus the boundary condition

$$\left.\frac{\partial u}{\partial r}\right|_{r=1} + hu(1,t) = hu_1$$

will be homogeneous when $hc_1 = hu_1$ or $c_1 = u_1$. Consequently $\psi(r) = u_1 r$. The initial condition

$$u(r,0) = \frac{1}{r}v(r,0) + \frac{1}{r}\psi(r) = \frac{1}{r}v(r,0) + u_1 = u_0$$

implies $v(r,0) = (u_0 - u_1)r$. We are thus led to solve the new boundary-value problem

$$\begin{gathered}
\frac{\partial^2 v}{\partial r^2} = \frac{\partial v}{\partial t}, \quad 0 < r < 1, \quad t > 0, \\
v_r(1,t) + (h-1)v(1,t) = 0, \quad t > 0, \\
\lim_{r\to 0}\frac{1}{r}v(r,t) < \infty, \\
v(r,0) = (u_0 - u_1)r.
\end{gathered}$$

Separating variables as in Problem 9 leads to

$$R(r) = c_3\cos\lambda r + c_4\sin\lambda r \quad \text{and} \quad T(t) = c_5 e^{-\lambda^2 t}.$$

The boundary conditions are equivalent to

$$R'(1) + (h-1)R(1) = 0 \quad \text{and} \quad \lim_{r\to 0}\frac{1}{r}R(r) < \infty.$$

As in Problem 6 we use the second condition to determine that $c_3 = 0$ and $R(r) = c_4\sin\lambda r$. Then

$$R'(1) + (h-1)R(1) = c_4\lambda\cos\lambda + c_4(h-1)\sin\lambda = 0$$

and the eigenvalues λ_n are the consecutive nonnegative roots of $\tan\lambda = \lambda/(1-h)$. Now

$$v(r,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 t}\sin\lambda_n r.$$

From

$$v(r,0) = (u_0 - u_1)r = \sum_{n=1}^{\infty} A_n \sin\lambda_n r$$

we obtain

$$A_n = \frac{\displaystyle\int_0^1 (u_0 - u_1)r\sin\lambda_n r\,dr}{\displaystyle\int_0^1 \sin^2\lambda_n r\,dr}.$$

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We compute the integrals

$$\int_0^1 r\sin\lambda_n r\,dr = \left(\frac{1}{\lambda_n^2}\sin\lambda_n r - \frac{1}{\lambda_n}\cos\lambda_n r\right)\Big|_0^1 = \frac{1}{\lambda_n^2}\sin\lambda_n - \frac{1}{\lambda_n}\cos\lambda_n$$

and

$$\int_0^1 \sin^2\lambda_n r\,dr = \left(\frac{1}{2}r - \frac{1}{4\lambda_n}\sin 2\lambda_n r\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{4\lambda_n}\sin 2\lambda_n.$$

Using $\lambda_n\cos\lambda_n = -(h-1)\sin\lambda_n$ we then have

$$A_n = (u_0-u_1)\frac{\frac{1}{\lambda_n^2}\sin\lambda_n - \frac{1}{\lambda_n}\cos\lambda_n}{\frac{1}{2} - \frac{1}{4\lambda_n}\sin 2\lambda_n} = (u_0-u_1)\frac{4\sin\lambda_n - 4\lambda_n\cos\lambda_n}{2\lambda_n^2 - \lambda_n 2\sin\lambda_n\cos\lambda_n}$$

$$= 2(u_0-u_1)\frac{\sin\lambda_n + (h-1)\sin\lambda_n}{\lambda_n^2 + (h-1)\sin\lambda_n\sin\lambda_n} = 2(u_0-u_1)h\frac{\sin\lambda_n}{\lambda_n^2 + (h-1)\sin^2\lambda_n}.$$

Therefore

$$u(r,t) = \frac{1}{r}v(r,t) + \frac{1}{r}\psi(r) = u_1 + 2(u_0-u_1)h\sum_{n=1}^{\infty}\frac{\sin\lambda_n\sin\lambda_n r}{r[\lambda_n^2 + (h-1)\sin^2\lambda_n]}\,e^{-\lambda_n^2 t}.$$

11. We write the differential equation in the form

$$a^2\,\frac{1}{r}\,\frac{\partial^2}{\partial r^2}\,(ru) = \frac{\partial^2 u}{\partial t^2} \quad \text{or} \quad a^2\,\frac{\partial^2}{\partial r^2}\,(ru) = r\,\frac{\partial^2 u}{\partial t^2},$$

and then let $v(r,t) = ru(r,t)$. The new boundary-value problem is

$$a^2\,\frac{\partial^2 v}{\partial r^2} = \frac{\partial^2 v}{\partial t^2}, \quad 0 < r < c, \quad t > 0$$

$$v(c,t) = 0, \quad t > 0$$

$$v(r,0) = rf(r), \quad \frac{\partial v}{\partial t}\,\Big|_{t=0} = rg(r).$$

Letting $v(r,t) = R(r)T(t)$ and separating variables we obtain

$$R'' + \lambda^2 R = 0$$

$$T'' + a^2\lambda^2 T = 0$$

and

$$R(r) = c_1\cos\lambda r + c_2\sin\lambda r$$

$$T(t) = c_3\cos a\lambda t + c_4\sin a\lambda t.$$

Since $u(r,t) = v(r,t)/r$, in order to insure boundedness at $r = 0$ we define $c_1 = 0$. Then $R(r) = c_2\sin\lambda r$ and the condition $R(c) = 0$ implies $\lambda = n\pi/c$. Thus

$$v(r,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{c}\,t + B_n\sin\frac{n\pi a}{c}\,t\right)\sin\frac{n\pi}{c}\,r.$$

From

$$v(r,0) = rf(r) = \sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{c}\,r$$

we see that

$$A_n = \frac{2}{c}\int_0^c rf(r)\sin\frac{n\pi}{c}\,r\,dr.$$

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From

$$\left.\frac{\partial v}{\partial t}\right|_{t=0} = rg(r) = \sum_{n=1}^{\infty}\left(B_n\,\frac{n\pi a}{c}\right)\sin\frac{n\pi}{c}\,r$$

we see that

$$B_n = \frac{c}{n\pi a}\cdot\frac{2}{c}\int_0^c rg(r)\sin\frac{n\pi}{c}\,r\,dr = \frac{2}{n\pi a}\int_0^c rg(r)\sin\frac{n\pi}{c}\,r\,dr.$$

12. Proceeding as in Example 1 we obtain

$$\Theta(\theta) = P_n(\cos\theta) \quad \text{and} \quad R(r) = c_1 r^n + c_2 r^{-(n+1)}$$

so that

$$u(r,\theta) = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-(n+1)})P_n(\cos\theta).$$

To satisfy $\lim_{r\to\infty} u(r,\theta) = -Er\cos\theta$ we must have $A_n = 0$ for $n = 2,\ 3,\ 4,\ \ldots$. Then

$$\lim_{r\to\infty} u(r,\theta) = -Er\cos\theta = A_0\cdot 1 + A_1 r\cos\theta,$$

so $A_0 = 0$ and $A_1 = -E$. Thus

$$u(r,\theta) = -Er\cos\theta + \sum_{n=0}^{\infty} B_n r^{-(n+1)}P_n(\cos\theta).$$

Now

$$u(c,\theta) = 0 = -Ec\cos\theta + \sum_{n=0}^{\infty} B_n c^{-(n+1)}P_n(\cos\theta)$$

so

$$\sum_{n=0}^{\infty} B_n c^{-(n+1)}P_n(\cos\theta) = Ec\cos\theta$$

and

$$B_n c^{-(n+1)} = \frac{2n+1}{2}\int_0^{\pi} Ec\cos\theta\,P_n(\cos\theta)\sin\theta\,d\theta.$$

Now $\cos\theta = P_1(\cos\theta)$ so, for $n \neq 1$,

$$\int_0^{\pi}\cos\theta\,P_n(\cos\theta)\sin\theta\,d\theta = 0$$

by orthogonality. Thus $B_n = 0$ for $n \neq 1$ and

$$B_1 = \frac{3}{2}\,Ec^3\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta = Ec^3.$$

Therefore,

$$u(r,\theta) = -Er\cos\theta + Ec^3 r^{-2}\cos\theta.$$

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Chapter 14 Review Exercises

1. We have

$$A_0 = \frac{1}{2\pi}\int_0^{\pi} u_0\,d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi}(-u_0)\,d\theta = 0$$

$$A_n = \frac{1}{c^n\pi}\int_0^{\pi} u_0\cos n\theta\,d\theta + \frac{1}{c^n\pi}\int_{\pi}^{2\pi}(-u_0)\cos n\theta\,d\theta = 0$$

$$B_n = \frac{1}{c^n\pi}\int_0^{\pi} u_0\sin n\theta\,d\theta + \frac{1}{c^n\pi}\int_{\pi}^{2\pi}(-u_0)\sin n\theta\,d\theta = \frac{2u_0}{c^n n\pi}[1-(-1)^n]$$

and so

$$u(r,\theta) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\left(\frac{r}{c}\right)^n \sin n\theta.$$

2. We have

$$A_0 = \frac{1}{2\pi}\int_0^{\pi/2} d\theta + \frac{1}{2\pi}\int_{3\pi/2}^{2\pi} d\theta = \frac{1}{2}$$

$$A_n = \frac{1}{c^n\pi}\int_0^{\pi/2}\cos n\theta\,d\theta + \frac{1}{c^n\pi}\int_{3\pi/2}^{2\pi}\cos n\theta\,d\theta = \frac{1}{c^n n\pi}\left[\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right]$$

$$B_n = \frac{1}{c^n\pi}\int_0^{\pi/2}\sin n\theta\,d\theta + \frac{1}{c^n\pi}\int_{3\pi/2}^{2\pi}\sin n\theta\,d\theta = \frac{1}{c^n n\pi}\left[\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right]$$

and so

$$u(r,\theta) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{c}\right)^n\left[\frac{\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}}{n}\cos n\theta + \frac{\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}}{n}\sin n\theta\right].$$

3. The conditions $\Theta(0) = 0$ and $\Theta(\pi) = 0$ applied to $\Theta = c_1\cos\lambda\theta + c_2\sin\lambda\theta$ give $c_1 = 0$ and $\lambda = n$, $n = 1, 2, 3, \ldots$, respectively. Thus we have the Fourier sine-series coefficients

$$A_n = \frac{2}{\pi}\int_0^{\pi} u_0(\pi\theta - \theta^2)\sin n\theta\,d\theta = \frac{4u_0}{n^3\pi}[1-(-1)^n].$$

Thus

$$u(r,\theta) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\,r^n\sin n\theta.$$

4. In this case

$$A_n = \frac{2}{\pi}\int_0^{\pi}\sin\theta\sin n\theta\,d\theta = \frac{1}{\pi}\int_0^{\pi}[\cos(1-n)\theta - \cos(1+n)\theta]\,d\theta = 0, \quad n \neq 1.$$

For $n = 1$,

$$A_1 = \frac{2}{\pi}\int_0^{\pi}\sin^2\theta\,d\theta = \frac{1}{\pi}\int_0^{\pi}(1-\cos 2\theta)\,d\theta = 1.$$

Thus

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta$$

reduces to

$$u(r,\theta) = r\sin\theta.$$

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5. We solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < \theta < \frac{\pi}{4}, \quad \frac{1}{2} < r < 1,$$

$$u(r,0) = 0, \quad u(r,\pi/4) = 0, \quad \frac{1}{2} < r < 1,$$

$$u(1/2,\theta) = u_0, \quad u_r(1,\theta) = 0, \quad 0 < \theta < \frac{\pi}{4}.$$

Proceeding as in Example 1 in Section 14.1 in the text we obtain the separated equations

$$r^2R'' + rR' - \lambda^2 R = 0$$

$$\Theta'' + \lambda^2\Theta = 0$$

with solutions

$$\Theta(\theta) = c_1\cos\lambda\theta + c_2\sin\lambda\theta$$

$$R(r) = c_3 r^{\lambda} + c_4 r^{-\lambda}.$$

Applying the boundary conditions $\Theta(0) = 0$ and $\Theta(\pi/4) = 0$ gives $c_1 = 0$ and $\lambda = 4n$ for $n = 1, 2, 3, \ldots$. From $R_r(1) = 0$ we obtain $c_3 = c_4$. Therefore

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n\left(r^{4n} + r^{-4n}\right)\sin 4n\theta.$$

From

$$u(1/2,\theta) = u_0 = \sum_{n=1}^{\infty} A_n\left(\frac{1}{2^{4n}} + \frac{1}{2^{-4n}}\right)\sin 4n\theta$$

we find

$$A_n\left(\frac{1}{2^{4n}} + \frac{1}{2^{-4n}}\right) = \frac{2}{\pi/4}\int_0^{\pi/4} u_0\sin 4n\theta\, d\theta = \frac{2u_0}{n\pi}[1-(-1)^n]$$

or

$$A_n = \frac{2u_0}{n\pi(2^{4n} + 2^{-4n})}[1-(-1)^n].$$

Thus the steady-state temperature in the plate is

$$u(r,\theta) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{[r^{4n} + r^{-4n}][1-(-1)^n]}{n[2^{4n} + 2^{-4n}]}\sin 4n\theta.$$

6. We solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad r > 1, \quad 0 < \theta < \pi,$$

$$u(r,0) = 0, \quad u(r,\pi) = 0, \quad r > 1,$$

$$u(1,\theta) = f(\theta), \quad 0 < \theta < \pi.$$

Separating variables we obtain

$$\Theta(\theta) = c_1\cos\lambda\theta + c_2\sin\lambda\theta$$

$$R(r) = c_3 r^{\lambda} + c_4 r^{-\lambda}.$$

Applying the boundary conditions $\Theta(0) = 0$, and $\Theta(\pi) = 0$ gives $c_1 = 0$ and $\lambda = n$ for $n = 1, 2, 3, \ldots$. Assuming $f(\theta)$ to be bounded, we expect the solution $u(r,\theta)$ to also be bounded as $r \to \infty$. This requires that $c_3 = 0$. Therefore

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{-n}\sin n\theta.$$

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From

$$u(1,\theta) = f(\theta) = \sum_{n=1}^{\infty} A_n \sin n\theta$$

we obtain

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta.$$

7. Letting $u(r,t) = R(r)T(t)$ and separating variables we obtain

$$\frac{R'' + \frac{1}{r}R' - hR}{R} = \frac{T'}{T} = \mu$$

so

$$R'' + \frac{1}{r}R' - (\mu + h)R = 0 \quad \text{and} \quad T' - \mu T = 0.$$

From the second equation we find $T(t) = c_1 e^{\mu t}$. If $\mu > 0$, $T(t)$ increases without bound as $t \to \infty$. Thus we assume $\mu \le 0$. Since $h > 0$ we can take $\mu = -\lambda^2 - h$. Then

$$R'' + \frac{1}{r}R' + \lambda^2 R = 0$$

is a parametric Bessel equation with solution

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r).$$

Since Y_0 is unbounded as $r \to 0$ we take $c_2 = 0$. Then $R(r) = c_1 J_0(\lambda r)$ and the boundary condition $u(1,t) = R(1)T(t) = 0$ implies $J_0(\lambda) = 0$. This latter equation defines the positive eigenvalues λ_n. Thus

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) e^{(-\lambda_n^2 - h)t}.$$

From

$$u(r,0) = 1 = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$$

we find

$$A_n = \frac{2}{J_1^2(\lambda_n)}\int_0^1 r J_0(\lambda_n r)\, dr \qquad \boxed{x = \lambda_n r,\ dx = \lambda_n, dr}$$

$$= \frac{2}{J_1^2(\lambda_n)}\int_0^{\lambda_n} \frac{1}{\lambda_n^2} x J_0(x)\, dx.$$

From recurrence relation (4) in Section 12.6 of the text we have

$$xJ_0(x) = \frac{d}{dx}[xJ_1(x)].$$

Then

$$A_n = \frac{2}{\lambda_n^2 J_1^2(\lambda_n)}\int_0^{\lambda_n} \frac{d}{dx}[xJ_1(x)]\, dx = \frac{2}{\lambda_n^2 J_1^2(\lambda_n)}\Big(xJ_1(x)\Big)\Big|_0^{\lambda_n} = \frac{2\lambda_1 J_1(\lambda_n)}{\lambda_n^2 J_1^2(\lambda_n)} = \frac{2}{\lambda_n J_1(\lambda_n)}$$

and

$$u(r,t) = 2e^{-ht}\sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)} e^{-\lambda_n^2 t}$$

8. Proceeding in the usual manner we find

$$u(r,t) = \sum_{n=1}^{\infty} A_n \cos a\lambda_n t J_0(\lambda_n r)$$

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where the eigenvalues are defined by $J_0(\lambda) = 0$. Thus the initial condition gives

$$u_0 J_0(x_k r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$$

and so

$$A_n = \frac{2}{J_1^2(\lambda_n)} \int_0^1 r\,(u_0 J_0(x_k r))\, J_0(\lambda_n r)\, dr.$$

But $J_0(\lambda) = 0$ implies that the eigenvalues are the positive zeros of J_0, that is, $\lambda_n = x_n$, $n = 1, 2, 3, \ldots$. Therefore

$$A_n = \frac{2u_0}{J_1^2(\lambda_n)} \int_0^1 r J_0(\lambda_k r) J_0(\lambda_n r)\, dr = 0, \quad n \neq k$$

by orthogonality. For $n = k$,

$$A_k = \frac{2u_0}{J_1^2(\lambda_k)} \int_0^1 r J_0^2(\lambda_k)\, dr = u_0$$

by (6) of Section 12.6. Thus the solution $u(r, t)$ reduces to one term when $n = k$, and

$$u(r, t) = u_0 \cos a\lambda_k t J_0(\lambda_k r) = u_0 \cos a x_k t J_0(x_k r).$$

9. Referring to Example 2 in Section 14.2 we have

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)$$
$$Z(z) = c_3 \cosh \lambda z + c_4 \sinh \lambda z$$

where $c_2 = 0$ and $J_0(2\lambda) = 0$ defines the positive eigenvalues λ_n. From $Z'(0) = 0$ we obtain $c_4 = 0$. Then

$$u(r, z) = \sum_{n=1}^{\infty} A_n \cosh \lambda_n z J_0(\lambda_n r).$$

From

$$u(r, 4) = 50 = \sum_{n=1}^{\infty} A_n \cosh 4\lambda_n J_0(\lambda_n r)$$

we obtain (as in Example 1 of Section 14.1)

$$A_n \cosh 4\lambda_n = \frac{2(50)}{4J_1^2(2\lambda_n)} \int_0^2 r J_0(\lambda_n r)\, dr = \frac{50}{\lambda_n J_1(2\lambda_n)}.$$

Thus the temperature in the cylinder is

$$u(r, z) = 50 \sum_{n=1}^{\infty} \frac{\cosh \lambda_n z J_0(\lambda_n r)}{\lambda_n \cosh 4\lambda_n J_1(2\lambda_n)}.$$

10. Using $u = RZ$ and $-\lambda^2$ as a separation constant leads to

$$r^2 R'' + rR' + \lambda^2 r^2 R = 0, \quad R'(1) = 0, \quad \text{and} \quad Z'' - \lambda^2 Z = 0.$$

Thus

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)$$
$$Z(z) = c_3 \cosh \lambda z + c_4 \sinh \lambda z$$

for $\lambda > 0$. Arguing that $u(r, z)$ is bounded as $r \to 0$ we defined $c_2 = 0$. Since the eigenvalues are defined by $J_0'(\lambda) = 0$ we know that $\lambda = 0$ is an eigenvalue. The solutions are then

$$R(r) = c_1 + c_2 \ln r \quad \text{and} \quad Z(z) = c_3 z + c_4$$

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where $c_2 = 0$. Thus a formal solution is

$$u(r,z) = A_0 z + B_0 + \sum_{n=1}^{\infty} (A_n \sinh \lambda_n z + B_n \cosh \lambda_n z) J_0(\lambda_n r).$$

Finally, the specified conditions $z = 0$ and $z = 1$ give, in turn,

$$B_0 = 2\int_0^1 r f(r)\, dr$$

$$B_n = \frac{2}{J_0^2(\lambda_n)} \int_0^1 r f(r) J_0(\lambda_n r)\, dr$$

$$A_0 = -B_0 + 2\int_0^1 r g(r)\, dr$$

$$A_n = \frac{1}{\sinh \lambda_n} \left[-B_n \cosh \lambda_n + \frac{2}{J_0^2(\lambda_n)} \int_0^1 r g(r) J_0(\lambda_n r)\, dr \right].$$

11. Referring to Example 1 in Section 14.3 of the text we have

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta).$$

For $x = \cos\theta$

$$u(1,\theta) = \begin{cases} 100 & 0 < \theta < \pi/2 \\ -100 & \pi/2 < \theta < \pi \end{cases} = 100 \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases} = g(x).$$

From Problem 22 in Exercise 12.6 we have

$$u(r,\theta) = 100 \left[\frac{3}{2} r P_1(\cos\theta) - \frac{7}{8} r^3 P_3(\cos\theta) + \frac{11}{16} r^5 P_5(\cos\theta) + \cdots \right].$$

12. Since

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r} + u \right] = \frac{1}{r}\left[r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} + \frac{\partial u}{\partial r} \right] = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{r}$$

the differential equation becomes

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) = \frac{\partial^2 u}{\partial t^2} \quad \text{or} \quad \frac{\partial^2}{\partial r^2}(ru) = r\frac{\partial^2 u}{\partial t^2}.$$

Letting $v(r,t) = ru(r,t)$ we obtain the boundary-value problem

$$\frac{\partial^2 v}{\partial r^2} = \frac{\partial^2 v}{\partial t^2}, \quad 0 < r < 1, \quad t > 0$$

$$\left.\frac{\partial v}{\partial r}\right|_{r=1} - v(1,t) = 0, \quad t > 0$$

$$v(r,0) = r f(r), \quad \left.\frac{\partial v}{\partial t}\right|_{t=0} = r g(r), \quad 0 < r < 1.$$

If we separate variables using $v(r,t) = R(r)T(t)$ then we obtain

$$R(r) = c_1 \cos\lambda r + c_2 \sin\lambda r$$

$$T(t) = c_3 \cos\lambda t + c_4 \sin\lambda t.$$

Since $u(r,t) = v(r,t)/r$, in order to insure boundedness at $r = 0$ we define $c_1 = 0$. Then $R(r) = c_2 \sin\lambda r$. Now the boundary condition $R'(1) - R(1) = 0$ implies $\lambda\cos\lambda - \sin\lambda = 0$. Thus, the eigenvalues λ_n are the positive solutions of $\tan\lambda = \lambda$. We now have

$$v_n(r,t) = (A_n \cos\lambda_n t + B_n \sin\lambda_n t) \sin\lambda_n r.$$

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For the eigenvalue $\lambda = 0$,

$$R(r) = c_1 r + c_2 \quad \text{and} \quad T(t) = c_3 t + c_4,$$

and boundedness at $r = 0$ implies $c_2 = 0$. We then take

$$v_0(r,t) = A_0 tr + B_0 r$$

so that

$$v(r,t) = A_0 tr + B_0 r + \sum_{n=1}^{\infty} (a_n \cos \lambda_n t + B_n \sin \lambda_n t) \sin \lambda_n r.$$

Now

$$v(r,0) = rf(r) = B_0 r + \sum_{n=1}^{\infty} A_n \sin \lambda_n r.$$

Since $\{r, \sin \lambda_n r\}$ is an orthogonal set on $[0,1]$,

$$\int_0^1 r \sin \lambda_n r \, dr = 0 \quad \text{and} \quad \int_0^1 \sin \lambda_n r \sin \lambda_n r \, dr = 0$$

for $m \neq n$. Therefore

$$\int_0^1 r^2 f(r) \, dr = B_0 \int_0^1 r^2 \, dr = \frac{1}{3} B_0$$

and

$$B_0 = 3 \int_0^1 r^2 f(r) \, dr.$$

Also

$$\int_0^1 rf(r) \sin \lambda_n r \, dr = A_n \int_0^1 \sin^2 \lambda_n r \, dr$$

and

$$A_n = \frac{\int_0^1 rf(r) \sin \lambda_n r \, dr}{\int_0^1 \sin^2 \lambda_n r \, dr}.$$

Now

$$\int_0^1 \sin^2 \lambda_n r \, dr = \frac{1}{2} \int_0^1 (1 - \cos 2\lambda_n r) \, dr = \frac{1}{2} \left[1 - \frac{\sin 2\lambda_n}{2\lambda_n} \right] = \frac{1}{2} [1 - \cos^2 \lambda_n].$$

Since $\tan \lambda_n = \lambda_n$,

$$1 + \lambda_n^2 = 1 + \tan^2 \lambda_n = \sec^2 \lambda_n = \frac{1}{\cos^2 \lambda_n}$$

and

$$\cos^2 \lambda_n = \frac{1}{1 + \lambda_n^2}.$$

Then

$$\int_0^1 \sin^2 \lambda_n r \, dr = \frac{1}{2} \left[1 - \frac{1}{1 + \lambda_n^2} \right] = \frac{\lambda_n^2}{2(1 + \lambda_n^2)}$$

and

$$A_n = \frac{2(1 + \lambda_n^2)}{\lambda_n^2} \int_0^1 rf(r) \sin \lambda_n r \, dr.$$

Similarly, setting

$$\left. \frac{\partial v}{\partial t} \right|_{t=0} = rg(r) = A_0 r + \sum_{n=1}^{\infty} B_n \lambda_n \sin \lambda_n r$$

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we obtain

$$A_0 = 3\int_0^1 r^2 g(r)\,dr$$

and

$$B_n = \frac{2(1+\lambda_n^2)}{\lambda_n^3}\int_0^1 rg(r)\sin\lambda_n r\,dr.$$

Therefore, since $v(r,t) = ru(r,t)$ we have

$$u(r,t) = A_0 t + B_0 + \sum_{n=1}^{\infty}(A_n\cos\lambda_n t + B_n\sin\lambda_n t)\frac{\sin\lambda_n r}{r},$$

where the λ_n are solutions of $\tan\lambda = \lambda$ and

$$A_0 = 3\int_0^1 r^2 g(r)\,dr$$

$$B_0 = 3\int_0^1 r^2 f(r)\,dr$$

$$A_n = \frac{2(1+\lambda_n^2)}{\lambda_n^2}\int_0^1 rf(r)\sin\lambda_n r\,dr$$

$$B_n = \frac{2(1+\lambda_n^2)}{\lambda_n^3}\int_0^1 rg(r)\sin\lambda_n r\,dr$$

for $n = 1, 2, 3, \ldots$.

13. We note that the differential equation can be expressed in the form

$$\frac{d}{dx}[xu'] = -\lambda^2 xu.$$

Thus

$$u_n\frac{d}{dx}[xu_m'] = -\lambda_m^2 xu_m u_n$$

and

$$u_m\frac{d}{dx}[xu_n'] = -\lambda_n^2 xu_n u_m.$$

Subtracting we obtain

$$u_n\frac{d}{dx}[xu_m'] - u_m\frac{d}{dx}[xu_n'] = (\lambda_n^2 - \lambda_m^2)xu_m u_n$$

and

$$\int_a^b u_n\frac{d}{dx}[xu_m']\,dx - \int_a^b u_m\frac{d}{dx}[xu_n'] = (\lambda_n^2 - \lambda_m^2)\int_a^b xu_m u_n\,dx.$$

Using integration by parts this becomes

$$\begin{aligned} u_n x u_m'\Big|_a^b - \int_a^b xu_m'u_n'\,dx - u_m x u_n'\Big|_a^b + \int_a^b xu_n'u_m'\,dx \\ &= b[u_n(b)u_m'(b) - u_m(b)u_n'(b)] - a[u_n(a)u_m'(a) - u_m(a)u_n'(a)] \\ &= (\lambda_n^2 - \lambda_m^2)\int_a^b xu_m u_n\,dx. \end{aligned}$$

Since

$$u(x) = Y_0(\lambda a)J_0(\lambda x) - J_0(\lambda a)Y_0(\lambda x)$$

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we have

$$u_n(b) = Y_0(\lambda_n a)J_0(\lambda_n b) - J_0(\lambda_n a)Y(\lambda_n b) = 0$$

by the definition of the λ_n. Similarly $u_m(b) = 0$. Also

$$u_n(a) = Y_0(\lambda a)J_0(\lambda_n a) - J_0(\lambda_n a)Y_0(\lambda_n a) = 0$$

and $u_m(a) = 0$. Therefore

$$\int_a^b x u_m u_n \, dx = \frac{1}{\lambda_n^2 - \lambda_m^2} \left(b[u_n(b)u_m'(b) - u_m(b)u_n'(b)] - a[u_n(a)u_m'(a) - u_m(a)u_n'(a)]\right) = 0$$

and the $u_n(x)$ are orthogonal with respect to the weight function x.

14. Letting $u(r, t) = R(r)T(t)$ and separating variables we obtain

$$rR'' + R' + \lambda^2 rR = 0$$
$$T' + \lambda^2 T = 0,$$

with solutions

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)$$
$$T(t) = c_3 e^{-\lambda^2 t}.$$

Now the boundary conditions imply

$$R(a) = 0 = c_1 J_0(\lambda a) + c_2 Y_0(\lambda a)$$
$$R(b) = 0 = c_1 J_0(\lambda b) + c_2 Y_0(\lambda b)$$

so that

$$c_2 = -\frac{c_1 J_0(\lambda a)}{Y_0(\lambda a)}$$

and

$$c_1 J_0(\lambda b) - \frac{c_1 J_0(\lambda a)}{Y(\lambda a)} Y_0(\lambda b) = 0$$

or

$$Y_0(\lambda a)J_0(\lambda b) - J_0(\lambda a)Y_0(\lambda b) = 0.$$

This equation defines λ_n for $n = 1, 2, 3, \ldots$. Now

$$R(r) = c_1 J_0(\lambda r) - c_1 \frac{J_0(\lambda a)}{Y_0(\lambda a)} Y_0(\lambda r) = \frac{c_1}{Y_0(\lambda a)} \left[Y_0(\lambda a)J_0(\lambda r) - J_0(\lambda a)Y_0(\lambda r)\right]$$

and

$$u_n(r, t) = A_n \left[Y_0(\lambda_n a)J_0(\lambda_n r) - J_0(\lambda_n a)Y_0(\lambda_n r)\right] e^{-\lambda_n^2 t} = A_n u_n(r) e^{-\lambda_n^2 t}.$$

Thus

$$u(r, t) = \sum_{n=1}^{\infty} A_n u_n(r) e^{-\lambda_n^2 t}.$$

From the initial condition

$$u(r, 0) = f(r) = \sum_{n=1}^{\infty} A_n u_n(r)$$

we obtain

$$A_n = \frac{\int_a^b r f(r) u_n(r) \, dr}{\int_a^b r u_n^2(r) \, dr}.$$

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15 Integral Transform Method

Exercises 15.1

1. **(a)** The result follows by letting $\tau = u^2$ or $u = \sqrt{\tau}$ in $\operatorname{erf}(\sqrt{t}\,) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^{\sqrt{t}} e^{-u^2} du$.

(b) Using $\mathscr{L}\{t^{-1/2}\} = \dfrac{\sqrt{\pi}}{s^{1/2}}$ and the first translation theorem, it follows from the convolution theorem that

$$\begin{aligned}\mathscr{L}\left\{\operatorname{erf}(\sqrt{t})\right\} &= \frac{1}{\sqrt{\pi}}\,\mathscr{L}\left\{\int_0^t \frac{e^{-\tau}}{\sqrt{\tau}}\,d\tau\right\} = \frac{1}{\sqrt{\pi}}\,\mathscr{L}\{1\}\,\mathscr{L}\left\{t^{-1/2}e^{-t}\right\} = \frac{1}{\sqrt{\pi}}\,\frac{1}{s}\,\mathscr{L}\left\{t^{-1/2}\right\}\Big|_{s\to s+1} \\ &= \frac{1}{\sqrt{\pi}}\,\frac{1}{s}\,\frac{\sqrt{\pi}}{\sqrt{s+1}} = \frac{1}{s\sqrt{s+1}}\,.\end{aligned}$$

2. Since $\operatorname{erfc}(\sqrt{t}\,) = 1 - \operatorname{erf}(\sqrt{t}\,)$ we have

$$\mathscr{L}\left\{\operatorname{erfc}(\sqrt{t}\,)\right\} = \mathscr{L}\{1\} - \mathscr{L}\left\{\operatorname{erf}(\sqrt{t}\,)\right\} = \frac{1}{s} - \frac{1}{s\sqrt{s+1}} = \frac{1}{s}\left[1 - \frac{1}{\sqrt{s+1}}\right].$$

3. By the first translation theorem,

$$\mathscr{L}\left\{e^t \operatorname{erf}(\sqrt{t}\,)\right\} = \mathscr{L}\left\{\operatorname{erf}(\sqrt{t}\,)\right\}\Big|_{s\to s-1} = \frac{1}{s\sqrt{s+1}}\Big|_{s\to s-1} = \frac{1}{\sqrt{s}\,(s-1)}\,.$$

4. By the first translation theorem and the result of Problem 2,

$$\begin{aligned}\mathscr{L}\left\{e^t \operatorname{erfc}(\sqrt{t}\,)\right\} &= \mathscr{L}\left\{\operatorname{erfc}(\sqrt{t}\,)\right\}\Big|_{s\to s-1} = \left(\frac{1}{s} - \frac{1}{s\sqrt{s+1}}\right)\Big|_{s\to s-1} = \frac{1}{s-1} - \frac{1}{\sqrt{s}\,(s-1)} \\ &= \frac{\sqrt{s}-1}{\sqrt{s}\,(s-1)} = \frac{\sqrt{s}-1}{\sqrt{s}\,(\sqrt{s}+1)(\sqrt{s}-1)} = \frac{1}{\sqrt{s}\,(\sqrt{s}+1)}\,.\end{aligned}$$

5. From table entry 3 and the first translation theorem we have

$$\begin{aligned}\mathscr{L}\left\{e^{-Gt/C}\operatorname{erf}\left(\frac{x}{2}\sqrt{\frac{RC}{t}}\right)\right\} &= \mathscr{L}\left\{e^{-Gt/C}\left[1 - \operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{RC}{t}}\right)\right]\right\} \\ &= \mathscr{L}\left\{e^{-Gt/C}\right\} - \mathscr{L}\left\{e^{-Gt/C}\operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{RC}{t}}\right)\right\} \\ &= \frac{1}{s+G/C} - \frac{e^{-x\sqrt{RC}\,\sqrt{s}}}{s}\Big|_{s\to s+G/C} \\ &= \frac{1}{s+G/C} - \frac{e^{-x\sqrt{RC}\,\sqrt{s+G/C}}}{s+G/C} = \frac{C}{Cs+G}\left(1 - e^{x\sqrt{RCs+RG}}\right).\end{aligned}$$

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6. We first compute

$$\begin{aligned}\frac{\sinh a\sqrt{s}}{s\sinh\sqrt{s}} &= \frac{e^{a\sqrt{s}}-e^{-a\sqrt{s}}}{s(e^{\sqrt{s}}-e^{-\sqrt{s}})} = \frac{e^{(a-1)\sqrt{s}}-e^{-(a+1)\sqrt{s}}}{s(1-e^{-2\sqrt{s}})}\\ &= \frac{e^{(a-1)\sqrt{s}}}{s}\left[1+e^{-2\sqrt{s}}+e^{-4\sqrt{s}}+\cdots\right]-\frac{e^{-(a+1)\sqrt{s}}}{s}\left[1+e^{-2\sqrt{s}}+e^{-4\sqrt{s}}+\cdots\right]\\ &= \left[\frac{e^{-(1-a)\sqrt{s}}}{s}+\frac{e^{-(3-a)\sqrt{s}}}{s}+\frac{e^{-(5-a)\sqrt{s}}}{s}+\cdots\right]\\ &\qquad -\left[\frac{e^{-(1+a)\sqrt{s}}}{s}+\frac{e^{-(3+a)\sqrt{s}}}{s}+\frac{e^{-(5+a)\sqrt{s}}}{s}+\cdots\right]\\ &= \sum_{n=0}^{\infty}\left[\frac{e^{-(2n+1-a)\sqrt{s}}}{s}-\frac{e^{-(2n+1+a)\sqrt{s}}}{s}\right].\end{aligned}$$

Then

$$\begin{aligned}\mathscr{L}\left\{\frac{\sinh a\sqrt{s}}{s\sinh\sqrt{s}}\right\} &= \sum_{n=0}^{\infty}\left[\mathscr{L}\left\{\frac{e^{-(2n+1-a)\sqrt{s}}}{s}\right\}-\mathscr{L}\left\{-\frac{e^{-(2n+1+a)\sqrt{s}}}{s}\right\}\right]\\ &= \sum_{n=0}^{\infty}\left[\operatorname{erfc}\left(\frac{2n+1-a}{2\sqrt{t}}\right)-\operatorname{erfc}\left(\frac{2n+1+a}{2\sqrt{t}}\right)\right]\\ &= \sum_{n=0}^{\infty}\left(\left[1-\operatorname{erf}\left(\frac{2n+1-a}{2\sqrt{t}}\right)\right]-\left[1-\operatorname{erf}\left(\frac{2n+1+a}{2\sqrt{t}}\right)\right]\right)\\ &= \sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2n+1+a}{2\sqrt{t}}\right)-\operatorname{erf}\left(\frac{2n+1-a}{2\sqrt{t}}\right)\right].\end{aligned}$$

7. Taking the Laplace transform of both sides of the equation we obtain

$$\begin{aligned}\mathscr{L}\{y(t)\} &= \mathscr{L}\{1\}-\mathscr{L}\left\{\int_0^t\frac{y(\tau)}{\sqrt{t-\tau}}\,d\tau\right\}\\ Y(s) &= \frac{1}{s}-Y(s)\frac{\sqrt{\pi}}{\sqrt{s}}\\ \frac{\sqrt{s}+\sqrt{\pi}}{\sqrt{s}}\,Y(s) &= \frac{1}{s}\\ Y(s) &= \frac{1}{\sqrt{s}\,(\sqrt{s}+\sqrt{\pi}\,)}.\end{aligned}$$

Thus

$$y(t) = \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}\,(\sqrt{s}+\sqrt{\pi}\,)}\right\} = e^{\pi t}\operatorname{erfc}(\sqrt{\pi t}\,). \qquad \boxed{\text{By entry 5 in the table}}$$

8. Using entries 3 and 5 in the table, we have

$$\begin{aligned}&\mathscr{L}\left\{-e^{ab}e^{b^2t}\operatorname{erfc}\left(b\sqrt{t}+\frac{a}{2\sqrt{t}}\right)+\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)\right\}\\ &\qquad = -\mathscr{L}\left\{e^{ab}e^{b^2t}\operatorname{erfc}\left(b\sqrt{t}+\frac{a}{2\sqrt{t}}\right)\right\}+\mathscr{L}\left\{\frac{a}{2\sqrt{t}}\right\}\\ &\qquad = -\frac{e^{-a\sqrt{s}}}{\sqrt{s}\,(\sqrt{s}+b)}+\frac{e^{-a\sqrt{s}}}{s}\end{aligned}$$

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$$= e^{-a\sqrt{s}}\left[\frac{1}{s} - \frac{1}{\sqrt{s}\,(\sqrt{s}+b)}\right] = e^{-a\sqrt{s}}\left[\frac{1}{s} - \frac{\sqrt{s}}{s\,(\sqrt{s}+b)}\right]$$
$$= e^{-a\sqrt{s}}\left[\frac{\sqrt{s}+b-\sqrt{s}}{s\,(\sqrt{s}+b)}\right] = \frac{be^{-a\sqrt{s}}}{s\,(\sqrt{s}+b)}.$$

9. $\displaystyle\int_a^b e^{-u^2}\,du = \int_a^0 e^{-u^2}\,du + \int_0^b e^{-u^2}\,du = \int_0^b e^{-u^2}\,du - \int_0^a e^{-u^2}\,du$
$$= \frac{\sqrt{\pi}}{2}\text{erf}(b) - \frac{\sqrt{\pi}}{2}\text{erf}(a) = \frac{\sqrt{\pi}}{2}[\text{erf}(b) - \text{erf}(a)]$$

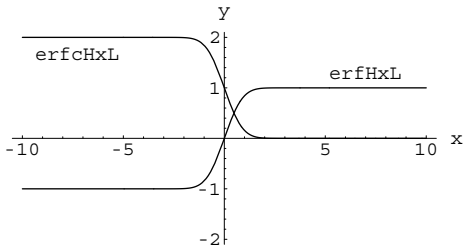
10. Since $f(x) = e^{-x^2}$ is an even function,

$$\int_{-a}^{a} e^{-u^2}\,du = 2\int_0^a e^{-u^2}\,du.$$

Therefore,

$$\int_{-a}^{a} e^{-u^2}\,du = \sqrt{\pi}\,\text{erf}(a).$$

11. The function $\text{erf}(x)$ is symmetric with respect to the origin, while $\text{erfc}(x)$ appears to be symmetric with respect to the point $(0, 1)$. From the graph it appears that $\lim_{x\to-\infty}\text{erf}(x) = -1$ and $\lim_{x\to-\infty}\text{erfc}(x) = 2$.



Exercises 15.2

1. The boundary-value problem is

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \quad t > 0,$$
$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0,$$
$$u(x,0) = A\sin\frac{\pi}{L}x, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

Transforming the partial differential equation gives

$$\frac{d^2U}{dx^2} - \left(\frac{s}{a}\right)^2 U = -\frac{s}{a^2}A\sin\frac{\pi}{L}x.$$

Using undetermined coefficients we obtain

$$U(x,s) = c_1\cosh\frac{s}{a}x + c_2\sinh\frac{s}{a}x + \frac{As}{s^2 + a^2\pi^2/L^2}\sin\frac{\pi}{L}x.$$

The transformed boundary conditions, $U(0,s) = 0$, $U(L,s) = 0$ give in turn $c_1 = 0$ and $c_2 = 0$. Therefore

$$U(x,s) = \frac{As}{s^2 + a^2\pi^2/L^2}\sin\frac{\pi}{L}x$$

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and

$$u(x,t) = A\,\mathscr{L}^{-1}\left\{\frac{s}{s^2 + a^2\pi^2/L^2}\right\}\sin\frac{\pi}{L}x = A\cos\frac{a\pi}{L}t\sin\frac{\pi}{L}x.$$

2. The transformed equation is

$$\frac{d^2U}{dx^2} - s^2U = -2\sin\pi x - 4\sin 3\pi x$$

and so

$$U(x,s) = c_1\cosh sx + c_2\sinh sx + \frac{2}{s^2+\pi^2}\sin\pi x + \frac{4}{s^2+9\pi^2}\sin 3\pi x.$$

The transformed boundary conditions, $U(0,s) = 0$ and $U(1,s) = 0$ give $c_1 = 0$ and $c_2 = 0$. Thus

$$U(x,s) = \frac{2}{s^2+\pi^2}\sin\pi x + \frac{4}{s^2+9\pi^2}\sin 3\pi x$$

and

$$\begin{aligned} u(x,t) &= 2\,\mathscr{L}^{-1}\left\{\frac{1}{s^2+\pi^2}\right\}\sin\pi x + 4\,\mathscr{L}^{-1}\left\{\frac{1}{s^2+9\pi^2}\right\}\sin 3\pi x \\ &= \frac{2}{\pi}\sin\pi t\sin\pi x + \frac{4}{3\pi}\sin 3\pi t\sin 3\pi x. \end{aligned}$$

3. The solution of

$$a^2\frac{d^2U}{dx^2} - s^2U = 0$$

is in this case

$$U(x,s) = c_1e^{-(x/a)s} + c_2e^{(x/a)s}.$$

Since $\lim_{x\to\infty} u(x,t) = 0$ we have $\lim_{x\to\infty} U(x,s) = 0$. Thus $c_2 = 0$ and

$$U(x,s) = c_1e^{-(x/a)s}.$$

If $\mathscr{L}\{u(0,t)\} = \mathscr{L}\{f(t)\} = F(s)$ then $U(0,s) = F(s)$. From this we have $c_1 = F(s)$ and

$$U(x,s) = F(s)e^{-(x/a)s}.$$

Hence, by the second translation theorem,

$$u(x,t) = f\left(t - \frac{x}{a}\right)\mathscr{U}\left(t - \frac{x}{a}\right).$$

4. Expressing $f(t)$ in the form $(\sin\pi t)[1 - \mathscr{U}(t-1)]$ and using the result of Problem 3 we find

$$\begin{aligned} u(x,t) &= f\left(t - \frac{x}{a}\right)\mathscr{U}\left(t - \frac{x}{a}\right) \\ &= \sin\pi\left(t - \frac{x}{a}\right)\left[1 - \mathscr{U}\left(t - \frac{x}{a} - 1\right)\right]\mathscr{U}\left(t - \frac{x}{a}\right) \\ &= \sin\pi\left(t - \frac{x}{a}\right)\left[\mathscr{U}\left(t - \frac{x}{a}\right) - \mathscr{U}\left(t - \frac{x}{a}\right)\mathscr{U}\left(t - \frac{x}{a} - 1\right)\right] \\ &= \sin\pi\left(t - \frac{x}{a}\right)\left[\mathscr{U}\left(t - \frac{x}{a}\right) - \mathscr{U}\left(t - \frac{x}{a} - 1\right)\right] \end{aligned}$$

Now

$$\mathscr{U}\left(t - \frac{x}{a}\right) - \mathscr{U}\left(t - \frac{x}{a} - 1\right) = \begin{cases} 0, & 0 \le t < x/a \\ 1, & x/a \le t \le x/a + 1 \\ 0, & t > x/a + 1 \end{cases}$$

$$= \begin{cases} 0, & x < a(t-1) \text{ or } x > at \\ 1, & a(t-1) \le x \le at \end{cases}$$

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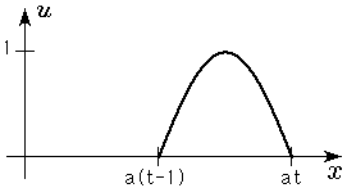
so

$$u(x,t) = \begin{cases} 0, & x < a(t-1) \quad \text{or} \quad x > at \\ \sin \pi(t - x/a), & a(t-1) \le x \le at. \end{cases}$$



The graph is shown for $t > 1$.

5. We use

$$U(x,s) = c_1 e^{-(x/a)s} - \frac{g}{s^3}.$$

Now

$$\mathscr{L}\{u(0,t)\} = U(0,s) = \frac{A\omega}{s^2 + \omega^2}$$

and so

$$U(0,s) = c_1 - \frac{g}{s^3} = \frac{A\omega}{s^2 + \omega^2} \quad \text{or} \quad c_1 = \frac{g}{s^3} + \frac{A\omega}{s^2 + \omega^2}.$$

Therefore

$$U(x,s) = \frac{A\omega}{s^2 + \omega^2} e^{-(x/a)s} + \frac{g}{s^3} e^{-(x/a)s} - \frac{g}{s^3}$$

and

$$\begin{aligned} u(x,t) &= A\mathscr{L}^{-1}\left\{\frac{\omega e^{-(x/a)s}}{s^2 + \omega^2}\right\} + g\mathscr{L}^{-1}\left\{\frac{e^{-(x/a)s}}{s^3}\right\} - g\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\} \\ &= A \sin\omega\left(t - \frac{x}{a}\right)\mathscr{U}\left(t - \frac{x}{a}\right) + \frac{1}{2}g\left(t - \frac{x}{a}\right)^2 \mathscr{U}\left(t - \frac{x}{a}\right) - \frac{1}{2}gt^2. \end{aligned}$$

6. Transforming the partial differential equation gives

$$\frac{d^2U}{dx^2} - s^2 U = -\frac{\omega}{s^2 + \omega^2} \sin \pi x.$$

Using undetermined coefficients we obtain

$$U(x,s) = c_1 \cosh sx + c_2 \sinh sx + \frac{\omega}{(s^2 + \pi^2)(s^2 + \omega^2)} \sin \pi x.$$

The transformed boundary conditions $U(0,s) = 0$ and $U(1,s) = 0$ give, in turn, $c_1 = 0$ and $c_2 = 0$. Therefore

$$U(x,s) = \frac{\omega}{(s^2 + \pi^2)(s^2 + \omega^2)} \sin \pi x$$

and

$$\begin{aligned} u(x,t) &= \omega \sin \pi x \mathscr{L}^{-1}\left\{\frac{1}{(s^2 + \pi^2)(s^2 + \omega^2)}\right\} \\ &= \frac{\omega}{\omega^2 - \pi^2} \sin \pi x \mathscr{L}^{-1}\left\{\frac{1}{\pi}\frac{\pi}{s^2 + \pi^2} - \frac{1}{\omega}\frac{\omega}{s^2 + \omega^2}\right\} \\ &= \frac{\omega}{\pi(\omega^2 - \pi^2)} \sin \pi t \sin \pi x - \frac{1}{\omega^2 - \pi^2} \sin \omega t \sin \pi x. \end{aligned}$$

7. We use

$$U(x,s) = c_1 \cosh \frac{s}{a} x +_2 \sinh \frac{s}{a} x.$$

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Now $U(0,s) = 0$ implies $c_1 = 0$, so $U(x,s) = c_2 \sinh(s/a)x$. The condition $E\dfrac{dU}{dx}\Big|_{x=L} = F_0$ then yields $c_2 = F_0 a/Es\cosh(s/a)L$ and so

$$\begin{aligned}
U(x,s) &= \frac{aF_0}{Es}\,\frac{\sinh(s/a)x}{\cosh(s/a)L} = \frac{aF_0}{Es}\,\frac{e^{(s/a)x} - e^{-(s/a)x}}{e^{(s/a)L} + e^{-(s/a)L}} \\
&= \frac{aF_0}{Es}\,\frac{e^{(s/a)(x-L)} - e^{-(s/a)(x+L)}}{1 + e^{-2sL/a}} \\
&= \frac{aF_0}{E}\left[\frac{e^{-(s/a)(L-x)}}{s} - \frac{e^{-(s/a)(3L-x)}}{s} + \frac{e^{-(s/a)(5L-x)}}{s} - \cdots\right] \\
&\qquad - \frac{aF_0}{E}\left[\frac{e^{-(s/a)(L+x)}}{s} - \frac{e^{-(s/a)(3L+x)}}{s} + \frac{e^{-(s/a)(5L+x)}}{s} - \cdots\right] \\
&= \frac{aF_0}{E}\sum_{n=0}^{\infty}(-1)^n\left[\frac{e^{-(s/a)(2nL+L-x)}}{s} - \frac{e^{-(s/a)(2nL+L+x)}}{s}\right]
\end{aligned}$$

and

$$\begin{aligned}
u(x,t) &= \frac{aF_0}{E}\sum_{n=0}^{\infty}(-1)^n\left[\mathscr{L}^{-1}\left\{\frac{e^{-(s/a)(2nL+L-x)}}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{e^{-(s/a)(2nL+L+x)}}{s}\right\}\right] \\
&= \frac{aF_0}{E}\sum_{n=0}^{\infty}(-1)^n\left[\left(t - \frac{2nL+L-x}{a}\right)\mathscr{U}\left(t - \frac{2nL+L-x}{a}\right)\right. \\
&\qquad \left. - \left(t - \frac{2nL+L+x}{a}\right)\mathscr{U}\left(t - \frac{2nL+L+x}{a}\right)\right].
\end{aligned}$$

8. We use

$$U(x,s) = c_1 e^{-(x/a)s} + c_2 e^{(x/a)s} - \frac{v_0}{s^2}\,.$$

Now $\lim\limits_{x\to\infty}\dfrac{dU}{dx} = 0$ implies $c_2 = 0$, and $U(0,s) = 0$ then gives $c_1 = v_0/s^2$. Hence

$$U(x,s) = \frac{v_0}{s^2}e^{-(x/a)s} - \frac{v_0}{s^2}$$

and

$$u(x,t) = v_0\left(t - \frac{x}{a}\right)\mathscr{U}\left(t - \frac{x}{a}\right) - v_0 t.$$

9. Transforming the partial differential equation gives

$$\frac{d^2U}{dx^2} - s^2U = -sxe^{-x}.$$

Using undetermined coefficients we obtain

$$U(x,s) = c_1 e^{-sx} + c_2 e^{sx} - \frac{2s}{(s^2-1)^2}\,e^{-x} + \frac{s}{s^2-1}\,xe^{-x}.$$

The transformed boundary conditions $\lim_{x\to\infty} U(x,s) = 0$ and $U(0,s) = 0$ give, in turn, $c_2 = 0$ and $c_1 = 2s/(s^2-1)^2$. Therefore

$$U(x,s) = \frac{2s}{(s^2-1)^2}\,e^{-sx} - \frac{2s}{(s^2-1)^2}\,e^{-x} + \frac{s}{s^2-1}\,xe^{-x}.$$

From entries (13) and (26) in the table we obtain

$$\begin{aligned}
u(x,t) &= \mathscr{L}^{-1}\left\{\frac{2s}{(s^2-1)^2}\,e^{-sx} - \frac{2s}{(s^2-1)^2}\,e^{-x} + \frac{s}{s^2-1}\,xe^{-x}\right\} \\
&= 2(t-x)\sinh(t-x)\,\mathscr{U}(t-x) - te^{-x}\sinh t + xe^{-x}\cosh t.
\end{aligned}$$

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10. We use

$$U(x,s) = c_1 e^{-xs} + c_2 e^{xs} + \frac{s}{s^2-1} e^{-x}.$$

Now $\lim_{x\to\infty} u(x,t) = 0$ implies $\lim_{x\to\infty} U(x,s) = 0$, so we define $c_2 = 0$. Then

$$U(x,s) = c_1 e^{-xs} + \frac{s}{s^2-1} e^{-x}.$$

Finally, $U(0,s) = 1/s$ gives $c_1 = 1/s - s/(s^2-1)$. Thus

$$U(x,s) = \frac{1}{s} - \frac{s}{s^2-1} e^{-xs} + \frac{s}{s^2-1} e^{-x}$$

and

$$\begin{aligned} u(x,t) &= -\mathscr{L}^{-1}\left\{\frac{s}{s^2-1} e^{-(x/a)s}\right\} + \mathscr{L}^{-1}\left\{\frac{s}{s^2-1}\right\} e^{-x} \\ &= -\cosh\left(t - \frac{x}{a}\right)\mathscr{U}\left(t - \frac{x}{a}\right) + e^{-x}\cosh t. \end{aligned}$$

11. (a) We use

$$U(x,s) = c_1 e^{-\sqrt{s/k}\,x} + c_2 e^{\sqrt{s/k}\,x}.$$

Now $\lim_{x\to\infty} u(x,t) = 0$ implies $\lim_{x\to\infty} U(x,s) = 0$, so we define $c_2 = 0$. Then

$$U(x,s) = c_1 e^{-\sqrt{s/k}\,x}.$$

Finally, from $U(0,s) = u_0/s$ we obtain $c_1 = u_0/s$. Thus

$$U(x,s) = u_0 \frac{e^{-\sqrt{s/k}\,x}}{s}$$

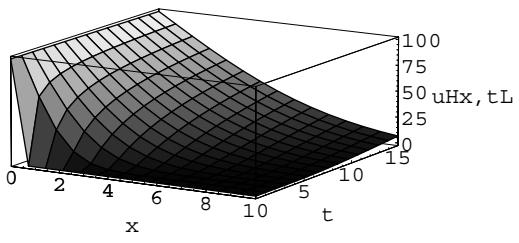
and

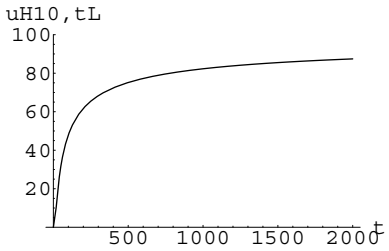
$$u(x,t) = u_0 \mathscr{L}^{-1}\left\{\frac{e^{-\sqrt{s/k}\,x}}{s}\right\} = u_0 \mathscr{L}^{-1}\left\{\frac{e^{-(x/\sqrt{k})\sqrt{s}}}{s}\right\} = u_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right).$$

Since $\operatorname{erfc}(0) = 1$,

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} u_0 \operatorname{erfc}(x/2\sqrt{kt}\,) = u_0.$$

(b)





12. (a) Transforming the partial differential equation and using the initial condition gives

$$k\frac{d^2U}{dx^2} - sU = 0.$$

Since the domain of the variable x is an infinite interval we write the general solution of this differential equation as

$$U(x,s) = c_1 e^{-\sqrt{s/k}\,x} + c_2 e^{-\sqrt{s/k}\,x}.$$

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Transforming the boundary conditions gives $U'(0,s) = -A/s$ and $\lim_{x\to\infty} U(x,s) = 0$. Hence we find $c_2 = 0$ and $c_1 = A\sqrt{k}/s\sqrt{s}$. From

$$U(x,s) = A\sqrt{k}\,\frac{e^{-\sqrt{s/k}\,x}}{s\sqrt{s}}$$

we see that

$$u(x,t) = A\sqrt{k}\,\mathscr{L}^{-1}\left\{\frac{e^{-\sqrt{s/k}\,x}}{s\sqrt{s}}\right\}.$$

With the identification $a = x/\sqrt{k}$ it follows from entry 49 of the table in Appendix III that

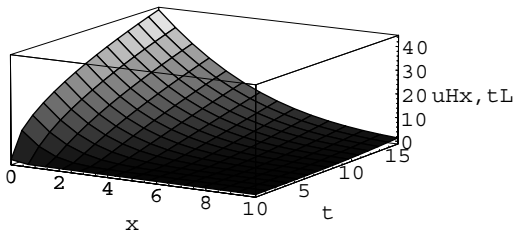
$$u(x,t) = A\sqrt{k}\left\{2\sqrt{\frac{t}{\pi}}\,e^{-x^2/4kt} - \frac{x}{\sqrt{k}}\Big(x/2\sqrt{kt}\Big)\right\}$$

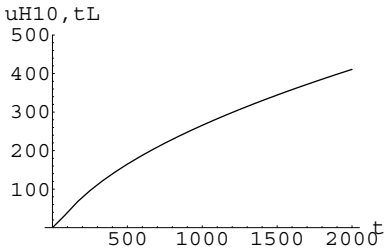
$$= 2A\sqrt{\frac{kt}{\pi}}\,e^{-x^2/4kt} - Ax\,\operatorname{erfc}\Big(x/2\sqrt{kt}\Big).$$

Since $\operatorname{erfc}(0) = 1$,

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}\left(2A\sqrt{\frac{kt}{\pi}}e^{-x^2/4kt} - Ax\operatorname{erfc}\frac{x}{2\sqrt{kt}}\right) = \infty.$$

(b)





13. We use

$$U(x,s) = c_1e^{-\sqrt{s}\,x} + c_2e^{\sqrt{s}\,x} + \frac{u_1}{s}.$$

The condition $\lim_{x\to\infty} u(x,t) = u_1$ implies $\lim_{x\to\infty} U(x,s) = u_1/s$, so we define $c_2 = 0$. Then

$$U(x,s) = c_1e^{-\sqrt{s}\,x} + \frac{u_1}{s}.$$

From $U(0,s) = u_0/s$ we obtain $c_1 = (u_0 - u_1)/s$. Thus

$$U(x,s) = (u_0 - u_1)\frac{e^{-\sqrt{s}\,x}}{s} + \frac{u_1}{s}$$

and

$$u(x,t) = (u_0 - u_1)\,\mathscr{L}^{-1}\left\{\frac{e^{-x\sqrt{s}}}{s}\right\} + u_1\,\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = (u_0 - u_1)\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) + u_1.$$

14. We use

$$U(x,s) = c_1e^{-\sqrt{s}\,x} + c_2e^{\sqrt{s}\,x} + \frac{u_1x}{s}.$$

The condition $\lim_{x\to\infty} u(x,t)/x = u_1$ implies $\lim_{x\to\infty} U(x,s)/x = u_1/s$, so we define $c_2 = 0$. Then

$$U(x,s) = c_1e^{-\sqrt{s}\,x} + \frac{u_1x}{s}.$$

From $U(0,s) = u_0/s$ we obtain $c_1 = u_0/s$. Hence

$$U(x,s) = u_0\,\frac{e^{-\sqrt{s}\,x}}{s} + \frac{u_1x}{s}$$

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and

$$u(x,t) = u_0\,\mathscr{L}^{-1}\left\{\frac{e^{-x\sqrt{s}}}{s}\right\} + u_1 x\,\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = u_0\,\text{erfc}\left(\frac{x}{2\sqrt{t}}\right) + u_1 x.$$

15. We use

$$U(x,s) = c_1 e^{-\sqrt{s}\,x} + c_2 e^{\sqrt{s}\,x} + \frac{u_0}{s}\,.$$

The condition $\lim_{x\to\infty} u(x,t) = u_0$ implies $\lim_{x\to\infty} U(x,s) = u_0/s$, so we define $c_2 = 0$. Then

$$U(x,s) = c_1 e^{-\sqrt{s}\,x} + \frac{u_0}{s}\,.$$

The transform of the remaining boundary conditions gives

$$\left.\frac{dU}{dx}\right|_{x=0} = U(0,s).$$

This condition yields $c_1 = -u_0/s(\sqrt{s}+1)$. Thus

$$U(x,s) = -u_0\,\frac{e^{-\sqrt{s}\,x}}{s(\sqrt{s}+1)} + \frac{u_0}{s}$$

and

$$\begin{aligned} u(x,t) &= -u_0\,\mathscr{L}^{-1}\left\{\frac{e^{-x\sqrt{s}}}{s(\sqrt{s}+1)}\right\} + u_0\,\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} \\ &= u_0 e^{x+t}\,\text{erfc}\left(\sqrt{t} + \frac{x}{2\sqrt{t}}\right) - u_0\,\text{erfc}\left(\frac{x}{2\sqrt{t}}\right) + u_0 \qquad \boxed{\text{By (5) in the table in 15.1.}} \end{aligned}$$

16. We use

$$U(x,s) = c_1 e^{-\sqrt{s}\,x} + c_2 e^{\sqrt{s}\,x}.$$

The condition $\lim_{x\to\infty} u(x,t) = 0$ implies $\lim_{x\to\infty} U(x,s) = 0$, so we define $c_2 = 0$. Hence

$$U(x,s) = c_1 e^{-\sqrt{s}\,x}.$$

The remaining boundary condition transforms into

$$\left.\frac{dU}{dx}\right|_{x=0} = U(0,s) - \frac{50}{s}\,.$$

This condition gives $c_1 = 50/s(\sqrt{s}+1)$. Therefore

$$U(x,s) = 50\,\frac{e^{-\sqrt{s}\,x}}{s(\sqrt{s}+1)}$$

and

$$u(x,t) = 50\,\mathscr{L}^{-1}\left\{\frac{e^{-x\sqrt{s}}}{s(\sqrt{s}+1)}\right\} = -50e^{x+t}\,\text{erfc}\left(\sqrt{t} + \frac{x}{2\sqrt{t}}\right) + 50\,\text{erfc}\left(\frac{x}{2\sqrt{t}}\right).$$

17. We use

$$U(x,s) = c_1 e^{-\sqrt{s}\,x} + c_2 e^{\sqrt{s}\,x}.$$

The condition $\lim_{x\to\infty} u(x,t) = 0$ implies $\lim_{x\to\infty} U(x,s) = 0$, so we define $c_2 = 0$. Hence

$$U(x,s) = c_1 e^{-\sqrt{s}\,x}.$$

The transform of $u(0,t) = f(t)$ is $U(0,s) = F(s)$. Therefore

$$U(x,s) = F(s)e^{-\sqrt{s}\,x}$$

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and

$$u(x,t) = \mathscr{L}^{-1}\left\{F(s)e^{-x\sqrt{s}}\right\} = \frac{x}{2\sqrt{\pi}}\int_0^t \frac{f(t-\tau)e^{-x^2/4\tau}}{\tau^{3/2}}\,d\tau.$$

18. We use

$$U(x,s) = c_1 e^{-\sqrt{s}\,x} + c_2 e^{\sqrt{s}\,x}.$$

The condition $\lim_{x\to\infty} u(x,t) = 0$ implies $\lim_{x\to\infty} U(x,s) = 0$, so we define $c_2 = 0$. Then $U(x,s) = c_1 e^{-\sqrt{s}\,x}$. The transform of the remaining boundary condition gives

$$\left.\frac{dU}{dx}\right|_{x=0} = -F(s)$$

where $F(s) = \mathscr{L}\{f(t)\}$. This condition yields $c_1 = F(s)/\sqrt{s}$. Thus

$$U(x,s) = F(s)\,\frac{e^{-\sqrt{s}\,x}}{\sqrt{s}}.$$

Using entry (44) of the table and the convolution theorem we obtain

$$u(x,t) = \mathscr{L}^{-1}\left\{F(s)\cdot\frac{e^{-\sqrt{s}\,x}}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi}}\int_0^t f(\tau)\,\frac{e^{-x^2/4(t-\tau)}}{\sqrt{t-\tau}}\,d\tau.$$

19. Transforming the partial differential equation gives

$$\frac{d^2U}{dx^2} - sU = -60.$$

Using undetermied coefficients we obtain

$$U(x,s) = c_1 e^{-\sqrt{s}\,x} + c_2 e^{\sqrt{s}\,x} + \frac{60}{s}.$$

The condition $\lim_{x\to\infty} u(x,t) = 60$ implies $\lim_{x\to\infty} U(x,s) = 60/s$, so we define $c_2 = 0$. The transform of the remaining boundary condition gives

$$U(0,s) = \frac{60}{s} + \frac{40}{s}\,e^{-2s}.$$

This condition yields $c_1 = \dfrac{40}{s}\,e^{-2s}$. Thus

$$U(x,s) = \frac{60}{s} + 40e^{-2s}\,\frac{e^{-\sqrt{s}\,x}}{s}.$$

Using entry (46) of the table in Appendix III and the second translation theorem we obtain

$$u(x,t) = \mathscr{L}^{-1}\left\{\frac{60}{s} + 40e^{-2s}\,\frac{e^{-\sqrt{s}\,x}}{s}\right\} = 60 + 40\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t-2}}\right)\mathscr{U}(t-2).$$

20. The solution of the transformed equation

$$\frac{d^2U}{dx^2} - sU = -100$$

by undetermined coefficients is

$$U(x,s) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{100}{s}.$$

From the fact that $\lim_{x\to\infty} U(x,s) = 100/s$ we see that $c_1 = 0$. Thus

$$U(x,s) = c_2 e^{-\sqrt{s}\,x} + \frac{100}{s}. \qquad (1)$$

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Now the transform of the boundary condition at $x = 0$ is

$$U(0, s) = 20\left[\frac{1}{s} - \frac{1}{s}e^{-s}\right].$$

It follows from (1) that

$$\frac{20}{s} - \frac{20}{s}e^{-s} = c_2 + \frac{100}{s} \qquad \text{or} \qquad c_2 = -\frac{80}{s} - \frac{20}{s}e^{-s}$$

and so

$$\begin{aligned} U(x, s) &= \left(-\frac{80}{s} - \frac{20}{s}e^{-s}\right)e^{-\sqrt{s}\,x} + \frac{100}{s} \\ &= \frac{100}{s} - \frac{80}{s}e^{-\sqrt{s}\,x} - \frac{20}{s}e^{-\sqrt{s}\,x}e^{-s}. \end{aligned}$$

Thus

$$\begin{aligned} u(x, t) &= 100\,\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 80\,\mathscr{L}^{-1}\left\{\frac{e^{-\sqrt{s}\,x}}{s}\right\} - 20\,\mathscr{L}^{-1}\left\{\frac{e^{-\sqrt{s}\,x}}{s}e^{-s}\right\} \\ &= 100 - 80\,\text{erfc}\left(x/2\sqrt{t}\right) - 20\,\text{erfc}\left(x/2\sqrt{t-1}\right)\mathcal{U}(t-1). \end{aligned}$$

21. Transforming the partial differential equation gives

$$\frac{d^2U}{dx^2} - sU = 0$$

and so

$$U(x, s) = c_1e^{-\sqrt{s}\,x} + c_2e^{\sqrt{s}\,x}.$$

The condition $\lim_{x\to-\infty} u(x, t) = 0$ implies $\lim_{x\to-\infty} U(x, s) = 0$, so we define $c_1 = 0$. The transform of the remaining boundary condition gives

$$\left.\frac{dU}{dx}\right|_{x=1} = \frac{100}{s} - U(1, s).$$

This condition yields

$$c_2\sqrt{s}\,e^{\sqrt{s}} = \frac{100}{s} - c_2e^{\sqrt{s}}$$

from which it follows that

$$c_2 = \frac{100}{s(\sqrt{s}+1)}\,e^{-\sqrt{s}}.$$

Thus

$$U(x, s) = 100\,\frac{e^{-(1-x)\sqrt{s}}}{s(\sqrt{s}+1)}.$$

Using entry (49) of the table in Appendix III we obtain

$$u(x, t) = 100\,\mathscr{L}^{-1}\left\{\frac{e^{-(1-x)\sqrt{s}}}{s(\sqrt{s}+1)}\right\} = 100\left[-e^{1-x+t}\,\text{erfc}\left(\sqrt{t} + \frac{1-x}{\sqrt{t}}\right) + \text{erfc}\left(\frac{1-x}{2\sqrt{t}}\right)\right].$$

22. Transforming the partial differential equation gives

$$k\,\frac{d^2U}{dx^2} - sU = -\frac{r}{s}.$$

Using undetermined coefficients we obtian

$$U(x, s) = c_1e^{-\sqrt{s/k}\,x} + c_2e^{\sqrt{s/k}\,x} + \frac{r}{s^2}.$$

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The condition $\lim_{x\to\infty}\dfrac{\partial u}{\partial x}=0$ implies $\lim_{x\to\infty}\dfrac{dU}{dx}=0$, so we define $c_2=0$. The transform of the remaining boundary condition gives $U(0,s)=0$. This condition yields $c_1=-r/s^2$. Thus

$$U(x,s)=r\left[\frac{1}{s^2}-\frac{e^{-\sqrt{s/k}\,x}}{s^2}\right].$$

Using entries (3) and (46) of the table in Appendix III and the convolution theorem we obtain

$$u(x,t)=r\,\mathscr{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s}\cdot\frac{e^{-\sqrt{s/k}\,x}}{s}\right\}=rt-r\int_0^t \operatorname{erfc}\left(\frac{x}{2\sqrt{k\tau}}\right)d\tau.$$

23. The solution of

$$\frac{d^2U}{dx^2}-sU=-u_0-u_0\sin\frac{\pi}{L}x$$

is

$$U(x,s)=c_1\cosh(\sqrt{s}\,x)+c_2\sinh(\sqrt{s}\,x)+\frac{u_0}{s}+\frac{u_0}{s+\pi^2/L^2}\sin\frac{\pi}{L}x.$$

The transformed boundary conditions $U(0,s)=u_0/s$ and $U(L,s)=u_0/s$ give, in turn, $c_1=0$ and $c_2=0$. Therefore

$$U(x,s)=\frac{u_0}{s}+\frac{u_0}{s+\pi^2/L^2}\sin\frac{\pi}{L}x$$

and

$$u(x,t)=u_0\,\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}+u_0\,\mathscr{L}^{-1}\left\{\frac{1}{s+\pi^2/L^2}\right\}\sin\frac{\pi}{L}x=u_0+u_0e^{-\pi^2t/L^2}\sin\frac{\pi}{L}x.$$

24. The transform of the partial differential equation is

$$k\frac{d^2U}{dx^2}-hU+h\frac{u_m}{s}=sU-u_0$$

or

$$k\frac{d^2U}{dx^2}-(h+s)U=-h\frac{u_m}{s}-u_0.$$

By undetermined coefficients we find

$$U(x,s)=c_1e^{\sqrt{(h+s)/k}\,x}+c_2e^{-\sqrt{(h+s)/k}\,x}+\frac{hu_m+u_0s}{s(s+h)}.$$

The transformed boundary conditions are $U'(0,s)=0$ and $U'(L,s)=0$. These conditions imply $c_1=0$ and $c_2=0$. By partial fractions we then get

$$U(x,s)=\frac{hu_m+u_0s}{s(s+h)}=\frac{u_m}{s}-\frac{u_m}{s+h}+\frac{u_0}{s+h}.$$

Therefore,

$$u(x,t)=u_m\,\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}-u_m\,\mathscr{L}^{-1}\left\{\frac{1}{s+h}\right\}+u_0\,\mathscr{L}^{-1}\left\{\frac{1}{s+h}\right\}=u_m-u_me^{-ht}+u_0e^{-ht}.$$

25. We use

$$U(x,s)=c_1\cosh\sqrt{\frac{s}{k}}\,x+c_2\sinh\sqrt{\frac{s}{k}}\,x+\frac{u_0}{s}.$$

The transformed boundary conditions $\left.\dfrac{dU}{dx}\right|_{x=0}=0$ and $U(1,s)=0$ give, in turn, $c_2=0$ and

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$c_1 = -u_0/s\cosh\sqrt{s/k}$. Therefore

$$U(x,s) = \frac{u_0}{s} - \frac{u_0\cosh\sqrt{s/k}\,x}{s\cosh\sqrt{s/k}} = \frac{u_0}{s} - u_0\,\frac{e^{\sqrt{s/k}}\,x + e^{-\sqrt{s/k}\,x}}{s(e^{\sqrt{s/k}} + e^{-\sqrt{s/k}})}$$

$$= \frac{u_0}{s} - u_0\,\frac{e^{\sqrt{s/k}\,(x-1)} + e^{-\sqrt{s/k}\,(x+1)}}{s(1+e^{-2\sqrt{s/k}})}$$

$$= \frac{u_0}{s} - u_0\left[\frac{e^{-\sqrt{s/k}\,(1-x)}}{s} - \frac{e^{-\sqrt{s/k}\,(3-x)}}{s} + \frac{e^{-\sqrt{s/k}\,(5-x)}}{s} - \cdots\right]$$

$$- u_0\left[\frac{e^{-\sqrt{s/k}\,(1+x)}}{s} - \frac{e^{-\sqrt{s/k}\,(3+x)}}{s} + \frac{e^{-\sqrt{s/k}\,(5+x)}}{s} - \cdots\right]$$

$$= \frac{u_0}{s} - u_0\sum_{n=0}^{\infty}(-1)^n\left[\frac{e^{-(2n+1-x)\sqrt{s}/\sqrt{k}}}{s} + \frac{e^{-(2n+1+x)\sqrt{s}/\sqrt{k}}}{s}\right]$$

and

$$u(x,t) = u_0\,\mathscr{L}\left\{\frac{1}{s}\right\} - u_0\sum_{n=0}^{\infty}(-1)^n\left[\mathscr{L}^{-1}\left\{\frac{e^{-(2n+1-x)\sqrt{s}/\sqrt{k}}}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{e^{-(2n+1+x)\sqrt{s}/\sqrt{k}}}{s}\right\}\right]$$

$$= u_0 - u_0\sum_{n=0}^{\infty}(-1)^n\left[\operatorname{erfc}\left(\frac{2n+1-x}{2\sqrt{kt}}\right) - \operatorname{erfc}\left(\frac{2n+1+x}{2\sqrt{kt}}\right)\right].$$

26. We use

$$c(x,s) = c_1\cosh\sqrt{\frac{s}{D}}\,x + c_2\sinh\sqrt{\frac{s}{D}}\,x.$$

The transform of the two boundary conditions are $c(0,s) = c_0/s$ and $c(1,s) = c_0/s$. From these conditions we obtain $c_1 = c_0/s$ and

$$c_2 = c_0(1-\cosh\sqrt{s/D})/s\sinh\sqrt{s/D}.$$

Therefore

$$c(x,s) = c_0\left[\frac{\cosh\sqrt{s/D}\,x}{s} + \frac{(1-\cosh\sqrt{s/D})}{s\sinh\sqrt{s/D}}\sinh\sqrt{s/D}\,x\right]$$

$$= c_0\left[\frac{\sinh\sqrt{s/D}\,(1-x)}{s\sinh\sqrt{s/D}} + \frac{\sin\sqrt{s/D}\,x}{s\sinh\sqrt{s/D}}\right]$$

$$= c_0\left[\frac{e^{\sqrt{s/D}\,(1-x)} - e^{-\sqrt{s/D}\,(1-x)}}{s(e^{\sqrt{s/D}} - e^{-\sqrt{s/D}})} + \frac{e^{\sqrt{s/D}\,x} - e^{-\sqrt{s/D}\,x}}{s(e^{\sqrt{s/D}} - e^{-\sqrt{s/D}})}\right]$$

$$= c_0\left[\frac{e^{-\sqrt{s/D}\,x} - e^{-\sqrt{s/D}\,(2-x)}}{s(1-e^{-2\sqrt{s/D}})} + \frac{e^{\sqrt{s/D}\,(x-1)} - e^{-\sqrt{s/D}\,(x+1)}}{s(1-e^{-2\sqrt{s/D}})}\right]$$

$$= c_0\,\frac{(e^{-\sqrt{s/D}\,x} - e^{-\sqrt{s/D}\,(2-x)})}{s}\left(1 + e^{-2\sqrt{s/D}} + e^{-4\sqrt{s/D}} + \cdots\right)$$

$$+ c_0\,\frac{(e^{\sqrt{s/D}\,(x-1)} - e^{-\sqrt{s/D}\,(x+1)}}{s}\left(1 + e^{-2\sqrt{s/D}} + e^{-4\sqrt{s/D}} + \cdots\right)$$

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$$= c_0 \sum_{n=0}^{\infty} \left[\frac{e^{-(2n+x)\sqrt{s/D}}}{s} - \frac{e^{-(2n+2-x)\sqrt{s/D}}}{s} \right]$$
$$+ c_0 \sum_{n=0}^{\infty} \left[\frac{e^{-(2n+1-x)\sqrt{s/D}}}{s} - \frac{e^{-(2n+1+x)\sqrt{s/D}}}{s} \right]$$

and

$$c(x,t) = c_0 \sum_{n=0}^{\infty} \left[\mathscr{L}^{-1}\left\{ \frac{e^{-\frac{(2n+x)}{\sqrt{D}}\sqrt{s}}}{s} \right\} - \mathscr{L}^{-1}\left\{ \frac{e^{-\frac{(2n+2-x)}{\sqrt{D}}\sqrt{s}}}{s} \right\} \right]$$
$$+ c_0 \sum_{n=0}^{\infty} \left[\mathscr{L}^{-1}\left\{ \frac{e^{-\frac{(2n+1-x)}{\sqrt{D}}\sqrt{s}}}{s} \right\} - \mathscr{L}^{-1}\left\{ \frac{e^{-\frac{(2n+1+x)}{\sqrt{D}}\sqrt{s}}}{s} \right\} \right]$$
$$= c_0 \sum_{n=0}^{\infty} \left[\operatorname{erfc}\left(\frac{2n+x}{2\sqrt{Dt}} \right) - \operatorname{erfc}\left(\frac{2n+2-x}{2\sqrt{Dt}} \right) \right]$$
$$+ c_0 \sum_{n=0}^{\infty} \left[\operatorname{erfc}\left(\frac{2n+1-x}{2\sqrt{Dt}} \right) - \operatorname{erfc}\left(\frac{2n+1+x}{2\sqrt{Dt}} \right) \right].$$

Now using $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$ we get

$$c(x,t) = c_0 \sum_{n=0}^{\infty} \left[\operatorname{erf}\left(\frac{2n+2-x}{2\sqrt{Dt}} \right) - \operatorname{erf}\left(\frac{2n+x}{2\sqrt{Dt}} \right) \right]$$
$$+ c_0 \sum_{n=0}^{\infty} \left[\operatorname{erf}\left(\frac{2n+1+x}{2\sqrt{Dt}} \right) - \operatorname{erf}\left(\frac{2n+1-x}{2\sqrt{Dt}} \right) \right].$$

27. We use

$$U(x,s) = c_1 e^{-\sqrt{RCs+RG}\,x} + c_2 e^{\sqrt{RCs+RG}} + \frac{Cu_0}{Cs+G}.$$

The condition $\lim_{x\to\infty} \partial u/\partial x = 0$ implies $\lim_{x\to\infty} dU/dx = 0$, so we define $c_2 = 0$. Applying $U(0,s) = 0$ to

$$U(x,s) = c_1 e^{-\sqrt{RCsRG}\,x} + \frac{Cu_0}{Cs+G}$$

gives $c_1 = -Cu_0/(Cs+G)$. Therefore

$$U(x,s) = -Cu_0 \frac{e^{-\sqrt{RCs+RG}\,x}}{Cs+G} + \frac{Cu_0}{Cs+G}$$

and

$$u(x,t) = u_0 \mathscr{L}^{-1}\left\{ \frac{1}{s+G/C} \right\} - u_0 \mathscr{L}^{-1}\left\{ \frac{e^{-x\sqrt{RC}\sqrt{s+G/C}}}{s+G/C} \right\}$$
$$= u_0 e^{-Gt/C} - u_0 e^{-Gt/C} \operatorname{erfc}\left(\frac{x\sqrt{RC}}{2\sqrt{t}} \right)$$
$$= u_0 e^{-Gt/C} \left[1 - \operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{RC}{t}} \right) \right]$$
$$= u_0 e^{-Gt/C} \operatorname{erf}\left(\frac{x}{2}\sqrt{\frac{RC}{t}} \right).$$

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28. We use

$$U(x,s) = c_1 e^{-\sqrt{s+h}\,x} + c_2 e^{\sqrt{s+h}\,x}.$$

The condition $\lim_{x\to\infty} u(x,t) = 0$ implies $\lim_{x\to\infty} U(x,s) = 0$, so we take $c_2 = 0$. Therefore

$$U(x,s) = c_1 e^{-\sqrt{s+h}\,x}.$$

The Laplace transform of $u(0,t) = u_0$ is $U(0,s) = u_0/s$ and so

$$U(x,s) = u_0\,\frac{e^{-\sqrt{s+h}\,x}}{s}$$

and

$$u(x,t) = u_0\,\mathscr{L}^{-1}\left\{\frac{e^{-\sqrt{s+h}\,x}}{s}\right\} = u_0\,\mathscr{L}^{-1}\left\{\frac{1}{s}\,e^{-\sqrt{s+h}\,x}\right\}.$$

From the first translation theorem,

$$\mathscr{L}^{-1}\left\{e^{-\sqrt{s+h}\,x}\right\} = e^{-ht}\,\mathscr{L}^{-1}\{e^{-x\sqrt{s}}\} = e^{-ht}\frac{x}{2\sqrt{\pi t^3}}\,e^{-x^2/4t}.$$

Thus, from the convolution theorem we obtain

$$u(x,s) = \frac{u_0 x}{2\sqrt{\pi}}\int_0^t \frac{e^{-h\tau - x^2/4\tau}}{\tau^{3/2}}\,d\tau.$$

29. **(a)** Letting $C(x,s) = \mathscr{L}\{c(x,t)\}$ we obtain

$$\frac{d^2C}{dx^2} - \frac{s}{k}C = 0 \qquad \text{subject to} \qquad \left.\frac{dC}{dx}\right|_{x=0} = -A.$$

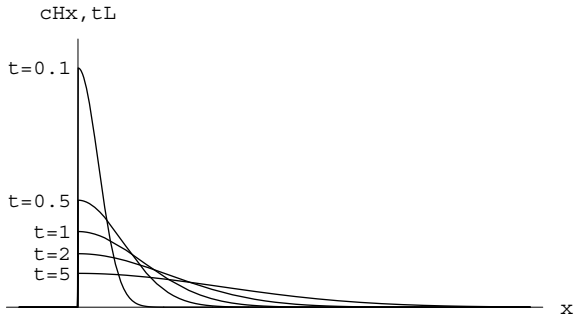
The solution of this initial-value problem is

$$C(x,s) = A\sqrt{k}\,\frac{e^{-(x/\sqrt{k})\sqrt{s}}}{\sqrt{s}},$$

so that

$$c(x,t) = A\sqrt{\frac{k}{\pi t}}\,e^{-x^2/4kt}.$$

(b)



(c) $\displaystyle\int_0^\infty c(x,t)dx = Ak\,\text{erf}\frac{x}{2\sqrt{kt}}\bigg|_0^\infty = Ak(1-0) = Ak$

30. **(a)** We use

$$U(x,s) = c_1 e^{-(s/a)x} + c_2 e^{(s/a)x} + \frac{v_0^2 F_0}{(a^2 - v_0^2)s^2}\,e^{-(s/v_0)x}.$$

The condition $\lim_{x\to\infty} u(x,t) = 0$ implies $\lim_{x\to\infty} U(x,s) = 0$, so we must define $c_2 = 0$. Consequently

$$U(x,s) = c_1 e^{-(s/a)x} + \frac{v_0^2 F_0}{(a^2 - v_0^2)s^2}\,e^{-(s/v_0)x}.$$

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The remaining boundary condition transforms into $U(0, s) = 0$. From this we find

$$c_1 = -v_0^2 F_0/(a^2 - v_0^2)s^2.$$

Therefore, by the second translation theorem

$$U(x, s) = -\frac{v_0^2 F_0}{(a^2 - v_0^2)s^2}\, e^{-(s/a)x} + \frac{v_0^2 F_0}{(a^2 - v_0^2)s^2}\, e^{-(s/v_0)x}$$

and

$$\begin{aligned} u(x, t) &= \frac{v_0^2 F_0}{a^2 - v_0^2}\left[\mathscr{L}^{-1}\left\{\frac{e^{-(x/v_0)s}}{s^2}\right\} - \mathscr{L}^{-1}\left\{\frac{e^{-(x/a)s}}{s^2}\right\}\right] \\ &= \frac{v_0^2 F_0}{a^2 - v_0^2}\left[\left(t - \frac{x}{v_0}\right)\mathscr{U}\left(t - \frac{x}{v_0}\right) - \left(t - \frac{x}{a}\right)\mathscr{U}\left(t - \frac{x}{a}\right)\right]. \end{aligned}$$

(b) In the case when $v_0 = a$ the solution of the transformed equation is

$$U(x, s) = c_1 e^{-(s/a)x} + c_2 e^{(s/a)x} - \frac{F_0}{2as} x e^{-(s/a)x}.$$

The usual analysis then leads to $c_1 = 0$ and $c_2 = 0$. Therefore

$$U(x, s) = -\frac{F_0}{2as} x e^{-(s/a)x}$$

and

$$u(x, t) = -\frac{xF_0}{2a}\,\mathscr{L}^{-1}\left\{\frac{e^{-(x/a)s}}{s}\right\} = -\frac{xF_0}{2a}\,\mathscr{U}\left(t - \frac{x}{a}\right).$$

Exercises 15.3

1. From formulas (5) and (6) in the text,

$$A(\alpha) = \int_{-1}^{0} (-1)\cos\alpha x\, dx + \int_{0}^{1} (2)\cos\alpha x\, dx = -\frac{\sin\alpha}{\alpha} + 2\frac{\sin\alpha}{\alpha} = \frac{\sin\alpha}{\alpha}$$

and

$$\begin{aligned} B(\alpha) &= \int_{-1}^{0} (-1)\sin\alpha x\, dx + \int_{0}^{1} (2)\sin\alpha x\, dx \\ &= \frac{1 - \cos\alpha}{\alpha} - 2\frac{\cos\alpha - 1}{\alpha} = \frac{3(1 - \cos\alpha)}{\alpha}. \end{aligned}$$

Hence

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\sin\alpha\cos\alpha x + 3(1 - \cos\alpha)\sin\alpha x}{\alpha}\, d\alpha.$$

2. From formulas (5) and (6) in the text,

$$A(\alpha) = \int_{\pi}^{2\pi} 4\cos\alpha x\, dx = 4\frac{\sin 2\pi\alpha - \sin\pi\alpha}{\alpha}$$

and

$$B(\alpha) = \int_{\pi}^{2\pi} 4\sin\alpha x\, dx = 4\frac{\cos\pi\alpha - \cos 2\pi\alpha}{\alpha}.$$

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Hence

$$f(x) = \frac{4}{\pi}\int_0^\infty \frac{(\sin 2\pi\alpha - \sin\pi\alpha)\cos\alpha x + (\cos\pi\alpha - \cos 2\pi\alpha)\sin\alpha x}{\alpha}\,d\alpha$$
$$= \frac{4}{\pi}\int_0^\infty \frac{\sin 2\pi\alpha\cos\alpha x - \cos 2\pi\alpha\sin\alpha x - \sin\pi\alpha\cos\alpha x + \cos\pi\alpha\sin\alpha x}{\alpha}\,d\alpha$$
$$= \frac{4}{\pi}\int_0^\infty \frac{\sin\alpha(2\pi - x) - \sin\alpha(\pi - x)}{\alpha}\,d\alpha.$$

3. From formulas (5) and (6) in the text,

$$A(\alpha) = \int_0^3 x\cos\alpha x\,dx = \frac{x\sin\alpha x}{\alpha}\,\Big|_0^3 - \frac{1}{\alpha}\int_0^3 \sin\alpha x\,dx$$
$$= \frac{3\sin 3\alpha}{\alpha} + \frac{\cos\alpha x}{\alpha^2}\,\Big|_0^3 = \frac{3\alpha\sin 3\alpha + \cos 3\alpha - 1}{\alpha^2}$$

and

$$B(\alpha) = \int_0^3 x\sin\alpha x\,dx = -\frac{x\cos\alpha x}{\alpha}\,\Big|_0^3 + \frac{1}{\alpha}\int_0^3 \cos\alpha x\,dx$$
$$= -\frac{3\cos 3\alpha}{\alpha} + \frac{\sin\alpha x}{\alpha^2}\,\Big|_0^3 = \frac{\sin 3\alpha - 3\alpha\cos 3\alpha}{\alpha^2}.$$

Hence

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{(3\alpha\sin 3\alpha + \cos 3\alpha - 1)\cos\alpha x + (\sin 3\alpha - 3\alpha\cos 3\alpha)\sin\alpha x}{\alpha^2}\,d\alpha$$
$$= \frac{1}{\pi}\int_0^\infty \frac{3\alpha(\sin 3\alpha\cos\alpha x - \cos 3\alpha\sin\alpha x) + \cos 3\alpha\cos\alpha x + \sin 3\alpha\sin\alpha x - \cos\alpha x}{\alpha^2}\,d\alpha$$
$$= \frac{1}{\pi}\int_0^\infty \frac{3\alpha\sin\alpha(3 - x) + \cos\alpha(3 - x) - \cos\alpha x}{\alpha^2}\,d\alpha.$$

4. From formulas (5) and (6) in the text,

$$A(\alpha) = \int_{-\infty}^\infty f(x)\cos\alpha x\,dx$$
$$= \int_{-\infty}^0 0\cdot\cos\alpha x\,dx + \int_0^\pi \sin x\cos\alpha x\,dx + \int_\pi^\infty 0\cdot\cos\alpha x\,dx$$
$$= \frac{1}{2}\int_0^\pi [\sin(1+\alpha)x + \sin(1-\alpha)x]\,dx$$
$$= \frac{1}{2}\left[-\frac{\cos(1+\alpha)x}{1+\alpha} - \frac{\cos(1-\alpha)x}{1-\alpha}\right]_0^\pi$$
$$= -\frac{1}{2}\left[\frac{\cos(1+\alpha)\pi - 1}{1+\alpha} + \frac{\cos(1-\alpha)\pi - 1}{1-\alpha}\right]$$
$$= -\frac{1}{2}\left[\frac{\cos(1+\alpha)\pi - \alpha\cos(1+\alpha)\pi + \cos(1-\alpha)\pi + \alpha\cos(1-\alpha)\pi - 2}{1-\alpha^2}\right]$$
$$= \frac{1 + \cos\alpha\pi}{1-\alpha^2},$$

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and

$$B(\alpha) = \int_0^{\pi} \sin x \sin \alpha x \, dx = \frac{1}{2} \int_0^{\pi} [\cos(1-\alpha)x - \cos(1+\alpha)] \, dx$$

$$= \frac{1}{2} \left[\frac{\sin(1-\alpha)\pi}{1-\alpha} - \frac{\sin(1+\alpha)\pi}{1+\alpha} \right] = \frac{\sin \alpha\pi}{1-\alpha^2} \, .$$

Hence

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos \alpha x + \cos \alpha x \cos \alpha\pi + \sin \alpha x \sin \alpha\pi}{1-\alpha^2} \, d\alpha$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{\cos \alpha x + \cos \alpha(x-\pi)}{1-\alpha^2} \, d\alpha.$$

5. From formula (5) in the text,

$$A(\alpha) = \int_0^{\infty} e^{-x} \cos \alpha x \, dx.$$

Recall $\mathscr{L}\{\cos kt\} = s/(s^2+k^2)$. If we set $s = 1$ and $k = \alpha$ we obtain

$$A(\alpha) = \frac{1}{1+\alpha^2} \, .$$

Now

$$B(\alpha) = \int_0^{\infty} e^{-x} \sin \alpha x \, dx.$$

Recall $\mathscr{L}\{\sin kt\} = k/(s^2+k^2)$. If we set $s = 1$ and $k = \alpha$ we obtain

$$B(\alpha) = \frac{\alpha}{1+\alpha^2} \, .$$

Hence

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos \alpha x + \alpha \sin \alpha x}{1+\alpha^2} \, d\alpha.$$

6. From formulas (5) and (6) in the text,

$$A(\alpha) = \int_{-1}^{1} e^x \cos \alpha x \, dx$$

$$= \frac{e(\cos \alpha + \alpha \sin \alpha) - e^{-1}(\cos \alpha - \alpha \sin \alpha)}{1+\alpha^2}$$

$$= \frac{2(\sinh 1)\cos \alpha - 2\alpha(\cosh 1)\sin \alpha}{1+\alpha^2}$$

and

$$B(\alpha) = \int_{-1}^{1} e^x \sin \alpha x \, dx$$

$$= \frac{e(\sin \alpha - \alpha \cos \alpha) - e^{-1}(-\sin \alpha - \alpha \cos \alpha)}{1+\alpha^2}$$

$$= \frac{2(\cosh 1)\sin \alpha - 2\alpha(\sinh 1)\cos \alpha}{1+\alpha^2} \, .$$

Hence

$$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\alpha)\cos \alpha x + B(\alpha)\sin \alpha x] \, d\alpha.$$

7. The function is odd. Thus from formula (11) in the text

$$B(\alpha) = 5 \int_0^{1} \sin \alpha x \, dx = \frac{5(1-\cos \alpha)}{\alpha} \, .$$

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Hence from formula (10) in the text,

$$f(x) = \frac{10}{\pi}\int_0^\infty \frac{(1-\cos\alpha)\sin\alpha x}{\alpha}\,d\alpha.$$

8. The function is even. Thus from formula (9) in the text

$$A(\alpha) = \pi\int_1^2 \cos\alpha x\,dx = \pi\left(\frac{\sin 2\alpha - \sin\alpha}{\alpha}\right).$$

Hence from formula (8) in the text,

$$f(x) = 2\int_0^\infty \frac{(\sin 2\alpha - \sin\alpha)\cos\alpha x}{\alpha}\,d\alpha.$$

9. The function is even. Thus from formula (9) in the text

$$\begin{aligned} A(\alpha) &= \int_0^\pi x\cos\alpha x\,dx = \frac{x\sin\alpha x}{\alpha}\,\Big|_0^\pi - \frac{1}{\alpha}\int_0^\pi \sin\alpha x\,dx \\ &= \frac{\pi\alpha\sin\pi\alpha}{\alpha} + \frac{1}{\alpha^2}\cos\alpha x\,\Big|_0^\pi = \frac{\pi\alpha\sin\pi\alpha + \cos\pi\alpha - 1}{\alpha^2}. \end{aligned}$$

Hence from formula (8) in the text

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{(\pi\alpha\sin\pi\alpha + \cos\pi\alpha - 1)\cos\alpha x}{\alpha^2}\,d\alpha.$$

10. The function is odd. Thus from formula (11) in the text

$$\begin{aligned} B(\alpha) &= \int_0^\pi x\sin\alpha x\,dx = -\frac{x\cos\alpha x}{\alpha}\,\Big|_0^\pi + \frac{1}{\alpha}\int_0^\pi \cos\alpha x\,dx \\ &= -\frac{\pi\cos\pi\alpha}{\alpha} + \frac{1}{\alpha^2}\sin\alpha x\,\Big|_0^\pi = \frac{-\pi\alpha\cos\pi\alpha + \sin\pi\alpha}{\alpha^2}. \end{aligned}$$

Hence from formula (10) in the text,

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{(-\pi\alpha\cos\pi\alpha + \sin\pi\alpha)\sin\alpha x}{\alpha^2}\,d\alpha.$$

11. The function is odd. Thus from formula (11) in the text

$$\begin{aligned} B(\alpha) &= \int_0^\infty (e^{-x}\sin x)\sin\alpha x\,dx \\ &= \frac{1}{2}\int_0^\infty e^{-x}[\cos(1-\alpha)x - \cos(1+\alpha)x]\,dx \\ &= \frac{1}{2}\int_0^\infty e^{-x}\cos(1-\alpha)x\,dx - \frac{1}{2}\int_0^\infty e^{-x}\cos(1+\alpha)x,dx. \end{aligned}$$

Now recall

$$\mathscr{L}\{\cos kt\} = \int_0^\infty e^{-st}\cos kt\,dt = s/(s^2+k^2).$$

If we set $s = 1$, and in turn, $k = 1-\alpha$ and then $k = 1+\alpha$, we obtain

$$B(\alpha) = \frac{1}{2}\,\frac{1}{1+(1-\alpha)^2} - \frac{1}{2}\,\frac{1}{1+(1+\alpha)^2} = \frac{1}{2}\,\frac{(1+\alpha)^2-(1-\alpha)^2}{[1+(1-\alpha)^2][1+(1+\alpha)^2]}.$$

Simplifying the last expression gives

$$B(\alpha) = \frac{2\alpha}{4+\alpha^4}.$$

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Hence from formula (10) in the text

$$f(x) = \frac{4}{\pi}\int_0^\infty \frac{\alpha \sin \alpha x}{4+\alpha^4}\, d\alpha.$$

12. The function is odd. Thus from formula (11) in the text

$$B(\alpha) = \int_0^\infty xe^{-x} \sin \alpha x\, dx.$$

Now recall

$$\mathscr{L}\{t \sin kt\} = -\frac{d}{ds}\mathscr{L}\{\sin kt\} = 2ks/(s^2+k^2)^2.$$

If we set $s = 1$ and $k = \alpha$ we obtain

$$B(\alpha) = \frac{2\alpha}{(1+\alpha^2)^2}\,.$$

Hence from formula (10) in the text

$$f(x) = \frac{4}{\pi}\int_0^\infty \frac{\alpha \sin \alpha x}{(1+\alpha^2)^2}\, d\alpha.$$

13. For the cosine integral,

$$A(\alpha) = \int_0^\infty e^{-kx} \cos \alpha x\, dx = \frac{k}{k^2+\alpha^2}\,.$$

Hence

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{k \cos \alpha x}{k^2+\alpha^2} = \frac{2k}{\pi}\int_0^\infty \frac{\cos \alpha x}{k^2+\alpha^2}\, d\alpha.$$

For the sine integral,

$$B(\alpha) = \int_0^\infty e^{-kx} \sin \alpha x\, dx = \frac{\alpha}{k^2+\alpha^2}\,.$$

Hence

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\alpha \sin \alpha x}{k^2+\alpha^2}\, d\alpha.$$

14. From Problem 13 the cosine and sine integral representations of $e^{-kx},\ k > 0$, are respectively,

$$e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos \alpha x}{k^2+\alpha^2}\, d\alpha \quad \text{and} \quad e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{\alpha \sin \alpha x}{k^2+\alpha^2}\, d\alpha.$$

Hence, the cosine integral representation of $f(x) = e^{-x} - e^{-3x}$ is

$$e^{-x} - e^{-3x} = \frac{2}{\pi}\int_0^\infty \frac{\cos \alpha x}{1+\alpha^2}\, d\alpha - \frac{2(3)}{\pi}\int_0^\infty \frac{\cos \alpha x}{9+\alpha^2}\, d\alpha = \frac{4}{\pi}\int_0^\infty \frac{3-\alpha^2}{(1+\alpha^2)\,(9+\alpha^2)} \cos \alpha x\, d\alpha.$$

The sine integral representation of f is

$$e^{-x} - e^{-3x} = \frac{2}{\pi}\int_0^\infty \frac{\alpha \sin \alpha x}{1+\alpha^2}\, d\alpha - \frac{2}{\pi}\int_0^\infty \frac{\alpha \sin \alpha x}{9+\alpha^2}\, d\alpha = \frac{16}{\pi}\int_0^\infty \frac{\alpha \sin \alpha x}{(1+\alpha^2)\,(9+\alpha^2)}\, d\alpha.$$

15. For the cosine integral,

$$A(\alpha) = \int_0^\infty xe^{-2x} \cos \alpha x\, dx.$$

But we know

$$\mathscr{L}\{t \cos kt\} = -\frac{d}{ds}\,\frac{s}{(s^2+k^2)} = \frac{(s^2-k^2)}{(s^2+k^2)^2}\,.$$

If we set $s = 2$ and $k = \alpha$ we obtain

$$A(\alpha) = \frac{4-\alpha^2}{(4+\alpha^2)^2}\,.$$

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Hence

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{(4-\alpha^2)\cos\alpha x}{(4+\alpha^2)^2}\,d\alpha.$$

For the sine integral,

$$B(\alpha) = \int_0^\infty xe^{-2x}\sin\alpha x\,dx.$$

From Problem 12, we know

$$\mathscr{L}\{t\sin kt\} = \frac{2ks}{(s^2+k^2)^2}\,.$$

If we set $s = 2$ and $k = \alpha$ we obtain

$$B(\alpha) = \frac{4\alpha}{(4+\alpha^2)^2}\,.$$

Hence

$$f(x) = \frac{8}{\pi}\int_0^\infty \frac{\alpha\sin\alpha x}{(4+\alpha^2)^2}\,d\alpha.$$

16. For the cosine integral,

$$\begin{aligned}
A(\alpha) &= \int_0^\infty e^{-x}\cos x\cos\alpha x\,dx \\
&= \frac{1}{2}\int_0^\infty e^{-x}[\cos(1+\alpha)x + \cos(1-\alpha)x]\,dx \\
&= \frac{1}{2}\,\frac{1}{1+(1+\alpha)^2} + \frac{1}{2}\,\frac{1}{1+(1-\alpha)^2} \\
&= \frac{1}{2}\,\frac{1+(1-\alpha)^2+1+(1+\alpha)^2}{[1+(1+\alpha)^2][1+(1-\alpha)^2]} \\
&= \frac{2+\alpha^2}{4+\alpha^4}\,.
\end{aligned}$$

Hence

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{(2+\alpha^2)\cos\alpha x}{4+\alpha^4}\,d\alpha.$$

For the sine integral,

$$\begin{aligned}
B(\alpha) &= \int_0^\infty e^{-x}\cos x\sin\alpha x\,dx \\
&= \frac{1}{2}\int_0^\infty e^{-x}[\sin(1+\alpha)x - \sin(1-\alpha)x]\,dx \\
&= \frac{1}{2}\,\frac{1+\alpha}{1+(1+\alpha)^2} - \frac{1}{2}\,\frac{1-\alpha}{1+(1-\alpha)^2} \\
&= \frac{1}{2}\left[\frac{(1+\alpha)[1+(1-\alpha)^2]-(1-\alpha)[1+(1+\alpha)^2]}{[1+(1+\alpha)^2][1+(1-\alpha)^2]}\right] \\
&= \frac{\alpha^3}{4+\alpha^4}\,.
\end{aligned}$$

Hence

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\alpha^3\sin\alpha x}{4+\alpha^4}\,d\alpha.$$

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17. By formula (8) in the text

$$f(x) = 2\pi \int_0^\infty e^{-\alpha} \cos \alpha x \, d\alpha = \frac{2}{\pi} \frac{1}{1+x^2}, \; x > 0.$$

18. From the formula for sine integral of $f(x)$ we have

$$\begin{aligned} f(x) &= \frac{2}{\pi} \int_0^\infty \left(\int_0^\infty f(x) \sin \alpha x \, dx \right) \sin \alpha x \, dx \\ &= \frac{2}{\pi} \left[\int_0^1 1 \cdot \sin \alpha x \, d\alpha + \int_1^\infty 0 \cdot \sin \alpha x \, d\alpha \right] \\ &= \frac{2}{\pi} \frac{(-\cos \alpha x)}{x} \Big|_0^1 = \frac{2}{\pi} \frac{1 - \cos x}{x}. \end{aligned}$$

19. **(a)** From formula (7) in the text with $x = 2$, we have

$$\frac{1}{2} = \frac{2}{\pi} \int_0^\infty \frac{\sin \alpha \cos \alpha}{\alpha} \, d\alpha = \frac{1}{\pi} \int_0^\infty \frac{\sin 2\alpha}{\alpha} \, d\alpha.$$

If we let $\alpha = x$ we obtain

$$\int_0^\infty \frac{\sin 2x}{x} \, dx = \frac{\pi}{2}.$$

(b) If we now let $2x = kt$ where $k > 0$, then $dx = (k/2)dt$ and the integral in part (a) becomes

$$\int_0^\infty \frac{\sin kt}{kt/2} (k/2) \, dt = \int_0^\infty \frac{\sin kt}{t} \, dt = \frac{\pi}{2}.$$

20. With $f(x) = e^{-|x|}$, formula (16) in the text is

$$C(\alpha) = \int_{-\infty}^\infty e^{-|x|} e^{i\alpha x} dx = \int_{-\infty}^\infty e^{-|x|} \cos \alpha x \, dx + i \int_{-\infty}^\infty e^{-|x|} \sin \alpha x \, dx.$$

The imaginary part in the last line is zero since the integrand is an odd function of x. Therefore,

$$C(\alpha) = \int_{-\infty}^\infty e^{-|x|} \cos \alpha x \, dx = 2 \int_0^\infty e^{-x} \cos \alpha x \, dx = \frac{2}{1+\alpha^2}$$

and so from formula (15) in the text,

$$f(x) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{\cos \alpha x}{1+\alpha^2} \, d\alpha = \frac{2}{\pi} \int_0^\infty \frac{\cos \alpha x}{1+\alpha^2} \, d\alpha.$$

This is the same result obtained from formulas (8) and (9) in the text.

Exercises 15.4

For the boundary-value problems in this section it is sometimes useful to note that the identities

$$e^{i\alpha} = \cos \alpha + i \sin \alpha \quad \text{and} \quad e^{-i\alpha} = \cos \alpha - i \sin \alpha$$

imply

$$e^{i\alpha} + e^{-i\alpha} = 2 \cos \alpha \quad \text{and} \quad e^{i\alpha} - e^{-i\alpha} = 2i \sin \alpha.$$

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1. Using the Fourier transform, the partial differential equation becomes

$$\frac{dU}{dt} + k\alpha^2 U = 0 \qquad \text{and so} \qquad U(\alpha, t) = ce^{-k\alpha^2 t}.$$

Now

$$\mathscr{F}\{u(x,0)\} = U(\alpha, 0) = \mathscr{F}\left\{e^{-|x|}\right\}.$$

We have

$$\mathscr{F}\left\{e^{-|x|}\right\} = \int_{-\infty}^{\infty} e^{-|x|} e^{i\alpha x} dx = \int_{-\infty}^{\infty} e^{-|x|}(\cos\alpha x + i\sin\alpha x)\, dx = \int_{-\infty}^{\infty} e^{-|x|}\cos\alpha x\, dx.$$

The integral

$$\int_{-\infty}^{\infty} e^{-|x|}\sin\alpha x\, dx = 0$$

since the integrand is an odd function of x. Continuing we obtain

$$\mathscr{F}\left\{e^{-|x|}\right\} = 2\int_0^{\infty} e^{-x}\cos\alpha x\, dx = \frac{2}{1+\alpha^2}.$$

But $U(\alpha, 0) = c = \dfrac{2}{1+\alpha^2}$ gives

$$U(\alpha, t) = \frac{2e^{-k\alpha^2 t}}{1+\alpha^2}$$

and so

$$u(x,t) = \frac{2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-k\alpha^2 t}e^{-i\alpha x}}{1+\alpha^2}\, d\alpha = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-k\alpha^2 t}}{1+\alpha^2}(\cos\alpha x - i\sin\alpha x) d\alpha$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-k\alpha^2 t}\cos\alpha x}{1+\alpha^2}\, d\alpha = \frac{2}{\pi}\int_0^{\infty} \frac{e^{-k\alpha^2 t}\cos\alpha x}{1+\alpha^2}\, d\alpha.$$

2. Since the domain of x is $(-\infty, \infty)$ we transform the differential equation using the Fourier transform:

$$-k\alpha^2 U(\alpha, t) = \frac{du}{dt}$$

$$\frac{du}{dt} + k\alpha^2 U(\alpha, t) = 0$$

$$U(\alpha, t) = ce^{-k\alpha^2 t}. \tag{1}$$

The transform of the initial condition is

$$\mathscr{F}\{u(x,0)\} = \int_{-\infty}^{\infty} u(x,0)e^{i\alpha x} dx = \int_{-1}^{0} (-100e^{i\alpha x}) dx + \int_0^1 100e^{i\alpha x} dx$$

$$= -100\frac{1-e^{-i\alpha}}{i\alpha} + 100\frac{e^{i\alpha}-1}{i\alpha} = 100\frac{e^{i\alpha}+e^{-i\alpha}-2}{i\alpha}$$

$$= 100\frac{2\cos\alpha - 2}{i\alpha} = 200\frac{\cos\alpha - 1}{i\alpha}.$$

Thus

$$U(\alpha, 0) = 200\frac{\cos\alpha - 1}{i\alpha},$$

and since $c = U(\alpha, 0)$ in (1) we have

$$U(\alpha, t) = 200\frac{\cos\alpha - 1}{i\alpha}e^{-k\alpha^2 t}.$$

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Applying the inverse Fourier transform we obtain

$$\begin{aligned} u(x,t) &= \mathscr{F}^{-1}\{U(\alpha,t)\} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 200\frac{\cos\alpha - 1}{i\alpha}e^{-k\alpha^2 t}e^{-i\alpha x}\,d\alpha \\ &= \frac{100}{\pi}\int_{-\infty}^{\infty} 200\frac{\cos\alpha - 1}{i\alpha}e^{-k\alpha^2 t}(\cos\alpha x - i\sin\alpha x)\,d\alpha \\ &= \frac{100}{\pi}\int_{-\infty}^{\infty} \underbrace{\frac{\cos\alpha x(\cos\alpha - 1)}{i\alpha}e^{-k\alpha^2 t}}_{\text{odd function}}\,d\alpha - \frac{100}{\pi}\int_{-\infty}^{\infty} \underbrace{\frac{\sin\alpha x(\cos\alpha - 1)}{\alpha}e^{-k\alpha^2 t}}_{\text{even function}}\,d\alpha \\ &= \frac{200}{\pi}\int_{0}^{\infty} \frac{\sin\alpha x(1-\cos\alpha)}{\alpha}e^{-k\alpha^2 t}d\alpha. \end{aligned}$$

3. Using the Fourier transform, the partial differential equation equation becomes

$$\frac{dU}{dt} + k\alpha^2 U = 0 \qquad \text{and so} \qquad U(\alpha,t) = ce^{-k\alpha^2 t}.$$

Now

$$\mathscr{F}\{u(x,0)\} = U(\alpha,0) = \sqrt{\pi}\,e^{-\alpha^2/4}$$

by the given result. This gives $c = \sqrt{\pi}\,e^{-\alpha^2/4}$ and so

$$U(\alpha,t) = \sqrt{\pi}\,e^{-(\frac{1}{4}+kt)\alpha^2}.$$

Using the given Fourier transform again we obtain

$$u(x,t) = \sqrt{\pi}\,\mathscr{F}^{-1}\{e^{-(1+4kt)\alpha^2/4}\} = \frac{1}{\sqrt{1+4kt}}\,e^{-x^2/(1+4kt)}.$$

4. **(a)** We use $U(\alpha,t) = ce^{-k\alpha^2 t}$. The Fourier transform of the boundary condition is $U(\alpha,0) = F(\alpha)$. This gives $c = F(\alpha)$ and so $U(\alpha,t) = F(\alpha)e^{-k\alpha^2 t}$. By the convolution theorem and the given result, we obtain

$$u(x,t) = \mathscr{F}^{-1}\{F(\alpha)\cdot e^{-k\alpha^2 t}\} = \frac{1}{2\sqrt{k\pi t}}\int_{-\infty}^{\infty} f(\tau)e^{-(x-\tau)^2/4kt}\,d\tau.$$

(b) Using the definition of f and the solution is part (a) we obtain

$$u(x,t) = \frac{u_0}{2\sqrt{k\pi t}}\int_{-1}^{1} e^{-(x-\tau)^2/4kt}d\tau.$$

If $u = \dfrac{x-\tau}{2\sqrt{kt}}$, then $d\tau = -2\sqrt{kt}\,du$ and the integral becomes

$$u(x,t) = \frac{u_0}{\sqrt{\pi}}\int_{(x-1)/2\sqrt{kt}}^{(x+1)/2\sqrt{\pi t}} e^{-u^2}\,du.$$

Using the result in Problem 9, Exercises 15.1, we have

$$u(x,t) = \frac{u_0}{2}\left[\operatorname{erf}\left(\frac{x+1}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{x-1}{2\sqrt{kt}}\right)\right].$$

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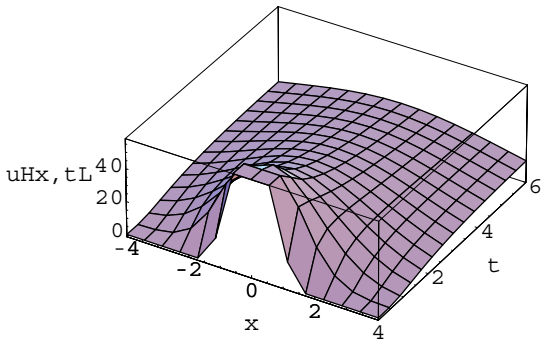
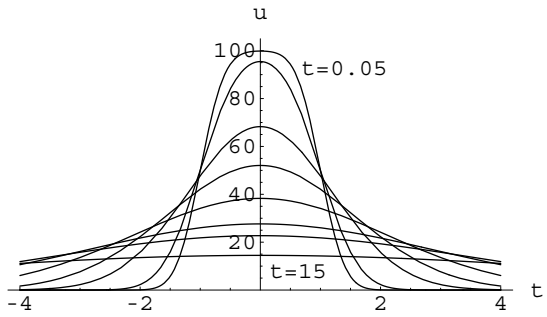
5.





Since $\operatorname{erf}(0) = 0$ and $\lim_{x\to\infty} \operatorname{erf}(x) = 1$, we have

$$\lim_{t\to\infty} u(x,t) = 50[\operatorname{erf}(0) - \operatorname{erf}(0)] = 0$$

and

$$\lim_{x\to\infty} u(x,t) = 50[\operatorname{erf}(\infty) - \operatorname{erf}(\infty)] = 50[1-1] = 0.$$

6. (a) Using the Fourier sine transform, the partial differential equation becomes

$$\frac{dU}{dt} + k\alpha^2 U = k\alpha u_0.$$

The general solution of this linear equation is

$$U(\alpha,t) = ce^{-k\alpha^2 t} + \frac{u_0}{\alpha}.$$

But $U(\alpha,0) = 0$ implies $c = -u_0/\alpha$ and so

$$U(\alpha,t) = u_0\,\frac{1-e^{-k\alpha^2 t}}{\alpha}$$

and

$$u(x,t) = \frac{2u_0}{\pi}\int_0^\infty \frac{1-e^{-k\alpha^2 t}}{\alpha}\sin\alpha x\,d\alpha.$$

(b) The solution of part (a) can be written

$$u(x,t) = \frac{2u_0}{\pi}\int_0^\infty \frac{\sin\alpha x}{\alpha}\,d\alpha - \frac{2u_0}{\pi}\int_0^\infty \frac{\sin\alpha x}{\alpha}\,e^{-k\alpha^2 t}\,d\alpha.$$

Using $\displaystyle\int_0^\infty \frac{\sin\alpha x}{\alpha}\,d\alpha = \pi/2$ the last line becomes

$$u(x,t) = u_0 - \frac{2u_0}{\pi}\int_0^\infty \frac{\sin\alpha x}{\alpha}\,e^{-k\alpha^2 t}\,d\alpha.$$

7. Using the Fourier sine transform we find

$$U(\alpha,t) = ce^{-k\alpha^2 t}.$$

Now

$$\mathscr{F}_S\{u(x,0)\} = U(\alpha,0) = \int_0^1 \sin\alpha x\,dx = \frac{1-\cos\alpha}{\alpha}.$$

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From this we find $c = (1 - \cos\alpha)/\alpha$ and so

$$U(\alpha, t) = \frac{1 - \cos\alpha}{\alpha}\, e^{-k\alpha^2 t}$$

and

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{1 - \cos\alpha}{\alpha}\, e^{-k\alpha^2 t} \sin\alpha x\, d\alpha.$$

8. Since the domain of x is $(0, \infty)$ and the condition at $x = 0$ involves $\partial u/\partial x$ we use the Fourier cosine transform:

$$-k\alpha^2 U(\alpha, t) - k u_x(0, t) = \frac{dU}{dt}$$

$$\frac{dU}{dt} + k\alpha^2 U(\alpha, t) = kA$$

$$U(\alpha, t) = ce^{-k\alpha^2 t} + \frac{A}{\alpha^2}.$$

Since

$$\mathscr{F}\{u(x, 0)\} = U(\alpha, 0) = 0$$

we find $c = -A/\alpha^2$, so that

$$U(\alpha, t) = A\frac{1 - e^{-k\alpha^2 t}}{\alpha^2}.$$

Applying the inverse Fourier cosine transform we obtain

$$u(x, t) = \mathscr{F}_C^{-1}\{U(\alpha, t)\} = \frac{2A}{\pi}\int_0^\infty \frac{1 - e^{-k\alpha^2 t}}{\alpha^2} \cos\alpha x\, d\alpha.$$

9. Using the Fourier cosine transform we find

$$U(\alpha, t) = ce^{-k\alpha^2 t}.$$

Now

$$\mathscr{F}_C\{u(x, 0)\} = \int_0^1 \cos\alpha x\, dx = \frac{\sin\alpha}{\alpha} = U(\alpha, 0).$$

From this we obtain $c = (\sin\alpha)/\alpha$ and so

$$U(\alpha, t) = \frac{\sin\alpha}{\alpha}\, e^{-k\alpha^2 t}$$

and

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha}{\alpha}\, e^{-k\alpha^2 t} \cos\alpha x\, d\alpha.$$

10. Using the Fourier sine transform we find

$$U(\alpha, t) = ce^{-k\alpha^2 t} + \frac{1}{\alpha}.$$

Now

$$\mathscr{F}_S\{u(x, 0)\} = \mathscr{F}_S\left\{e^{-x}\right\} = \int_0^\infty e^{-x} \sin\alpha x\, dx = \frac{\alpha}{1 + \alpha^2} = U(\alpha, 0).$$

From this we obtain $c = \alpha/(1 + \alpha^2) - 1/\alpha$. Therefore

$$U(\alpha, t) = \left(\frac{\alpha}{1 + \alpha^2} - \frac{1}{\alpha}\right) e^{-k\alpha^2 t} + \frac{1}{\alpha} = \frac{1}{\alpha} - \frac{e^{-k\alpha^2 t}}{\alpha(1 + \alpha^2)}$$

and

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \left(\frac{1}{\alpha} - \frac{e^{-k\alpha^2 t}}{\alpha(1 + \alpha^2)}\right) \sin\alpha x\, d\alpha.$$

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11. **(a)** Using the Fourier transform we obtain

$$U(\alpha, t) = c_1 \cos \alpha a t + c_2 \sin \alpha a t.$$

If we write

$$\mathscr{F}\{u(x,0)\} = \mathscr{F}\{f(x)\} = F(\alpha)$$

and

$$\mathscr{F}\{u_t(x,0)\} = \mathscr{F}\{g(x)\} = G(\alpha)$$

we first obtain

$c_1 = F(\alpha)$ from $U(\alpha, 0) = F(\alpha)$ and then $c_2 = G(\alpha)/\alpha a$ from $\left.\dfrac{dU}{dt}\right|_{t=0} = G(\alpha)$. Thus

$$U(\alpha, t) = F(\alpha) \cos \alpha a t + \frac{G(\alpha)}{\alpha a} \sin \alpha a t$$

and

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(F(\alpha) \cos \alpha a t + \frac{G(\alpha)}{\alpha a} \sin \alpha a t \right) e^{-i\alpha x} d\alpha.$$

(b) If $g(x) = 0$ then $c_2 = 0$ and

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) \cos \alpha a t e^{-i\alpha x} d\alpha \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) \left(\frac{e^{\alpha a t i} + e^{-\alpha a t i}}{2} \right) e^{-i\alpha x} d\alpha \\ &= \frac{1}{2} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) e^{-i(x-at)\alpha} d\alpha + \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) e^{-i(x+at)\alpha} d\alpha \right] \\ &= \frac{1}{2} \left[f(x-at) + f(x+at) \right]. \end{aligned}$$

12. Using the Fourier sine transform we obtain

$$U(\alpha, t) = c_1 \cos \alpha a t + c_2 \sin \alpha a t.$$

Now

$$\mathscr{F}_S\{u(x,0)\} = \mathscr{F}\left\{x e^{-x}\right\} = \int_0^{\infty} x e^{-x} \sin \alpha x \, dx = \frac{2\alpha}{(1+\alpha^2)^2} = U(\alpha, 0).$$

Also,

$$\mathscr{F}_S\{u_t(x,0)\} = \left.\frac{dU}{dt}\right|_{t=0} = 0.$$

This last condition gives $c_2 = 0$. Then $U(\alpha, 0) = 2\alpha/(1+\alpha^2)^2$ yields $c_1 = 2\alpha/(1+\alpha^2)^2$. Therefore

$$U(\alpha, t) = \frac{2\alpha}{(1+\alpha^2)^2} \cos \alpha a t$$

and

$$u(x,t) = \frac{4}{\pi} \int_0^{\infty} \frac{\alpha \cos \alpha a t}{(1+\alpha^2)^2} \sin \alpha x \, d\alpha.$$

13. Using the Fourier cosine transform we obtain

$$U(x, \alpha) = c_1 \cosh \alpha x + c_2 \sinh \alpha x.$$

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Now the Fourier cosine transforms of $u(0, y) = e^{-y}$ and $u(\pi, y) = 0$ are, respectively, $U(0, \alpha) = 1/(1 + \alpha^2)$ and $U(\pi, \alpha) = 0$. The first of these conditions gives $c_1 = 1/(1 + \alpha^2)$. The second condition gives

$$c_2 = -\frac{\cosh \alpha\pi}{(1 + \alpha^2) \sinh \alpha\pi}.$$

Hence

$$U(x, \alpha) = \frac{\cosh \alpha x}{1 + \alpha^2} - \frac{\cosh \alpha\pi \sinh \alpha x}{(1 + \alpha^2) \sinh \alpha\pi} = \frac{\sinh \alpha\pi \cosh \alpha\pi - \cosh \alpha\pi \sinh \alpha x}{(1 + \alpha^2) \sinh \alpha\pi} = \frac{\sinh \alpha(\pi - x)}{(1 + \alpha^2) \sinh \alpha\pi}$$

and

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \frac{\sinh \alpha(\pi - x)}{(1 + \alpha^2) \sinh \alpha\pi} \cos \alpha y \, d\alpha.$$

14. Since the boundary condition at $y = 0$ now involves $u(x, 0)$ rather than $u'(x, 0)$, we use the Fourier sine transform. The transform of the partial differential equation is then

$$\frac{d^2U}{dx^2} - \alpha^2 U + \alpha u(x, 0) = 0 \quad \text{or} \quad \frac{d^2U}{dx^2} - \alpha^2 U = -\alpha.$$

The solution of this differential equation is

$$U(x, \alpha) = c_1 \cosh \alpha x + c_2 \sinh \alpha x + \frac{1}{\alpha}.$$

The transforms of the boundary conditions at $x = 0$ and $x = \pi$ in turn imply that $c_1 = 1/\alpha$ and

$$c_2 = \frac{\cosh \alpha\pi}{\alpha \sinh \alpha\pi} - \frac{1}{\alpha \sinh \alpha\pi} + \frac{\alpha}{(1 + \alpha^2) \sinh \alpha\pi}.$$

Hence

$$U(\alpha, x) = \frac{1}{\alpha} - \frac{\cosh \alpha x}{\alpha} + \frac{\cosh \alpha\pi}{\alpha \sinh \alpha\pi} \sinh \alpha x - \frac{\sinh \alpha x}{\alpha \sinh \alpha\pi} + \frac{\alpha \sinh \alpha x}{(1 + \alpha^2) \sinh \alpha\pi}$$

$$= \frac{1}{\alpha} - \frac{\sinh \alpha(\pi - x)}{\alpha \sinh \alpha\pi} - \frac{\sinh \alpha x}{\alpha(1 + \alpha^2) \sinh \alpha\pi}.$$

Taking the inverse transform it follows that

$$u(x, y) = \frac{2}{\pi} \int_0^\infty \left(\frac{1}{\alpha} - \frac{\sinh \alpha(\pi - x)}{\alpha \sinh \alpha\pi} - \frac{\sinh \alpha x}{\alpha(1 + \alpha^2) \sinh \alpha\pi} \right) \sin \alpha y \, d\alpha.$$

15. Using the Fourier cosine transform with respect to x gives

$$U(\alpha, y) = c_1 e^{-\alpha y} + c_2 e^{\alpha y}.$$

Since we expect $u(x, y)$ to be bounded as $y \to \infty$ we define $c_2 = 0$. Thus

$$U(\alpha, y) = c_1 e^{-\alpha y}.$$

Now

$$\mathscr{F}_C\{u(x, 0)\} = \int_0^1 50 \cos \alpha x \, dx = 50 \frac{\sin \alpha}{\alpha}$$

and so

$$U(\alpha, y) = 50 \frac{\sin \alpha}{\alpha} e^{-\alpha y}$$

and

$$u(x, y) = \frac{100}{\pi} \int_0^\infty \frac{\sin \alpha}{\alpha} e^{-\alpha y} \cos \alpha x \, d\alpha.$$

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16. The boundary condition $u(0, y) = 0$ indicates that we now use the Fourier sine transform. We still have $U(\alpha, y) = c_1 e^{-\alpha y}$, but

$$\mathscr{F}_S\{u(x,0)\} = \int_0^1 50 \sin \alpha x \, dx = 50(1 - \cos \alpha)/\alpha = U(\alpha, 0).$$

This gives $c_1 = 50(1 - \cos \alpha)/\alpha$ and so

$$U(\alpha, y) = 50 \, \frac{1 - \cos \alpha}{\alpha} \, e^{-\alpha y}$$

and

$$u(x, y) = \frac{100}{\pi} \int_0^\infty \frac{1 - \cos \alpha}{\alpha} \, e^{-\alpha y} \sin \alpha x \, d\alpha.$$

17. We use the Fourier sine transform with respect to x to obtain

$$U(\alpha, y) = c_1 \cosh \alpha y + c_2 \sinh \alpha y.$$

The transforms of $u(x, 0) = f(x)$ and $u(x, 2) = 0$ give, in turn, $U(\alpha, 0) = F(\alpha)$ and $U(\alpha, 2) = 0$. The first condition gives $c_1 = F(\alpha)$ and the second condition then yields

$$c_2 = -\frac{F(\alpha) \cosh 2\alpha}{\sinh 2\alpha}.$$

Hence

$$\begin{aligned} U(\alpha, y) &= F(\alpha) \cosh \alpha y - \frac{F(\alpha) \cosh 2\alpha \sinh \alpha y}{\sinh 2\alpha} \\ &= F(\alpha) \frac{\sinh 2\alpha \cosh \alpha y - \cosh 2\alpha \sinh \alpha y}{\sinh 2\alpha} \\ &= F(\alpha) \frac{\sinh \alpha(2 - y)}{\sinh 2\alpha} \end{aligned}$$

and

$$u(x, y) = \frac{2}{\pi} \int_0^\infty F(\alpha) \frac{\sinh \alpha(2 - y)}{\sinh 2\alpha} \sin \alpha x \, d\alpha.$$

18. The domain of y and the boundary condition at $y = 0$ suggest that we use a Fourier cosine transform. The transformed equation is

$$\frac{d^2U}{dx^2} - \alpha^2 U - u_y(x, 0) = 0 \quad \text{or} \quad \frac{d^2U}{dx^2} - \alpha^2 U = 0.$$

Because the domain of the variable x is a finite interval we choose to write the general solution of the latter equation as

$$U(x, \alpha) = c_1 \cosh \alpha x + c_2 \sinh \alpha x.$$

Now $U(0, \alpha) = F(\alpha)$, where $F(\alpha)$ is the Fourier cosine transform of $f(y)$, and $U'(\pi, \alpha) = 0$ imply $c_1 = F(\alpha)$ and $c_2 = -F(\alpha) \sinh \alpha\pi / \cosh \alpha\pi$. Thus

$$U(x, \alpha) = F(\alpha) \cosh \alpha x - F(\alpha) \frac{\sinh \alpha\pi}{\cosh \alpha\pi} \sinh \alpha x = F(\alpha) \frac{\cosh \alpha(\pi - x)}{\cosh \alpha\pi}.$$

Using the inverse transform we find that a solution to the problem is

$$u(x, y) = \frac{2}{\pi} \int_0^\infty F(\alpha) \frac{\cosh \alpha(\pi - x)}{\cosh \alpha\pi} \cos \alpha y \, d\alpha.$$

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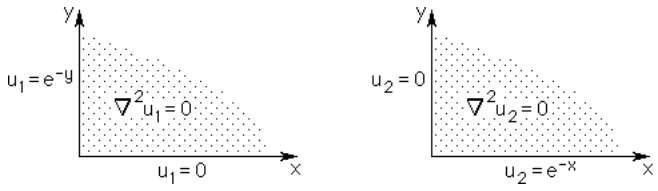
19. We solve two boundary-value problems:



Using the Fourier sine transform with respect to y gives

$$u_1(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\alpha e^{-\alpha x}}{1 + \alpha^2} \sin \alpha y \, d\alpha.$$

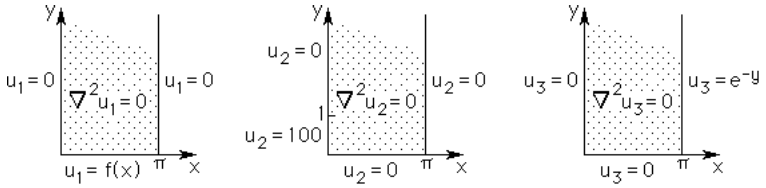
The Fourier sine transform with respect to x yields the solution to the second problem:

$$u_2(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\alpha e^{-\alpha y}}{1 + \alpha^2} \sin \alpha x \, d\alpha.$$

We define the solution of the original problem to be

$$u(x, y) = u_1(x, y) + u_2(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\alpha}{1 + \alpha^2} \left[e^{-\alpha x} \sin \alpha y + e^{-\alpha y} \sin \alpha x \right] d\alpha.$$

20. We solve the three boundary-value problems:



Using separation of variables we find the solution of the first problem is

$$u_1(x, y) = \sum_{n=1}^{\infty} A_n e^{-ny} \sin nx \quad \text{where} \quad A_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx.$$

Using the Fourier sine transform with respect to y gives the solution of the second problem:

$$u_2(x, y) = \frac{200}{\pi} \int_0^\infty \frac{(1 - \cos \alpha) \sinh \alpha(\pi - x)}{\alpha \sinh \alpha\pi} \sin \alpha y \, d\alpha.$$

Also, the Fourier sine transform with respect to y gives the solution of the third problem:

$$u_3(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\alpha \sinh \alpha x}{(1 + \alpha^2) \sinh \alpha\pi} \sin \alpha y \, d\alpha.$$

The solution of the original problem is

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y).$$

21. Using the Fourier transform with respect to x gives

$$U(\alpha, y) = c_1 \cosh \alpha y + c_2 \sinh \alpha y.$$

The transform of the boundary condition $\left.\dfrac{\partial u}{\partial y}\right|_{y=0} = 0$ is $\left.\dfrac{dU}{dy}\right|_{y=0} = 0$. This condition gives $c_2 = 0$. Hence

$$U(\alpha, y) = c_1 \cosh \alpha y.$$

Now by the given information the transform of the boundary condition $u(x, 1) = e^{-x^2}$ is

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$U(\alpha, 1) = \sqrt{\pi}\, e^{-\alpha^2/4}$. This condition then gives $c_1 = \sqrt{\pi}\, e^{-\alpha^2/4} \cosh \alpha$. Therefore

$$U(\alpha, y) = \sqrt{\pi}\, \frac{e^{-\alpha^2/4} \cosh \alpha y}{\cosh \alpha}$$

and

$$\begin{aligned} U(x, y) &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\alpha^2/4} \cosh \alpha y}{\cosh \alpha}\, e^{-i\alpha x} d\alpha \\ &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\alpha^2/4} \cosh \alpha y}{\cosh \alpha}\, \cos \alpha x\, d\alpha \\ &= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-\alpha^2/4} \cosh \alpha y}{\cosh \alpha}\, \cos \alpha x\, d\alpha. \end{aligned}$$

22. Entries 42 and 43 of the table in Appendix III imply

$$\int_0^{\infty} e^{-st} \frac{\sin at}{t} dt = \arctan \frac{a}{s}$$

and

$$\int_0^{\infty} e^{-st} \frac{\sin at \cos bt}{t} dt = \frac{1}{2} \arctan \frac{a+b}{s} + \frac{1}{2} \arctan \frac{a-b}{s}.$$

Identifying $\alpha = t, x = a$, and $y = s$, the solution of Problem 16 is

$$\begin{aligned} u(x, y) &= \frac{100}{\pi} \int_0^{\infty} \frac{1 - \cos \alpha}{\alpha} e^{-\alpha y} \sin \alpha x\, d\alpha \\ &= \frac{100}{\pi} \left[\int_0^{\infty} \frac{\sin \alpha x}{\alpha} e^{-\alpha y} d\alpha - \int_0^{\infty} \frac{\sin \alpha x \cos \alpha}{\alpha} e^{-\alpha y} d\alpha \right] \\ &= \frac{100}{\pi} \left[\arctan \frac{x}{y} - \frac{1}{2} \arctan \frac{x+1}{y} - \frac{1}{2} \arctan \frac{x-1}{y} \right]. \end{aligned}$$

Exercises 15.5

1. We show that $\frac{1}{4}\overline{F}_4 F_4 = I$:

$$\frac{1}{4}\overline{F}_4 F_4 = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = I.$$

Thus $F_4^{-1} = \frac{1}{4}\overline{F}_4$.

2. We have

$$\int_{-\infty}^{\infty} f(x)\delta_\epsilon(x-a)dx = \frac{1}{2\epsilon} \int_{a-\epsilon}^{a+\epsilon} f(x)dx = \frac{1}{2\epsilon} f(c)(2\epsilon) = f(c)$$

by the mean value theorem for integrals.

3. By the sifting property,

$$\mathscr{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x) e^{i\alpha x} dx = e^{i\alpha 0} = 1.$$

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4. We already know that $f * \delta = \delta * f$. Then, by the sifting property,

$$(f * \delta)(x) = \int_{\infty}^{\infty} f(\tau)\delta(x-\tau)d\tau = \int_{-\infty}^{\infty} f(\tau)\delta(\tau - x)d\tau = f(x).$$

5. Using integration by parts with $u = f(x)$ and $dv = \delta'(x-a)$ we find

$$\int_{-\infty}^{\infty} f(x)\delta'(x-a)dx = -\int_{-\infty}^{\infty} f'(x)\delta(x-a)dx = -f'(a)$$

by the sifting property.

6. Using a CAS we find

$$\mathscr{F}\{g(x)\} = \frac{1}{2}[\text{sign}(A-\alpha) + \text{sign}(A+\alpha)]$$

where $\text{sign}(t) = 1$ if $t > 0$ and $\text{sign}\, t = -1$ if $t < 0$. Thus

$$\mathscr{F}\{g(x)\} = \begin{cases} 1, & -A < \alpha < A \\ 0, & \text{elsewhere.} \end{cases}$$

7. Using

$$\begin{aligned} \omega_8 &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & \omega_8^5 &= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\ \omega_8^2 &= i & \omega_8^6 &= -i \\ \omega_8^3 &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & \omega_8^7 &= \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\ \omega_8^4 &= -1 & \omega_8^8 &= 1 \end{aligned}$$

we have

$$F8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & i & -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} & -i & \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & -i & \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} & i & -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} & i & \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & -i & -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} & -i & -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & i & \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \end{pmatrix}.$$

In factored form

$$F8 = \begin{pmatrix} I_4 & D_4 \\ I_4 & -D_4 \end{pmatrix} \begin{pmatrix} F_4 & 0 \\ 0 & F_4 \end{pmatrix} P,$$

where I_4 is the 4×4 identity matrix.

$$D_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2}/2 + i\sqrt{2}/2 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -\sqrt{2}/2 + i\sqrt{2}/2 \end{pmatrix},$$

and P is the 8×8 matrix with 1 in positions $(1,1)$, $(2,3)$, $(3,5)$, $(4,7)$, $(5,2)$, $(6,4)$, $(7,6)$, and $(8,8)$.

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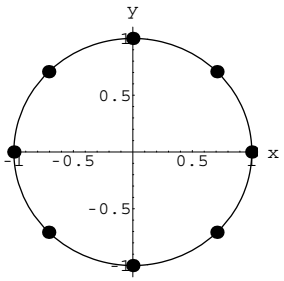
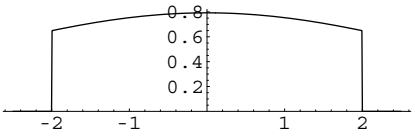
8. The 8th roots of unity, ω_8^1, ω_8^2, …, ω_8^8 are shown in the solution of Problem 7 above. The points in the complex plane are equally spaced on the perimeter of the unit circle.



9. The Fourier transform of $g(x) = (\sin 2x)/\pi x$ is

$$G(\alpha) = \begin{cases} 1, & -2 < \alpha < 2 \\ 0, & \text{elsewhere.} \end{cases}$$



This implies that $(f * g)(x) = \mathscr{F}^{-1}\{F(\alpha)G(\alpha)\}$ is band-limited. The graph of $F(\alpha)G(\alpha)$, which is identical to the graph of $\mathscr{F}(f * g)$, is shown.

10. For $N = 6$,

$$F_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1/2 + \sqrt{3}i/2 & -1/2 + \sqrt{3}i/2 & -1 & -1/2 - \sqrt{3}i/2 & 1/2 - \sqrt{3}i/2 \\ 1 & -1/2 + \sqrt{3}i/2 & -1/2 - \sqrt{3}i/2 & 1 & -1/2 + \sqrt{3}i/2 & -1/2 - \sqrt{3}i/2 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1/2 - \sqrt{3}i/2 & -1/2 + \sqrt{3}i/2 & 1 & -1/2 - \sqrt{3}i/2 & -1/2 + \sqrt{3}i/2 \\ 1 & 1/2 - \sqrt{3}i/2 & -1/2 - \sqrt{3}i/2 & -1 & -1/2 + \sqrt{3}i/2 & 1/2 + \sqrt{3}i/2 \end{pmatrix}.$$

If, for example, $f = (2, 0, 1, 6, 2, 3)$, then

$$c = \frac{1}{6}\overline{F}_6 f = \begin{pmatrix} 7/3 \\ -2/3 + \sqrt{3}i/3 \\ 5/6 - \sqrt{3}i/6 \\ -2/3 \\ 5/6 - \sqrt{3}i/6 \\ -2/3 - i/\sqrt{3} \end{pmatrix}.$$

Chapter 5 Review Exercises

1. The partial differential equation and the boundary conditions indicate that the Fourier cosine transform is appropriate for the problem. We find in this case

$$u(x, y) = \frac{2}{\pi}\int_0^\infty \frac{\sinh \alpha y}{\alpha(1 + \alpha^2)\cosh \alpha\pi} \cos \alpha x \, d\alpha.$$

2. We use the Laplace transform and undetermined coefficients to obtain

$$U(x, s) = c_1 \cosh \sqrt{s}\, x + c_2 \sinh \sqrt{s}\, x + \frac{50}{s + 4\pi^2} \sin 2\pi x.$$

The transformed boundary conditions $U(0, s) = 0$ and $U(1, s) = 0$ give, in turn, $c_1 = 0$ and $c_2 = 0$. Hence

$$U(x, s) = \frac{50}{s + 4\pi^2} \sin 2\pi x$$

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and

$$u(x,t) = 50 \sin 2\pi x \, \mathscr{L}^{-1}\left\{\frac{1}{s+4\pi^2}\right\} = 50e^{-4\pi^2 t}\sin 2\pi x.$$

3. The Laplace transform gives

$$U(x,s) = c_1 e^{-\sqrt{s+h}\,x} + c_2 e^{\sqrt{s+h}\,x} + \frac{u_0}{s+h}.$$

The condition $\lim_{x\to\infty} \partial u/\partial x = 0$ implies $\lim_{x\to\infty} dU/dx = 0$ and so we define $c_2 = 0$. Thus

$$U(x,s) = c_1 e^{-\sqrt{s+h}\,x} + \frac{u_0}{s+h}.$$

The condition $U(0,s) = 0$ then gives $c_1 = -u_0/(s+h)$ and so

$$U(x,s) = \frac{u_0}{s+h} - u_0\,\frac{e^{-\sqrt{s+h}\,x}}{s+h}.$$

With the help of the first translation theorem we then obtain

$$u(x,t) = u_0\,\mathscr{L}^{-1}\left\{\frac{1}{s+h}\right\} - u_0\,\mathscr{L}^{-1}\left\{\frac{e^{-\sqrt{s+h}\,x}}{s+h}\right\} = u_0e^{-ht} - u_0e^{-ht}\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$$

$$= u_0e^{-ht}\left[1 - \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)\right] = u_0e^{-ht}\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right).$$

4. Using the Fourier transform and the result $\mathscr{F}\left\{e^{-|x|}\right\} = 1/(1+\alpha^2)$ we find

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-e^{-\alpha^2 t}}{\alpha^2(1+\alpha^2)}\, e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-e^{-\alpha^2 t}}{\alpha^2(1+\alpha^2)}\cos\alpha x\, d\alpha$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{1-e^{-\alpha^2 t}}{\alpha^2(1+\alpha^2)}\cos\alpha x\, d\alpha.$$

5. The Laplace transform gives

$$U(x,s) = c_1 e^{-\sqrt{s}\,x} + c_2 e^{\sqrt{s}\,x}.$$

The condition $\lim_{x\to\infty} u(x,t) = 0$ implies $\lim_{x\to\infty} U(x,s) = 0$ and so we define $c_2 = 0$. Thus

$$U(x,s) = c_1 e^{-\sqrt{s}\,x}.$$

The transform of the remaining boundary condition is $U(0,s) = 1/s^2$. This gives $c_1 = 1/s^2$. Hence

$$U(x,s) = \frac{e^{-\sqrt{s}\,x}}{s^2} \quad \text{and} \quad u(x,t) = \mathscr{L}^{-1}\left\{\frac{1}{s}\,\frac{e^{-\sqrt{s}\,x}}{s}\right\}.$$

Using

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \quad \text{and} \quad \mathscr{L}^{-1}\left\{\frac{e^{-\sqrt{s}\,x}}{s}\right\} = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right),$$

it follows from the convolution theorem that

$$u(x,t) = \int_0^t \operatorname{erfc}\left(\frac{x}{2\sqrt{\tau}}\right) d\tau.$$

6. The Laplace transform and undetermined coefficients gives

$$U(x,s) = c_1\cosh sx + c_2\sinh sx + \frac{s-1}{s^2+\pi^2}\sin\pi x.$$

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The conditions $U(0,s)=0$ and $U(1,s)=0$ give, in turn, $c_1=0$ and $c_2=0$. Thus

$$U(x,s)=\frac{s-1}{s^2+\pi^2}\sin\pi x$$

and

$$u(x,t)=\sin\pi x\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\}-\frac{1}{\pi}\sin\pi x\,\mathscr{L}^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\}$$

$$=(\sin\pi x)\cos\pi t-\frac{1}{\pi}(\sin\pi x)\sin\pi t.$$

7. The Fourier transform gives the solution

$$u(x,t)=\frac{u_0}{2\pi}\int_{-\infty}^{\infty}\left(\frac{e^{i\alpha\pi}-1}{i\alpha}\right)e^{-i\alpha x}e^{-k\alpha^2t}d\alpha$$

$$=\frac{u_0}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\alpha(\pi-x)}-e^{-i\alpha x}}{i\alpha}\,e^{-k\alpha^2t}d\alpha$$

$$=\frac{u_0}{2\pi}\int_{-\infty}^{\infty}\frac{\cos\alpha(\pi-x)+i\sin\alpha(\pi-x)-\cos\alpha x+i\sin\alpha x}{i\alpha}\,e^{-k\alpha^2t}d\alpha.$$

Since the imaginary part of the integrand of the last integral is an odd function of α, we obtain

$$u(x,t)=\frac{u_0}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\alpha(\pi-x)+\sin\alpha x}{\alpha}\,e^{-k\alpha^2t}d\alpha.$$

8. Using the Fourier cosine transform we obtain

$$U(x,\alpha)=c_1\cosh\alpha x+c_2\sinh\alpha x.$$

The condition $U(0,\alpha)=0$ gives $c_1=0$. Thus

$$U(x,\alpha)=c_2\sinh\alpha x.$$

Now

$$\mathscr{F}_C\{u(\pi,y)\}=\int_1^2\cos\alpha y\,dy=\frac{\sin2\alpha-\sin\alpha}{\alpha}=U(\pi,\alpha).$$

This last condition gives $c_2=(\sin2\alpha-\sin\alpha)/\alpha\sinh\alpha\pi$. Hence

$$U(x,\alpha)=\frac{\sin2\alpha-\sin\alpha}{\alpha\sinh\alpha\pi}\,\sinh\alpha x$$

and

$$u(x,y)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin2\alpha-\sin\alpha}{\alpha\sinh\alpha\pi}\sinh\alpha x\cos\alpha y\,d\alpha.$$

9. We solve the two problems

$$\frac{\partial^2u_1}{\partial x^2}+\frac{\partial^2u_1}{\partial y^2}=0,\quad x>0,\quad y>0,$$

$$u_1(0,y)=0,\quad y>0,$$

$$u_1(x,0)=\begin{cases}100, & 0<x<1\\ 0, & x>1\end{cases}$$

and

$$\frac{\partial^2u_2}{\partial x^2}+\frac{\partial^2u_2}{\partial y^2}=0,\quad x>0,\quad y>0,$$

$$u_2(0,y)=\begin{cases}50, & 0<y<1\\ 0, & y>1\end{cases}$$

$$u_2(x,0)=0.$$

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Using the Fourier sine transform with respect to x we find

$$u_1(x,y) = \frac{200}{\pi}\int_0^\infty \left(\frac{1-\cos\alpha}{\alpha}\right) e^{-\alpha y}\sin\alpha x\, d\alpha.$$

Using the Fourier sine transform with respect to y we find

$$u_2(x,y) = \frac{100}{\pi}\int_0^\infty \left(\frac{1-\cos\alpha}{\alpha}\right) e^{-\alpha x}\sin\alpha y\, d\alpha.$$

The solution of the problem is then

$$u(x,y) = u_1(x,y) + u_2(x,y).$$

10. The Laplace transform gives

$$U(x,s) = c_1\cosh\sqrt{s}\,x + c_2\sinh\sqrt{s}\,x + \frac{r}{s^2}.$$

The condition $\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0$ transforms into $\left.\dfrac{dU}{dx}\right|_{x=0} = 0$. This gives $c_2 = 0$. The remaining condition $u(1,t) = 0$ transforms into $U(1,s) = 0$. This condition then implies $c_1 = -r/s^2\cosh\sqrt{s}$. Hence

$$U(x,s) = \frac{r}{s^2} - r\,\frac{\cosh\sqrt{s}\,x}{s^2\cosh\sqrt{s}}.$$

Using geometric series and the convolution theorem we obtain

$$\begin{aligned} u(x,t) &= r\,\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} - r\,\mathscr{L}^{-1}\left\{\frac{\cosh\sqrt{s}\,x}{s^2\cosh\sqrt{s}}\right\} \\ &= rt - r\sum_{n=0}^{\infty}(-1)^n\left[\int_0^t \operatorname{erfc}\left(\frac{2n+1-x}{2\sqrt{\tau}}\right)d\tau + \int_0^t \operatorname{erfc}\left(\frac{2n+1+x}{2\sqrt{\tau}}\right)d\tau\right]. \end{aligned}$$

11. The Fourier sine transform with respect to x and undetermined coefficients give

$$U(\alpha,y) = c_1\cosh\alpha y + c_2\sinh\alpha y + \frac{A}{\alpha}.$$

The transforms of the boundary conditions are

$$\left.\frac{dU}{dy}\right|_{y=0} = 0 \quad \text{and} \quad \left.\frac{dU}{dy}\right|_{y=\pi} = \frac{B\alpha}{1+\alpha^2}.$$

The first of these conditions gives $c_2 = 0$ and so

$$U(\alpha,y) = c_1\cosh\alpha y + \frac{A}{\alpha}.$$

The second transformed boundary condition yields $c_1 = B/(1+\alpha^2)\sinh\alpha\pi$. Therefore

$$U(\alpha,y) = \frac{B\cosh\alpha y}{(1+\alpha^2)\sinh\alpha\pi} + \frac{A}{\alpha}$$

and

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left(\frac{B\cosh\alpha y}{(1+\alpha^2)\sinh\alpha\pi} + \frac{A}{\alpha}\right)\sin\alpha x\, d\alpha.$$

12. Using the Laplace transform gives

$$U(x,s) = c_1\cosh\sqrt{s}\,x + c_2\sinh\sqrt{s}\,x.$$

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The condition $u(0,t) = u_0$ transforms into $U(0,s) = u_0/s$. This gives $c_1 = u_0/s$. The condition $u(1,t) = u_0$ transforms into $U(1,s) = u_0/s$. This implies that $c_2 = u_0(1 - \cosh\sqrt{s}\,)/s\sinh\sqrt{s}\,$. Hence

$$\begin{aligned} U(x,s) &= \frac{u_0}{s}\cosh\sqrt{s}\,x + u_0\left[\frac{1-\cosh\sqrt{s}}{s\sinh\sqrt{s}}\right]\sinh\sqrt{s}\,x \\ &= u_0\left[\frac{\sinh\sqrt{s}\,\cosh\sqrt{s}\,x - \cosh\sinh\sqrt{s}\,\sinh\sqrt{s}\,x + \sinh\sqrt{s}\,x}{s\sinh\sqrt{s}}\right] \\ &= u_0\left[\frac{\sinh\sqrt{s}\,(1-x) + \sinh\sqrt{s}\,x}{s\sinh\sqrt{s}}\right] \\ &= u_0\left[\frac{\sinh\sqrt{s}\,(1-x)}{s\sinh\sqrt{s}} + \frac{\sinh\sqrt{s}\,x}{s\sinh\sqrt{s}}\right] \end{aligned}$$

and

$$\begin{aligned} u(x,t) &= u_0\left[\mathscr{L}^{-1}\left\{\frac{\sinh\sqrt{s}\,(1-x)}{s\sinh\sqrt{s}}\right\} + \mathscr{L}^{-1}\left\{\frac{\sinh\sqrt{s}\,x}{s\sinh\sqrt{s}}\right\}\right] \\ &= u_0\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2n+2-x}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{2n+x}{2\sqrt{t}}\right)\right] \\ &\quad + u_0\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2n+1+x}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{2n+1-x}{2\sqrt{t}}\right)\right]. \end{aligned}$$

13. Using the Fourier transform gives

$$U(\alpha,t) = c_1 e^{-k\alpha^2 t}.$$

Now

$$u(\alpha,0) = \int_0^{\infty} e^{-x}e^{i\alpha x}\,dx = \frac{e^{(i\alpha-1)x}}{i\alpha-1}\,\Big|_0^{\infty} = 0 - \frac{1}{i\alpha-1} = \frac{1}{1-i\alpha} = c_1$$

so

$$U(\alpha,t) = \frac{1+i\alpha}{1+\alpha^2}\,e^{-k\alpha^2 t}$$

and

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1+i\alpha}{1+\alpha^2}\,e^{-k\alpha^2 t}e^{-i\alpha x}\,d\alpha.$$

Since

$$\frac{1+i\alpha}{1+\alpha^2}\,(\cos\alpha x - i\sin\alpha x) = \frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2} + \frac{i(\alpha\cos\alpha x - \sin\alpha x)}{1+\alpha^2}$$

and the integral of the product of the second term with $e^{-k\alpha^2 t}$ is 0 (it is an odd function), we have

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2}\,e^{-k\alpha^2 t}\,d\alpha.$$

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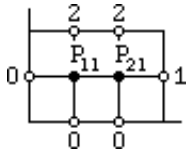
16 Numerical Solutions of Partial Differential Equations

Exercises 16.1

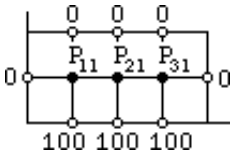
1. The figure shows the values of $u(x, y)$ along the boundary. We need to determine u_{11} and u_{21}. The system is



$$\begin{aligned} u_{21} + 2 + 0 + 0 - 4u_{11} &= 0 \\ 1 + 2 + u_{11} + 0 - 4u_{21} &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} -4u_{11} + u_{21} &= -2 \\ u_{11} - 4u_{21} &= -3. \end{aligned}$$

Solving we obtain $u_{11} = 11/15$ and $u_{21} = 14/15$.

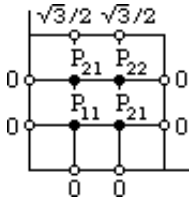
2. The figure shows the values of $u(x, y)$ along the boundary. We need to determine u_{11}, u_{21}, and u_{31}. By symmetry $u_{11} = u_{31}$ and the system is



$$\begin{aligned} u_{21} + 0 + 0 + 100 - 4u_{11} &= 0 \\ u_{31} + 0 + u_{11} + 100 - 4u_{21} &= 0 \\ 0 + 0 + u_{21} + 100 - 4u_{31} &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} -4u_{11} + u_{21} &= -100 \\ 2u_{11} - 4u_{21} &= -100. \end{aligned}$$

Solving we obtain $u_{11} = u_{31} = 250/7$ and $u_{21} = 300/7$.

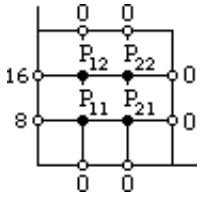
3. The figure shows the values of $u(x, y)$ along the boundary. We need to determine u_{11}, u_{21}, u_{12}, and u_{22}. By symmetry $u_{11} = u_{21}$ and $u_{12} = u_{22}$. The system is



$$\begin{aligned} u_{21} + u_{12} + 0 + 0 - 4u_{11} &= 0 \\ 0 + u_{22} + u_{11} + 0 - 4u_{21} &= 0 \\ u_{22} + \sqrt{3}/2 + 0 + u_{11} - 4u_{12} &= 0 \\ 0 + \sqrt{3}/2 + u_{12} + u_{21} - 4u_{22} &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} 3u_{11} + u_{12} &= 0 \\ u_{11} - 3u_{12} &= -\frac{\sqrt{3}}{2}. \end{aligned}$$

Solving we obtain $u_{11} = u_{21} = \sqrt{3}/16$ and $u_{12} = u_{22} = 3\sqrt{3}/16$.

4. The figure shows the values of $u(x, y)$ along the boundary. We need to determine u_{11}, u_{21}, u_{12}, and u_{22}. The system is



$$\begin{aligned} u_{21} + u_{12} + 8 + 0 - 4u_{11} &= 0 \\ 0 + u_{22} + u_{11} + 0 - 4u_{21} &= 0 \\ u_{22} + 0 + 16 + u_{11} - 4u_{12} &= 0 \\ 0 + 0 + u_{12} + u_{21} - 4u_{22} &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} -4u_{11} + u_{21} + u_{12} &= -8 \\ u_{11} - 4u_{21} + u_{22} &= 0 \\ u_{11} - 4u_{12} + u_{22} &= -16 \\ u_{21} + u_{12} - 4u_{22} &= 0. \end{aligned}$$

Solving we obtain $u_{11} = 11/3$, $u_{21} = 4/3$, $u_{12} = 16/3$, and $u_{22} = 5/3$.

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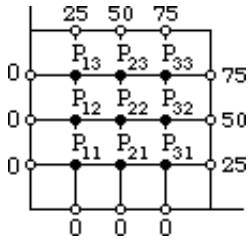
5. The figure shows the values of $u(x, y)$ along the boundary. For Gauss-Seidel the coefficients of the unknowns u_{11}, u_{21}, u_{31}, u_{12}, u_{22}, u_{32}, u_{13}, u_{23}, u_{33} are shown in the matrix



$$\begin{bmatrix} 0 & .25 & 0 & .25 & 0 & 0 & 0 & 0 & 0 \\ .25 & 0 & .25 & 0 & .25 & 0 & 0 & 0 & 0 \\ 0 & .25 & 0 & 0 & 0 & .25 & 0 & 0 & 0 \\ .25 & 0 & 0 & 0 & .25 & 0 & .25 & 0 & 0 \\ 0 & .25 & 0 & .25 & 0 & .25 & 0 & .25 & 0 \\ 0 & 0 & .25 & 0 & .25 & 0 & 0 & 0 & .25 \\ 0 & 0 & 0 & .25 & 0 & 0 & 0 & .25 & 0 \\ 0 & 0 & 0 & 0 & .25 & 0 & .25 & 0 & .25 \\ 0 & 0 & 0 & 0 & 0 & .25 & 0 & .25 & 0 \end{bmatrix}$$

The constant terms in the equations are 0, 0, 6.25, 0, 0, 12.5, 6.25, 12.5, 37.5. We use 25 as the initial guess for each variable. Then $u_{11} = 6.25$, $u_{21} = u_{12} = 12.5$, $u_{31} = u_{13} = 18.75$, $u_{22} = 25$, $u_{32} = u_{23} = 37.5$, and $u_{33} = 56.25$

6. The coefficients of the unknowns are the same as shown above in Problem 5. The constant terms are 7.5, 5, 20, 10, 0, 15, 17.5, 5, 27.5. We use 32.5 as the initial guess for each variable. Then $u_{11} = 21.92$, $u_{21} = 28.30$, $u_{31} = 38.17$, $u_{12} = 29.38$, $u_{22} = 33.13$, $u_{32} = 44.38$, $u_{13} = 22.46$, $u_{23} = 30.45$, and $u_{33} = 46.21$.

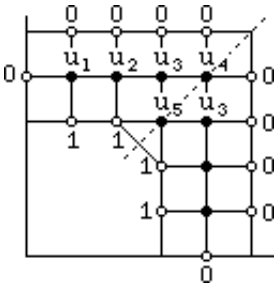
7. **(a)** Using the difference approximations for u_{xx} and u_{yy} we obtain

$$u_{xx} + u_{yy} = \frac{1}{h^2}(u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij}) = f(x, y)$$

so that

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = h^2 f(x, y).$$

(b) By symmetry, as shown in the figure, we need only solve for u_1, u_2, u_3, u_4, and u_5. The difference equations are



$$u_2 + 0 + 0 + 1 - 4u_1 = \frac{1}{4}(-2)$$
$$u_3 + 0 + u_1 + 1 - 4u_2 = \frac{1}{4}(-2)$$
$$u_4 + 0 + u_2 + u_5 - 4u_3 = \frac{1}{4}(-2)$$
$$0 + 0 + u_3 + u_3 - 4u_4 = \frac{1}{4}(-2)$$
$$u_3 + u_3 + 1 + 1 - 4u_5 = \frac{1}{4}(-2)$$

or

$$u_1 = 0.25u_2 + 0.375$$
$$u_2 = 0.25u_1 + 0.25u_3 + 0.375$$
$$u_3 = 0.25u_2 + 0.25u_4 + 0.25u_5 + 0.125$$
$$u_4 = 0.5u_3 + 0.125$$
$$u_5 = 0.5u_3 + 0.625.$$

Using Gauss-Seidel iteration we find $u_1 = 0.5427$, $u_2 = 0.6707$, $u_3 = 0.6402$, $u_4 = 0.4451$, and $u_5 = 0.9451$.

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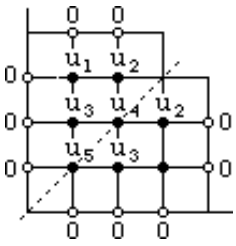
8. By symmetry, as shown in the figure, we need only solve for u_1, u_2, u_3, u_4, and u_5. The difference equations are



$$\begin{aligned} u_2 + 0 + 0 + u_3 - 4u_1 &= -1 & u_1 &= 0.25u_2 + 0.25u_3 + 0.25 \\ 0 + 0 + u_1 + u_4 - 4u_2 &= -1 & u_2 &= 0.25u_1 + 0.25u_4 + 0.25 \\ u_4 + u_1 + 0 + u_5 - 4u_3 &= -1 \quad \text{or} & u_3 &= 0.25u_1 + 0.25u_4 + 0.25u_5 + 0.25 \\ u_2 + u_2 + u_3 + u_3 - 4u_4 &= -1 & u_4 &= 0.5u_2 + 0.5u_3 + 0.25 \\ u_3 + u_3 + 0 + 0 - 4u_5 &= -1 & u_5 &= 0.5u_3 + 0.25. \end{aligned}$$

Using Gauss-Seidel iteration we find $u_1 = 0.6157$, $u_2 = 0.6493$, $u_3 = 0.8134$, $u_4 = 0.9813$, and $u_5 = 0.6567$.

Exercises 16.2

1. We identify $c = 1$, $a = 2$, $T = 1$, $n = 8$, and $m = 40$. Then $h = 2/8 = 0.25$, $k = 1/40 = 0.025$, and $\lambda = 2/5 = 0.4$.

TIME	X=0.25	X=0.50	X=0.75	X=1.00	X=1.25	X=1.50	X=1.75
0.000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.025	0.6000	1.0000	1.0000	0.6000	0.4000	0.0000	0.0000
0.050	0.5200	0.8400	0.8400	0.6800	0.3200	0.1600	0.0000
0.075	0.4400	0.7120	0.7760	0.6000	0.4000	0.1600	0.0640
0.100	0.3728	0.6288	0.6800	0.5904	0.3840	0.2176	0.0768
0.125	0.3261	0.5469	0.6237	0.5437	0.4000	0.2278	0.1024
0.150	0.2840	0.4893	0.5610	0.5182	0.3886	0.2465	0.1116
0.175	0.2525	0.4358	0.5152	0.4835	0.3836	0.2494	0.1209
0.200	0.2248	0.3942	0.4708	0.4562	0.3699	0.2517	0.1239
0.225	0.2027	0.3571	0.4343	0.4275	0.3571	0.2479	0.1255
0.250	0.1834	0.3262	0.4007	0.4021	0.3416	0.2426	0.1242
0.275	0.1672	0.2989	0.3715	0.3773	0.3262	0.2348	0.1219
0.300	0.1530	0.2752	0.3448	0.3545	0.3101	0.2262	0.1183
0.325	0.1407	0.2541	0.3209	0.3329	0.2943	0.2166	0.1141
0.350	0.1298	0.2354	0.2990	0.3126	0.2787	0.2067	0.1095
0.375	0.1201	0.2186	0.2790	0.2936	0.2635	0.1966	0.1046
0.400	0.1115	0.2034	0.2607	0.2757	0.2488	0.1865	0.0996
0.425	0.1036	0.1895	0.2438	0.2589	0.2347	0.1766	0.0945
0.450	0.0965	0.1769	0.2281	0.2432	0.2211	0.1670	0.0896
0.475	0.0901	0.1652	0.2136	0.2283	0.2083	0.1577	0.0847
0.500	0.0841	0.1545	0.2002	0.2144	0.1961	0.1487	0.0800

(continued)

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(continued)

TIME	X=0.25	X=0.50	X=0.75	X=1.00	X=1.25	X=1.50	X=1.75
0.525	0.0786	0.1446	0.1876	0.2014	0.1845	0.1402	0.0755
0.550	0.0736	0.1354	0.1759	0.1891	0.1735	0.1320	0.0712
0.575	0.0689	0.1269	0.1650	0.1776	0.1632	0.1243	0.0670
0.600	0.0645	0.1189	0.1548	0.1668	0.1534	0.1169	0.0631
0.625	0.0605	0.1115	0.1452	0.1566	0.1442	0.1100	0.0594
0.650	0.0567	0.1046	0.1363	0.1471	0.1355	0.1034	0.0559
0.675	0.0532	0.0981	0.1279	0.1381	0.1273	0.0972	0.0525
0.700	0.0499	0.0921	0.1201	0.1297	0.1196	0.0914	0.0494
0.725	0.0468	0.0864	0.1127	0.1218	0.1124	0.0859	0.0464
0.750	0.0439	0.0811	0.1058	0.1144	0.1056	0.0807	0.0436
0.775	0.0412	0.0761	0.0994	0.1074	0.0992	0.0758	0.0410
0.800	0.0387	0.0715	0.0933	0.1009	0.0931	0.0712	0.0385
0.825	0.0363	0.0671	0.0876	0.0948	0.0875	0.0669	0.0362
0.850	0.0341	0.0630	0.0823	0.0890	0.0822	0.0628	0.0340
0.875	0.0320	0.0591	0.0772	0.0836	0.0772	0.0590	0.0319
0.900	0.0301	0.0555	0.0725	0.0785	0.0725	0.0554	0.0300
0.925	0.0282	0.0521	0.0681	0.0737	0.0681	0.0521	0.0282
0.950	0.0265	0.0490	0.0640	0.0692	0.0639	0.0489	0.0265
0.975	0.0249	0.0460	0.0601	0.0650	0.0600	0.0459	0.0249
1.000	0.0234	0.0432	0.0564	0.0610	0.0564	0.0431	0.0233

2.

(x,y)	exact	approx	abs error
(0.25,0.1)	0.3794	0.3728	0.0066
(1,0.5)	0.1854	0.2144	0.0290
(1.5,0.8)	0.0623	0.0712	0.0089

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3. We identify $c = 1$, $a = 2$, $T = 1$, $n = 8$, and $m = 40$. Then $h = 2/8 = 0.25$, $k = 1/40 = 0.025$, and $\lambda = 2/5 = 0.4$.

TIME	X=0.25	X=0.50	X=0.75	X=1.00	X=1.25	X=1.50	X=1.75
0.000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.025	0.7074	0.9520	0.9566	0.7444	0.2545	0.0371	0.0053
0.050	0.5606	0.8499	0.8685	0.6633	0.3303	0.1034	0.0223
0.075	0.4684	0.7473	0.7836	0.6191	0.3614	0.1529	0.0462
0.100	0.4015	0.6577	0.7084	0.5837	0.3753	0.1871	0.0684
0.125	0.3492	0.5821	0.6428	0.5510	0.3797	0.2101	0.0861
0.150	0.3069	0.5187	0.5857	0.5199	0.3778	0.2247	0.0990
0.175	0.2721	0.4652	0.5359	0.4901	0.3716	0.2329	0.1078
0.200	0.2430	0.4198	0.4921	0.4617	0.3622	0.2362	0.1132
0.225	0.2186	0.3809	0.4533	0.4348	0.3507	0.2358	0.1160
0.250	0.1977	0.3473	0.4189	0.4093	0.3378	0.2327	0.1166
0.275	0.1798	0.3181	0.3881	0.3853	0.3240	0.2275	0.1157
0.300	0.1643	0.2924	0.3604	0.3626	0.3097	0.2208	0.1136
0.325	0.1507	0.2697	0.3353	0.3412	0.2953	0.2131	0.1107
0.350	0.1387	0.2495	0.3125	0.3211	0.2808	0.2047	0.1071
0.375	0.1281	0.2313	0.2916	0.3021	0.2666	0.1960	0.1032
0.400	0.1187	0.2150	0.2725	0.2843	0.2528	0.1871	0.0989
0.425	0.1102	0.2002	0.2549	0.2675	0.2393	0.1781	0.0946
0.450	0.1025	0.1867	0.2387	0.2517	0.2263	0.1692	0.0902
0.475	0.0955	0.1743	0.2236	0.2368	0.2139	0.1606	0.0858
0.500	0.0891	0.1630	0.2097	0.2228	0.2020	0.1521	0.0814
0.525	0.0833	0.1525	0.1967	0.2096	0.1906	0.1439	0.0772
0.550	0.0779	0.1429	0.1846	0.1973	0.1798	0.1361	0.0731
0.575	0.0729	0.1339	0.1734	0.1856	0.1696	0.1285	0.0691
0.600	0.0683	0.1256	0.1628	0.1746	0.1598	0.1214	0.0653
0.625	0.0641	0.1179	0.1530	0.1643	0.1506	0.1145	0.0617
0.650	0.0601	0.1106	0.1438	0.1546	0.1419	0.1080	0.0582
0.675	0.0564	0.1039	0.1351	0.1455	0.1336	0.1018	0.0549
0.700	0.0530	0.0976	0.1270	0.1369	0.1259	0.0959	0.0518
0.725	0.0497	0.0917	0.1194	0.1288	0.1185	0.0904	0.0488
0.750	0.0467	0.0862	0.1123	0.1212	0.1116	0.0852	0.0460
0.775	0.0439	0.0810	0.1056	0.1140	0.1050	0.0802	0.0433
0.800	0.0413	0.0762	0.0993	0.1073	0.0989	0.0755	0.0408
0.825	0.0388	0.0716	0.0934	0.1009	0.0931	0.0711	0.0384
0.850	0.0365	0.0674	0.0879	0.0950	0.0876	0.0669	0.0362
0.875	0.0343	0.0633	0.0827	0.0894	0.0824	0.0630	0.0341
0.900	0.0323	0.0596	0.0778	0.0841	0.0776	0.0593	0.0321
0.925	0.0303	0.0560	0.0732	0.0791	0.0730	0.0558	0.0302
0.950	0.0285	0.0527	0.0688	0.0744	0.0687	0.0526	0.0284
0.975	0.0268	0.0496	0.0647	0.0700	0.0647	0.0495	0.0268
1.000	0.0253	0.0466	0.0609	0.0659	0.0608	0.0465	0.0252

(x,y)	exact	approx	abs error
(0.25,0.1)	0.3794	0.4015	0.0221
(1,0.5)	0.1854	0.2228	0.0374
(1.5,0.8)	0.0623	0.0755	0.0132

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4. We identify $c = 1$, $a = 2$, $T = 1$, $n = 8$, and $m = 20$. Then $h = 2/8 = 0.25$, $h = 1/20 = 0.05$, and $\lambda = 4/5 = 0.8$.

TIME	X=0.25	X=0.50	X=0.75	X=1.00	X=1.25	X=1.50	X=1.75
0.00	1.00	1.00	1.00	1.00	0.00	0.00	0.00
0.05	0.20	1.00	1.00	0.20	0.80	0.00	0.00
0.10	0.68	0.36	0.36	1.32	-0.32	0.64	0.00
0.15	-0.12	0.62	1.13	-0.76	1.76	-0.64	0.51
0.20	0.56	0.44	-0.79	2.77	-2.18	2.20	-0.82
0.25	0.01	-0.44	3.04	-4.03	5.28	-3.72	2.25
0.30	-0.36	2.70	-5.41	9.07	-9.37	8.26	-4.33
0.35	2.38	-6.24	12.67	-17.26	19.49	-15.91	9.20
0.40	-6.42	15.78	-26.40	36.08	-38.23	32.50	-18.25
0.45	16.47	-35.72	57.33	-73.35	77.80	-64.68	36.94
0.50	-38.46	80.48	-121.66	152.12	-157.11	130.60	-73.91
0.55	87.46	-176.38	259.07	-314.28	320.44	-263.18	148.83
0.60	-193.58	383.05	-547.97	652.17	-654.23	533.32	-299.84
0.65	422.59	-823.07	1156.96	-1353.07	1340.93	-1083.25	606.56
0.70	-912.01	1757.48	-2435.09	2810.16	-2753.61	2207.94	-1230.53
0.75	1953.19	-3732.17	5115.16	-5837.05	5666.65	-4512.08	2504.67
0.80	-4157.65	7893.99	-10724.47	12127.68	-11679.29	9244.30	-5112.47
0.85	8809.78	-16642.09	22452.02	-25199.62	24105.16	-18979.99	10462.92
0.90	-18599.54	34994.69	-46944.58	52365.51	-49806.79	39042.46	-21461.75
0.95	39155.48	-73432.11	98054.91	-108820.40	103010.45	-80440.31	44111.02
1.00	-82238.97	153827.58	-204634.95	226144.53	-213214.84	165961.36	-90818.86

(x,y)	exact	approx	abs error
(0.25,0.1)	0.3794	0.6800	0.3006
(1,0.5)	0.1854	152.1152	151.9298
(1.5,0.8)	0.0623	9244.3042	9244.2419

In this case $\lambda = 0.8$ is greater than 0.5 and the procedure is unstable.

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5. We identify $c = 1$, $a = 2$, $T = 1$, $n = 8$, and $m = 20$. Then $h = 2/8 = 0.25$, $k = 1/20 = 0.05$, and $\lambda = 4/5 = 0.8$.

TIME	X=0.25	X=0.50	X=0.75	X=1.00	X=1.25	X=1.50	X=1.75
0.00	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.05	0.5265	0.8693	0.8852	0.6141	0.3783	0.0884	0.0197
0.10	0.3972	0.6551	0.7043	0.5883	0.3723	0.1955	0.0653
0.15	0.3042	0.5150	0.5844	0.5192	0.3812	0.2261	0.1010
0.20	0.2409	0.4171	0.4901	0.4620	0.3636	0.2385	0.1145
0.25	0.1962	0.3452	0.4174	0.4092	0.3391	0.2343	0.1178
0.30	0.1631	0.2908	0.3592	0.3624	0.3105	0.2220	0.1145
0.35	0.1379	0.2482	0.3115	0.3208	0.2813	0.2056	0.1077
0.40	0.1181	0.2141	0.2718	0.2840	0.2530	0.1876	0.0993
0.45	0.1020	0.1860	0.2381	0.2514	0.2265	0.1696	0.0904
0.50	0.0888	0.1625	0.2092	0.2226	0.2020	0.1523	0.0816
0.55	0.0776	0.1425	0.1842	0.1970	0.1798	0.1361	0.0732
0.60	0.0681	0.1253	0.1625	0.1744	0.1597	0.1214	0.0654
0.65	0.0599	0.1104	0.1435	0.1544	0.1418	0.1079	0.0582
0.70	0.0528	0.0974	0.1268	0.1366	0.1257	0.0959	0.0518
0.75	0.0466	0.0860	0.1121	0.1210	0.1114	0.0851	0.0460
0.80	0.0412	0.0760	0.0991	0.1071	0.0987	0.0754	0.0408
0.85	0.0364	0.0672	0.0877	0.0948	0.0874	0.0668	0.0361
0.90	0.0322	0.0594	0.0776	0.0839	0.0774	0.0592	0.0320
0.95	0.0285	0.0526	0.0687	0.0743	0.0686	0.0524	0.0284
1.00	0.0252	0.0465	0.0608	0.0657	0.0607	0.0464	0.0251

(x,y)	exact	approx	abs error
(0.25,0.1)	0.3794	0.3972	0.0178
(1,0.5)	0.1854	0.2226	0.0372
(1.5,0.8)	0.0623	0.0754	0.0131

6. We identify $c = 15/88 \approx 0.1705$, $a = 20$, $T = 10$, $n = 10$, and $m = 10$. Then $h = 2$, $k = 1$, and $\lambda = 15/352 \approx 0.0426$.

TIME	X=2	X=4	X=6	X=8	X=10	X=12	X=14	X=16	X=18
0	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
1	28.7216	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	28.7216
2	27.5521	29.9455	30.0000	30.0000	30.0000	30.0000	30.0000	29.9455	27.5521
3	26.4800	29.8459	29.9977	30.0000	30.0000	30.0000	29.9977	29.8459	26.4800
4	25.4951	29.7089	29.9913	29.9999	30.0000	29.9999	29.9913	29.7089	25.4951
5	24.5882	29.5414	29.9796	29.9995	30.0000	29.9995	29.9796	29.5414	24.5882
6	23.7515	29.3490	29.9618	29.9987	30.0000	29.9987	29.9618	29.3490	23.7515
7	22.9779	29.1365	29.9373	29.9972	29.9998	29.9972	29.9373	29.1365	22.9779
8	22.2611	28.9082	29.9057	29.9948	29.9996	29.9948	29.9057	28.9082	22.2611
9	21.5958	28.6675	29.8670	29.9912	29.9992	29.9912	29.8670	28.6675	21.5958
10	20.9768	28.4172	29.8212	29.9862	29.9985	29.9862	29.8212	28.4172	20.9768

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7. We identify $c = 15/88 \approx 0.1705$, $a = 50$, $T = 10$, $n = 10$, and $m = 10$. Then $h = 5$, $k = 1$, and $\lambda = 3/440 \approx 0.0068$.

TIME	X=5	X=10	X=15	X=20	X=25	X=30	X=35	X=40	X=45
0	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
1	29.7955	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	29.7955
2	29.5937	29.9986	30.0000	30.0000	30.0000	30.0000	30.0000	29.9986	29.5937
3	29.3947	29.9959	30.0000	30.0000	30.0000	30.0000	30.0000	29.9959	29.3947
4	29.1984	29.9918	30.0000	30.0000	30.0000	30.0000	30.0000	29.9918	29.1984
5	29.0047	29.9864	29.9999	30.0000	30.0000	30.0000	29.9999	29.9864	29.0047
6	28.8136	29.9798	29.9998	30.0000	30.0000	30.0000	29.9998	29.9798	28.8136
7	28.6251	29.9720	29.9997	30.0000	30.0000	30.0000	29.9997	29.9720	28.6251
8	28.4391	29.9630	29.9995	30.0000	30.0000	30.0000	29.9995	29.9630	28.4391
9	28.2556	29.9529	29.9992	30.0000	30.0000	30.0000	29.9992	29.9529	28.2556
10	28.0745	29.9416	29.9989	30.0000	30.0000	30.0000	29.9989	29.9416	28.0745

8. We identify $c = 50/27 \approx 1.8519$, $a = 20$, $T = 10$, $n = 10$, and $m = 10$. Then $h = 2$, $k = 1$, and $\lambda = 25/54 \approx 0.4630$.

TIME	X=2	X=4	X=6	X=8	X=10	X=12	X=14	X=16	X=18
0	18.0000	32.0000	42.0000	48.0000	50.0000	48.0000	42.0000	32.0000	18.0000
1	16.1481	30.1481	40.1481	46.1481	48.1481	46.1481	40.1481	30.1481	16.1481
2	15.1536	28.2963	38.2963	44.2963	46.2963	44.2963	38.2963	28.2963	15.1536
3	14.2226	26.8414	36.4444	42.4444	44.4444	42.4444	36.4444	26.8414	14.2226
4	13.4801	25.4452	34.7764	40.5926	42.5926	40.5926	34.7764	25.4452	13.4801
5	12.7787	24.2258	33.1491	38.8258	40.7407	38.8258	33.1491	24.2258	12.7787
6	12.1622	23.0574	31.6460	37.0842	38.9677	37.0842	31.6460	23.0574	12.1622
7	11.5756	21.9895	30.1875	35.4385	37.2238	35.4385	30.1875	21.9895	11.5756
8	11.0378	20.9636	28.8232	33.8340	35.5707	33.8340	28.8232	20.9636	11.0378
9	10.5230	20.0070	27.5043	32.3182	33.9626	32.3182	27.5043	20.0070	10.5230
10	10.0420	19.0872	26.2620	30.8509	32.4400	30.8509	26.2620	19.0872	10.0420

9. We identify $c = 260/159 \approx 1.6352$, $a = 100$, $T = 10$, $n = 10$, and $m = 10$. Then $h = 10$, $k = 1$, and $\lambda = 13/795 \approx 0.0164$.

TIME	X=10	X=20	X=30	X=40	X=50	X=60	X=70	X=80	X=90
0	8.0000	16.0000	24.0000	32.0000	40.0000	32.0000	24.0000	16.0000	8.0000
1	8.0000	16.0000	23.6075	31.3459	39.2151	31.6075	23.7384	15.8692	8.0000
2	8.0000	15.9936	23.2279	30.7068	38.4452	31.2151	23.4789	15.7384	7.9979
3	7.9999	15.9812	22.8606	30.0824	37.6900	30.8229	23.2214	15.6076	7.9937
4	7.9996	15.9631	22.5050	29.4724	36.9492	30.4312	22.9660	15.4769	7.9874
5	7.9990	15.9399	22.1606	28.8765	36.2228	30.0401	22.7125	15.3463	7.9793
6	7.9981	15.9118	21.8270	28.2945	35.5103	29.6500	22.4610	15.2158	7.9693
7	7.9967	15.8791	21.5037	27.7261	34.8117	29.2610	22.2112	15.0854	7.9575
8	7.9948	15.8422	21.1902	27.1709	34.1266	28.8733	21.9633	14.9553	7.9439
9	7.9924	15.8013	20.8861	26.6288	33.4548	28.4870	21.7172	14.8253	7.9287
10	7.9894	15.7568	20.5911	26.0995	32.7961	28.1024	21.4727	14.6956	7.9118

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10. **(a)** With $n = 4$ we have $h = 1/2$ so that $\lambda = 1/100 = 0.01$.

(b) We observe that $\alpha = 2(1 + 1/\lambda) = 202$ and $\beta = 2(1 - 1/\lambda) = -198$. The system of equations is

$$\begin{aligned} -u_{01} + \alpha u_{11} - u_{21} &= u_{20} - \beta u_{10} + u_{00} \\ -u_{11} + \alpha u_{21} - u_{31} &= u_{30} - \beta u_{20} + u_{10} \\ -u_{21} + \alpha u_{31} - u_{41} &= u_{40} - \beta u_{30} + u_{20}. \end{aligned}$$

Now $u_{00} = u_{01} = u_{40} = u_{41} = 0$, so the system is

$$\begin{aligned} \alpha u_{11} - u_{21} &= u_{20} - \beta u_{10} \\ -u_{11} + \alpha u_{21} - u_{31} &= u_{30} - \beta u_{20} + u_{10} \\ -u_{21} + \alpha u_{31} &= -\beta u_{30} + u_{20} \end{aligned}$$

or

$$\begin{aligned} 202u_{11} - u_{21} &= \sin\pi + 198\sin\frac{\pi}{2} = 198 \\ -u_{11} + 202u_{21} - u_{31} &= \sin\frac{3\pi}{2} + 198\sin\pi + \sin\frac{\pi}{2} = 0 \\ -u_{21} + 202u_{31} &= 198\sin\frac{3\pi}{2} + \sin\pi = -198. \end{aligned}$$

(c) The solution of this system is $u_{11} \approx 0.9802$, $u_{21} = 0$, $u_{31} \approx -0.9802$. The corresponding entries in Table 16.3 in the text are 0.9768, 0, and -0.9768.

11. **(a)** The differential equation is $k\,\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ where $k = K/\gamma\rho$. If we let $u(x,t) = v(x,t) + \psi(x)$, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \psi'' \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t}\,.$$

Substituting into the differential equation gives

$$k\,\frac{\partial^2 v}{\partial x^2} + k\psi'' = \frac{\partial v}{\partial t}\,.$$

Requiring $k\psi'' = 0$ we have $\psi(x) = c_1 x + c_2$. The boundary conditions become

$$u(0,t) = v(0,t) + \psi(0) = 20 \quad \text{and} \quad u(20,t) = v(20,t) + \psi(20) = 30.$$

Letting $\psi(0) = 20$ and $\psi(20) = 30$ we obtain the homogeneous boundary conditions in v: $v(0,t) = v(20,t) = 0$. Now $\psi(0) = 20$ and $\psi(20) = 30$ imply that $c_1 = 1/2$ and $c_2 = 20$. The steady-state solution is $\psi(x) = \frac{1}{2}\,x + 20$.

(b) To use the Crank-Nicholson method we identify $c = 375/212 \approx 1.7689$, $a = 20$, $T = 400$, $n = 5$, and $m = 40$. Then $h = 4$, $k = 10$, and $\lambda = 1875/1696 \approx 1.1055$.

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TIME	X=4.00	X=8.00	X=12.00	X=16.00
0.00	50.0000	50.0000	50.0000	50.0000
10.00	32.7433	44.2679	45.4228	38.2971
20.00	29.9946	36.2354	38.3148	35.8160
30.00	26.9487	32.1409	34.0874	32.9644
40.00	25.2691	29.2562	31.2704	31.2580
50.00	24.1178	27.4348	29.4296	30.1207
60.00	23.3821	26.2339	28.2356	29.3810
70.00	22.8995	25.4560	27.4554	28.8998
80.00	22.5861	24.9481	26.9482	28.5859
90.00	22.3817	24.6176	26.6175	28.3817
100.00	22.2486	24.4022	26.4023	28.2486
110.00	22.1619	24.2620	26.2620	28.1619
120.00	22.1055	24.1707	26.1707	28.1055
130.00	22.0687	24.1112	26.1112	28.0687
140.00	22.0447	24.0724	26.0724	28.0447
150.00	22.0291	24.0472	26.0472	28.0291
160.00	22.0190	24.0307	26.0307	28.0190
170.00	22.0124	24.0200	26.0200	28.0124
180.00	22.0081	24.0130	26.0130	28.0081
190.00	22.0052	24.0085	26.0085	28.0052
200.00	22.0034	24.0055	26.0055	28.0034
210.00	22.0022	24.0036	26.0036	28.0022
220.00	22.0015	24.0023	26.0023	28.0015
230.00	22.0009	24.0015	26.0015	28.0009
240.00	22.0006	24.0010	26.0010	28.0006
250.00	22.0004	24.0007	26.0007	28.0004
260.00	22.0003	24.0004	26.0004	28.0003
270.00	22.0002	24.0003	26.0003	28.0002
280.00	22.0001	24.0002	26.0002	28.0001
290.00	22.0001	24.0001	26.0001	28.0001
300.00	22.0000	24.0001	26.0001	28.0000
310.00	22.0000	24.0001	26.0001	28.0000
320.00	22.0000	24.0000	26.0000	28.0000
330.00	22.0000	24.0000	26.0000	28.0000
340.00	22.0000	24.0000	26.0000	28.0000
350.00	22.0000	24.0000	26.0000	28.0000

We observe that the approximate steady-state temperatures agree exactly with the corresponding values of $\psi(x)$.

12. We identify $c = 1$, $a = 1$, $T = 1$, $n = 5$, and $m = 20$. Then $h = 0.2$, $k = 0.04$, and $\lambda = 1$. The values below were obtained using *Excel*, which carries more than 12 significant digits. In order to see evidence of instability use $0 \le t \le 2$.

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TIME	X=0.2	X=0.4	X=0.6	X=0.8
0.00	0.5878	0.9511	0.9511	0.5878
0.04	0.3633	0.5878	0.5878	0.3633
0.08	0.2245	0.3633	0.3633	0.2245
0.12	0.1388	0.2245	0.2245	0.1388
0.16	0.0858	0.1388	0.1388	0.0858
0.20	0.0530	0.0858	0.0858	0.0530
0.24	0.0328	0.0530	0.0530	0.0328
0.28	0.0202	0.0328	0.0328	0.0202
0.32	0.0125	0.0202	0.0202	0.0125
0.36	0.0077	0.0125	0.0125	0.0077
0.40	0.0048	0.0077	0.0077	0.0048
0.44	0.0030	0.0048	0.0048	0.0030
0.48	0.0018	0.0030	0.0030	0.0018
0.52	0.0011	0.0018	0.0018	0.0011
0.56	0.0007	0.0011	0.0011	0.0007
0.60	0.0004	0.0007	0.0007	0.0004
0.64	0.0003	0.0004	0.0004	0.0003
0.68	0.0002	0.0003	0.0003	0.0002
0.72	0.0001	0.0002	0.0002	0.0001
0.76	0.0001	0.0001	0.0001	0.0001
0.80	0.0000	0.0001	0.0001	0.0000
0.84	0.0000	0.0000	0.0000	0.0000
0.88	0.0000	0.0000	0.0000	0.0000
0.92	0.0000	0.0000	0.0000	0.0000
0.96	0.0000	0.0000	0.0000	0.0000
1.00	0.0000	0.0000	0.0000	0.0000

TIME	X=0.2	X=0.4	X=0.6	X=0.8
1.04	0.0000	0.0000	0.0000	0.0000
1.08	0.0000	0.0000	0.0000	0.0000
1.12	0.0000	0.0000	0.0000	0.0000
1.16	0.0000	0.0000	0.0000	0.0000
1.20	-0.0001	0.0001	-0.0001	0.0001
1.24	0.0001	-0.0002	0.0002	-0.0001
1.28	-0.0004	0.0006	-0.0006	0.0004
1.32	0.0010	-0.0015	0.0015	-0.0010
1.36	-0.0025	0.0040	-0.0040	0.0025
1.40	0.0065	-0.0106	0.0106	-0.0065
1.44	-0.0171	0.0277	-0.0277	0.0171
1.48	0.0448	-0.0724	0.0724	-0.0448
1.52	-0.1172	0.1897	-0.1897	0.1172
1.56	0.3069	-0.4965	0.4965	-0.3069
1.60	-0.8034	1.2999	-1.2999	0.8034
1.64	2.1033	-3.4032	3.4032	-2.1033
1.68	-5.5064	8.9096	-8.9096	5.5064
1.72	14.416	-23.326	23.326	-14.416
1.76	-37.742	61.067	-61.067	37.742
1.80	98.809	-159.88	159.88	-98.809
1.84	-258.68	418.56	-418.56	258.685
1.88	677.24	-1095.8	1095.8	-677.245
1.92	-1773.1	2868.9	-2868.9	1773.1
1.96	4641.9	-7510.8	7510.8	-4641.9
2.00	-12153	19663	-19663	12153

Exercises 16.3

1. **(a)** Identifying $h = 1/4$ and $k = 1/10$ we see that $\lambda = 2/5$.

TIME	X=0.25	X=0.5	X=0.75
0.00	0.1875	0.2500	0.1875
0.10	0.1775	0.2400	0.1775
0.20	0.1491	0.2100	0.1491
0.30	0.1066	0.1605	0.1066
0.40	0.0556	0.0938	0.0556
0.50	0.0019	0.0148	0.0019
0.60	-0.0501	-0.0682	-0.0501
0.70	-0.0970	-0.1455	-0.0970
0.80	-0.1361	-0.2072	-0.1361
0.90	-0.1648	-0.2462	-0.1648
1.00	-0.1802	-0.2591	-0.1802

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(b) Identifying $h = 2/5$ and $k = 1/10$ we see that $\lambda = 1/4$.

TIME	X=0.4	X=0.8	X=1.2	X=1.6
0.00	0.0032	0.5273	0.5273	0.0032
0.10	0.0194	0.5109	0.5109	0.0194
0.20	0.0652	0.4638	0.4638	0.0652
0.30	0.1318	0.3918	0.3918	0.1318
0.40	0.2065	0.3035	0.3035	0.2065
0.50	0.2743	0.2092	0.2092	0.2743
0.60	0.3208	0.1190	0.1190	0.3208
0.70	0.3348	0.0413	0.0413	0.3348
0.80	0.3094	-0.0180	-0.0180	0.3094
0.90	0.2443	-0.0568	-0.0568	0.2443
1.00	0.1450	-0.0768	-0.0768	0.1450

(c) Identifying $h = 1/10$ and $k = 1/25$ we see that $\lambda = 2\sqrt{2}/5$.

TIME	X=0.1	X=0.2	X=0.3	X=0.4	X=0.5	X=0.6	X=0.7	X=0.8	X=0.9
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000	0.5000	0.5000	0.5000
0.04	0.0000	0.0000	0.0000	0.0000	0.0800	0.4200	0.5000	0.5000	0.4200
0.08	0.0000	0.0000	0.0000	0.0256	0.2432	0.2568	0.4744	0.4744	0.2312
0.12	0.0000	0.0000	0.0082	0.1126	0.3411	0.1589	0.3792	0.3710	0.0462
0.16	0.0000	0.0026	0.0472	0.2394	0.3076	0.1898	0.2108	0.1663	-0.0496
0.20	0.0008	0.0187	0.1334	0.3264	0.2146	0.2651	0.0215	-0.0933	-0.0605
0.24	0.0071	0.0657	0.2447	0.3159	0.1735	0.2463	-0.1266	-0.3056	-0.0625
0.28	0.0299	0.1513	0.3215	0.2371	0.2013	0.0849	-0.2127	-0.3829	-0.1223
0.32	0.0819	0.2525	0.3168	0.1737	0.2033	-0.1345	-0.2580	-0.3223	-0.2264
0.36	0.1623	0.3197	0.2458	0.1657	0.0877	-0.2853	-0.2843	-0.2104	-0.2887
0.40	0.2412	0.3129	0.1727	0.1583	-0.1223	-0.3164	-0.2874	-0.1473	-0.2336
0.44	0.2657	0.2383	0.1399	0.0658	-0.3046	-0.2761	-0.2549	-0.1565	-0.0761
0.48	0.1965	0.1410	0.1149	-0.1216	-0.3593	-0.2381	-0.1977	-0.1715	0.0800
0.52	0.0466	0.0531	0.0225	-0.3093	-0.2992	-0.2260	-0.1451	-0.1144	0.1300
0.56	-0.1161	-0.0466	-0.1662	-0.3876	-0.2188	-0.2114	-0.1085	0.0111	0.0602
0.60	-0.2194	-0.2069	-0.3875	-0.3411	-0.1901	-0.1662	-0.0666	0.1140	-0.0446
0.64	-0.2485	-0.4290	-0.5362	-0.2611	-0.2021	-0.0969	0.0012	0.1084	-0.0843
0.68	-0.2559	-0.6276	-0.5625	-0.2503	-0.1993	-0.0298	0.0720	0.0068	-0.0354
0.72	-0.3003	-0.6865	-0.5097	-0.3230	-0.1585	0.0156	0.0893	-0.0874	0.0384
0.76	-0.3722	-0.5652	-0.4538	-0.4029	-0.1147	0.0289	0.0265	-0.0849	0.0596
0.80	-0.3867	-0.3464	-0.4172	-0.4068	-0.1172	-0.0046	-0.0712	-0.0005	0.0155
0.84	-0.2647	-0.1633	-0.3546	-0.3214	-0.1763	-0.0954	-0.1249	0.0665	-0.0386
0.88	-0.0254	-0.0738	-0.2202	-0.2002	-0.2559	-0.2215	-0.1079	0.0385	-0.0468
0.92	0.2064	-0.0157	-0.0325	-0.1032	-0.3067	-0.3223	-0.0804	-0.0636	-0.0127
0.96	0.3012	0.1081	0.1380	-0.0487	-0.2974	-0.3407	-0.1250	-0.1548	0.0092
1.00	0.2378	0.3032	0.2392	-0.0141	-0.2223	-0.2762	-0.2481	-0.1840	-0.0244

2. (a) In Section 13.4 the solution of the wave equation is shown to be

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos n\pi t + B_n \sin n\pi t) \sin n\pi x$$

where

$$A_n = 2\int_0^1 \sin \pi x \sin n\pi x \, dx = \begin{cases} 1, & n = 1 \\ 0, & n = 2,\ 3,\ 4,\ \ldots \end{cases}$$

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and

$$B_n = \frac{2}{n\pi}\int_0^1 0\,dx = 0.$$

Thus $u(x,t) = \cos\pi t \sin\pi x$.

(b) We have $h = 1/4$, $k = 0.5/5 = 0.1$ and $\lambda = 0.4$. Now $u_{0,j} = u_{4,j} = 0$ or $j = 0, 1, \ldots, 5$, and the initial values of u are $u_{1,0} = u(1/4, 0) = \sin\pi/4 \approx 0.7071$, $u_{2,0} = u(1/2, 0) = \sin\pi/2 = 1$, $u_{3,0} = u(3/4, 0) = \sin 3\pi/4 \approx 0.7071$. From equation (6) in the text we have

$$u_{i,1} = 0.8(u_{i+1,0} + u_{i-1,0}) + 0.84u_{i,0} + 0.1(0).$$

Then $u_{1,1} \approx 0.6740$, $u_{2,1} = 0.9531$, $u_{3,1} = 0.6740$. From equation (3) in the text we have for $j = 1, 2, 3, \ldots$

$$u_{i,j+1} = 0.16u_{i+1,j} + 2(0.84)u_{i,j} + 0.16u_{i-1,j} - u_{i,j-1}.$$

The results of the calculations are given in the table.

TIME	x=0.25	x=0.50	x=0.75
0.0	0.7071	1.0000	0.7071
0.1	0.6740	0.9531	0.6740
0.2	0.5777	0.8169	0.5777
0.3	0.4272	0.6042	0.4272
0.4	0.2367	0.3348	0.2367
0.5	0.0241	0.0340	0.0241

(c)

i,j	approx	exact	error
1,1	0.6740	0.6725	0.0015
1,2	0.5777	0.5721	0.0056
1,3	0.4272	0.4156	0.0116
1,4	0.2367	0.2185	0.0182
1,5	0.0241	0.0000	0.0241
2,1	0.9531	0.9511	0.0021
2,2	0.8169	0.8090	0.0079
2,3	0.6042	0.5878	0.0164
2,4	0.3348	0.3090	0.0258
2,5	0.0340	0.0000	0.0340
3,1	0.6740	0.6725	0.0015
3,2	0.5777	0.5721	0.0056
3,3	0.4272	0.4156	0.0116
3,4	0.2367	0.2185	0.0182
3,5	0.0241	0.0000	0.0241

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3. **(a)** Identifying $h = 1/5$ and $k = 0.5/10 = 0.05$ we see that $\lambda = 0.25$.

TIME	X=0.2	X=0.4	X=0.6	X=0.8
0.00	0.5878	0.9511	0.9511	0.5878
0.05	0.5808	0.9397	0.9397	0.5808
0.10	0.5599	0.9059	0.9059	0.5599
0.15	0.5256	0.8505	0.8505	0.5256
0.20	0.4788	0.7748	0.7748	0.4788
0.25	0.4206	0.6806	0.6806	0.4206
0.30	0.3524	0.5701	0.5701	0.3524
0.35	0.2757	0.4460	0.4460	0.2757
0.40	0.1924	0.3113	0.3113	0.1924
0.45	0.1046	0.1692	0.1692	0.1046
0.50	0.0142	0.0230	0.0230	0.0142

(b) Identifying $h = 1/5$ and $k = 0.5/20 = 0.025$ we see that $\lambda = 0.125$.

TIME	X=0.2	X=0.4	X=0.6	X=0.8
0.00	0.5878	0.9511	0.9511	0.5878
0.03	0.5860	0.9482	0.9482	0.5860
0.05	0.5808	0.9397	0.9397	0.5808
0.08	0.5721	0.9256	0.9256	0.5721
0.10	0.5599	0.9060	0.9060	0.5599
0.13	0.5445	0.8809	0.8809	0.5445
0.15	0.5257	0.8507	0.8507	0.5257
0.18	0.5039	0.8153	0.8153	0.5039
0.20	0.4790	0.7750	0.7750	0.4790
0.23	0.4513	0.7302	0.7302	0.4513
0.25	0.4209	0.6810	0.6810	0.4209
0.28	0.3879	0.6277	0.6277	0.3879
0.30	0.3527	0.5706	0.5706	0.3527
0.33	0.3153	0.5102	0.5102	0.3153
0.35	0.2761	0.4467	0.4467	0.2761
0.38	0.2352	0.3806	0.3806	0.2352
0.40	0.1929	0.3122	0.3122	0.1929
0.43	0.1495	0.2419	0.2419	0.1495
0.45	0.1052	0.1701	0.1701	0.1052
0.48	0.0602	0.0974	0.0974	0.0602
0.50	0.0149	0.0241	0.0241	0.0149

4. We have $\lambda = 1$. The initial values of n are $u_{1,0} = u(0.2, 0) = 0.16$, $u_{2,0} = u(0.4) = 0.24$, $u_{3,0} = 0.24$, and $u_{4,0} = 0.16$. From equation (6) in the text we have

$$u_{i,1} = \frac{1}{2}(u_{i+1,0} + u_{i-1,0}) + 0u_{i,0} + k \cdot 0 = \frac{1}{2}(u_{i+1,0} + u_{i-1,0}).$$

Then, using $u_{0,0} = u_{5,0} = 0$, we find $u_{1,1} = 0.12$, $u_{2,1} = 0.2$, $u_{3,1} = 0.2$, and $u_{4,1} = 0.12$.

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5. We identify $c = 24944.4$, $k = 0.00020045$ seconds $= 0.20045$ milliseconds, and $\lambda = 0.5$. Time in the table is expressed in milliseconds.

TIME	X=10	X=20	X=30	X=40	X=50
0.00000	0.1000	0.2000	0.3000	0.2000	0.1000
0.20045	0.1000	0.2000	0.2750	0.2000	0.1000
0.40089	0.1000	0.1938	0.2125	0.1938	0.1000
0.60134	0.0984	0.1688	0.1406	0.1688	0.0984
0.80178	0.0898	0.1191	0.0828	0.1191	0.0898
1.00223	0.0661	0.0531	0.0432	0.0531	0.0661
1.20268	0.0226	-0.0121	0.0085	-0.0121	0.0226
1.40312	-0.0352	-0.0635	-0.0365	-0.0635	-0.0352
1.60357	-0.0913	-0.1011	-0.0950	-0.1011	-0.0913
1.80401	-0.1271	-0.1347	-0.1566	-0.1347	-0.1271
2.00446	-0.1329	-0.1719	-0.2072	-0.1719	-0.1329
2.20491	-0.1153	-0.2081	-0.2402	-0.2081	-0.1153
2.40535	-0.0920	-0.2292	-0.2571	-0.2292	-0.0920
2.60580	-0.0801	-0.2230	-0.2601	-0.2230	-0.0801
2.80624	-0.0838	-0.1903	-0.2445	-0.1903	-0.0838
3.00669	-0.0932	-0.1445	-0.2018	-0.1445	-0.0932
3.20713	-0.0921	-0.1003	-0.1305	-0.1003	-0.0921
3.40758	-0.0701	-0.0615	-0.0440	-0.0615	-0.0701
3.60803	-0.0284	-0.0205	0.0336	-0.0205	-0.0284
3.80847	0.0224	0.0321	0.0842	0.0321	0.0224
4.00892	0.0700	0.0953	0.1087	0.0953	0.0700
4.20936	0.1064	0.1555	0.1265	0.1555	0.1064
4.40981	0.1285	0.1962	0.1588	0.1962	0.1285
4.61026	0.1354	0.2106	0.2098	0.2106	0.1354
4.81070	0.1273	0.2060	0.2612	0.2060	0.1273
5.01115	0.1070	0.1955	0.2851	0.1955	0.1070
5.21159	0.0821	0.1853	0.2641	0.1853	0.0821
5.41204	0.0625	0.1689	0.2038	0.1689	0.0625
5.61249	0.0539	0.1347	0.1260	0.1347	0.0539
5.81293	0.0520	0.0781	0.0526	0.0781	0.0520
6.01338	0.0436	0.0086	-0.0080	0.0086	0.0436
6.21382	0.0156	-0.0564	-0.0604	-0.0564	0.0156
6.41427	-0.0343	-0.1043	-0.1107	-0.1043	-0.0343
6.61472	-0.0931	-0.1364	-0.1578	-0.1364	-0.0931
6.81516	-0.1395	-0.1630	-0.1942	-0.1630	-0.1395
7.01561	-0.1568	-0.1915	-0.2150	-0.1915	-0.1568
7.21605	-0.1436	-0.2173	-0.2240	-0.2173	-0.1436
7.41650	-0.1129	-0.2263	-0.2297	-0.2263	-0.1129
7.61695	-0.0824	-0.2078	-0.2336	-0.2078	-0.0824
7.81739	-0.0625	-0.1644	-0.2247	-0.1644	-0.0625
8.01784	-0.0526	-0.1106	-0.1856	-0.1106	-0.0526
8.21828	-0.0440	-0.0611	-0.1091	-0.0611	-0.0440
8.41873	-0.0287	-0.0192	-0.0085	-0.0192	-0.0287
8.61918	-0.0038	0.0229	0.0867	0.0229	-0.0038
8.81962	0.0287	0.0743	0.1500	0.0743	0.0287
9.02007	0.0654	0.1332	0.1755	0.1332	0.0654
9.22051	0.1027	0.1858	0.1799	0.1858	0.1027
9.42096	0.1352	0.2160	0.1872	0.2160	0.1352
9.62140	0.1540	0.2189	0.2089	0.2189	0.1540
9.82185	0.1506	0.2030	0.2356	0.2030	0.1506
10.02230	0.1226	0.1822	0.2461	0.1822	0.1226

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6. We identify $c = 24944.4$, $k = 0.00010022$ seconds $= 0.10022$ milliseconds, and $\lambda = 0.25$. Time in the table is expressed in milliseconds.

TIME	X=10	X=20	X=30	X=40	X=50
0.00000	0.2000	0.2667	0.2000	0.1333	0.0667
0.10022	0.1958	0.2625	0.2000	0.1333	0.0667
0.20045	0.1836	0.2503	0.1997	0.1333	0.0667
0.30067	0.1640	0.2307	0.1985	0.1333	0.0667
0.40089	0.1384	0.2050	0.1952	0.1332	0.0667
0.50111	0.1083	0.1744	0.1886	0.1328	0.0667
0.60134	0.0755	0.1407	0.1777	0.1318	0.0666
0.70156	0.0421	0.1052	0.1615	0.1295	0.0665
0.80178	0.0100	0.0692	0.1399	0.1253	0.0661
0.90201	-0.0190	0.0340	0.1129	0.1184	0.0654
1.00223	-0.0435	0.0004	0.0813	0.1077	0.0638
1.10245	-0.0626	-0.0309	0.0464	0.0927	0.0610
1.20268	-0.0758	-0.0593	0.0095	0.0728	0.0564
1.30290	-0.0832	-0.0845	-0.0278	0.0479	0.0493
1.40312	-0.0855	-0.1060	-0.0639	0.0184	0.0390
1.50334	-0.0837	-0.1237	-0.0974	-0.0150	0.0250
1.60357	-0.0792	-0.1371	-0.1275	-0.0511	0.0069
1.70379	-0.0734	-0.1464	-0.1533	-0.0882	-0.0152
1.80401	-0.0675	-0.1515	-0.1747	-0.1249	-0.0410
1.90424	-0.0627	-0.1528	-0.1915	-0.1595	-0.0694
2.00446	-0.0596	-0.1509	-0.2039	-0.1904	-0.0991
2.10468	-0.0585	-0.1467	-0.2122	-0.2165	-0.1283
2.20491	-0.0592	-0.1410	-0.2166	-0.2368	-0.1551
2.30513	-0.0614	-0.1349	-0.2175	-0.2507	-0.1772
2.40535	-0.0643	-0.1294	-0.2154	-0.2579	-0.1929
2.50557	-0.0672	-0.1251	-0.2105	-0.2585	-0.2005
2.60580	-0.0696	-0.1227	-0.2033	-0.2524	-0.1993
2.70602	-0.0709	-0.1219	-0.1942	-0.2399	-0.1889
2.80624	-0.0710	-0.1225	-0.1833	-0.2214	-0.1699
2.90647	-0.0699	-0.1236	-0.1711	-0.1972	-0.1435
3.00669	-0.0678	-0.1244	-0.1575	-0.1681	-0.1115
3.10691	-0.0649	-0.1237	-0.1425	-0.1348	-0.0761
3.20713	-0.0617	-0.1205	-0.1258	-0.0983	-0.0395
3.30736	-0.0583	-0.1139	-0.1071	-0.0598	-0.0042
3.40758	-0.0547	-0.1035	-0.0859	-0.0209	0.0279
3.50780	-0.0508	-0.0889	-0.0617	0.0171	0.0552
3.60803	-0.0460	-0.0702	-0.0343	0.0525	0.0767
3.70825	-0.0399	-0.0478	-0.0037	0.0840	0.0919
3.80847	-0.0318	-0.0221	0.0297	0.1106	0.1008
3.90870	-0.0211	0.0062	0.0648	0.1314	0.1041
4.00892	-0.0074	0.0365	0.1005	0.1464	0.1025
4.10914	0.0095	0.0680	0.1350	0.1558	0.0973
4.20936	0.0295	0.1000	0.1666	0.1602	0.0897
4.30959	0.0521	0.1318	0.1937	0.1606	0.0808
4.40981	0.0764	0.1625	0.2148	0.1581	0.0719
4.51003	0.1013	0.1911	0.2291	0.1538	0.0639
4.61026	0.1254	0.2164	0.2364	0.1485	0.0575
4.71048	0.1475	0.2373	0.2369	0.1431	0.0532
4.81070	0.1659	0.2526	0.2315	0.1379	0.0512
4.91093	0.1794	0.2611	0.2217	0.1331	0.0514
5.01115	0.1867	0.2620	0.2087	0.1288	0.0535

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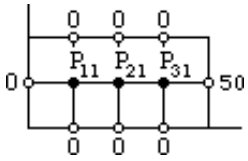
Chapter 16 Review Exercises

1. Using the figure we obtain the system



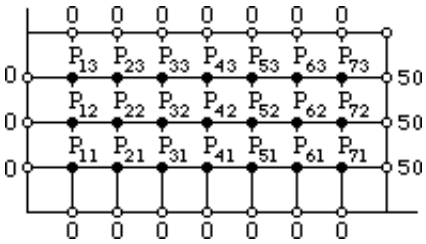
$$\begin{aligned} u_{21} + 0 + 0 + 0 - 4u_{11} &= 0 \\ u_{31} + 0 + u_{11} + 0 - 4u_{21} &= 0 \\ 50 + 0 + u_{21} + 0 - 4u_{31} &= 0. \end{aligned}$$

By Gauss-Elimination then,

$$\left[\begin{array}{rrr|r} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & -50 \end{array}\right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{rrr|r} 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & -50 \\ 0 & 0 & 1 & 13.3928 \end{array}\right].$$

The solution is $u_{11} = 0.8929$, $u_{21} = 3.5714$, $u_{31} = 13.3928$.

2. By symmetry we observe that $u_{i,1} = u_{i,3}$ for $i = 1, 2, \ldots, 7$. We then use Gauss-Seidel iteration with an initial guess of 7.5 for all variables to solve the system



$$\begin{aligned} u_{11} &= 0.25u_{21} + 0.25u_{12} \\ u_{21} &= 0.25u_{31} + 0.25u_{22} + 0.25u_{11} \\ u_{31} &= 0.25u_{41} + 0.25u_{32} + 0.25u_{21} \\ u_{41} &= 0.25u_{51} + 0.25u_{42} + 0.25u_{31} \\ u_{51} &= 0.25u_{61} + 0.25u_{52} + 0.25u_{41} \\ u_{61} &= 0.25u_{71} + 0.25u_{62} + 0.25u_{51} \\ u_{71} &= 12.5 + 0.25u_{72} + 0.25u_{61} \\ u_{12} &= 0.25u_{22} + 0.5u_{11} \\ u_{22} &= 0.25u_{32} + 0.5u_{21} + 0.25u_{12} \\ u_{32} &= 0.25u_{42} + 0.5u_{31} + 0.25u_{22} \\ u_{42} &= 0.25u_{52} + 0.5u_{41} + 0.25u_{32} \\ u_{52} &= 0.25u_{62} + 0.5u_{51} + 0.25u_{42} \\ u_{62} &= 0.25u_{72} + 0.5u_{61} + 0.25u_{52} \\ u_{72} &= 12.5 + 0.5u_{71} + 0.25u_{62}. \end{aligned}$$

After 30 iterations we obtain $u_{11} = u_{13} = 0.1765$, $u_{21} = u_{23} = 0.4566$, $u_{31} = u_{33} = 1.0051$, $u_{41} = u_{43} = 2.1479$, $u_{51} = u_{53} = 4.5766$, $u_{61} = u_{63} = 9.8316$, $u_{71} = u_{73} = 21.6051$, $u_{12} = 0.2494$, $u_{22} = 0.6447$, $u_{32} = 1.4162$, $u_{42} = 3.0097$, $u_{52} = 6.3269$, $u_{62} = 13.1447$, $u_{72} = 26.5887$.

3. **(a)**

TIME	X=0.0	X=0.2	X=0.4	X=0.6	X=0.8	X=1.0
0.00	0.0000	0.2000	0.4000	0.6000	0.8000	0.0000
0.01	0.0000	0.2000	0.4000	0.6000	0.5500	0.0000
0.02	0.0000	0.2000	0.4000	0.5375	0.4250	0.0000
0.03	0.0000	0.2000	0.3844	0.4750	0.3469	0.0000
0.04	0.0000	0.1961	0.3609	0.4203	0.2922	0.0000
0.05	0.0000	0.1883	0.3346	0.3734	0.2512	0.0000

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(b)

TIME	X=0.0	X=0.2	X=0.4	X=0.6	X=0.8	X=1.0
0.00	0.0000	0.2000	0.4000	0.6000	0.8000	0.0000
0.01	0.0000	0.2000	0.4000	0.6000	0.8000	0.0000
0.02	0.0000	0.2000	0.4000	0.6000	0.5500	0.0000
0.03	0.0000	0.2000	0.4000	0.5375	0.4250	0.0000
0.04	0.0000	0.2000	0.3844	0.4750	0.3469	0.0000
0.05	0.0000	0.1961	0.3609	0.4203	0.2922	0.0000

(c) The table in part (b) is the same as the table in part (a) shifted downward one row.

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17 Functions of a Complex Variable

Exercises 17.1

1. $3 + 3i$

2. $-4i$

3. $i^8 = (i^2)^4 = (-1)^4 = 1$

4. $i^{11} = i(i^2)^5 = i(-1)^5 = -i$

5. $7 - 13i$

6. $-3 - 9i$

7. $-7 + 5i$

8. $-7 + 8i$

9. $11 - 10i$

10. $\dfrac{3}{4} + \dfrac{2}{3}\,i$

11. $-5 + 12i$

12. $-2 - 2i$

13. $-2i$

14. $\dfrac{i}{1+i} \cdot \dfrac{1-i}{1-i} = \dfrac{i+1}{2} = \dfrac{1}{2} + \dfrac{1}{2}\,i$

15. $\dfrac{2-4i}{3+5i} \cdot \dfrac{3-5i}{3-5i} = \dfrac{-14-22i}{34} = -\dfrac{7}{17} - \dfrac{11}{17}\,i$

16. $\dfrac{10-5i}{6+2i} \cdot \dfrac{6-2i}{6-2i} = \dfrac{50-50i}{40} = \dfrac{5}{4} - \dfrac{5}{4}\,i$

17. $\dfrac{9+7i}{1+i} \cdot \dfrac{1-i}{1-i} = \dfrac{16-2i}{2} = 8 - i$

18. $\dfrac{3-i}{11-2i} \cdot \dfrac{11+2i}{11+2i} = \dfrac{35-5i}{125} = \dfrac{7}{25} - \dfrac{1}{25}\,i$

19. $\dfrac{2-11i}{6-i} \cdot \dfrac{6+i}{6+i} = \dfrac{23-64i}{37} = \dfrac{23}{37} - \dfrac{64}{37}\,i$

20. $\dfrac{4+3i}{3+4i} \cdot \dfrac{3-4i}{3-4i} = \dfrac{24-7i}{25} = \dfrac{24}{25} - \dfrac{7}{25}\,i$

21. $(1+i)(10+10i) = 10(1+i)^2 = 20i$

22. $[(1+i)(1-i)]^2(1-i) = 4 - 4i$

23. $20 + 23i + \dfrac{1}{2-i} \cdot \dfrac{2+i}{2+i} = 20 + 23i + \dfrac{2}{5} + \dfrac{1}{5}\,i = \dfrac{102}{5} + \dfrac{116}{5}\,i$

24. $(2+3i)(-i)^2 = -2 - 3i$

25. $\dfrac{i}{9+7i} \cdot \dfrac{9-7i}{9-7i} = \dfrac{7+9i}{130} = \dfrac{7}{130} + \dfrac{9}{130}\,i$

26. $\dfrac{1}{6+8i} \cdot \dfrac{6-8i}{6-8i} = \dfrac{6-8i}{84} = \dfrac{1}{14} - \dfrac{2}{21}\,i$

27. $\dfrac{x}{x^2+y^2}$

28. $x^2 - y^2$

29. $-2y - 4$

30. 0

31. $\sqrt{(x-1)^2 + (y-3)^2}$

32. $\sqrt{36x^2 + 16y^2}$

33. $2x + 2yi = -9 + 2i$ implies $2x = -9$ and $2y = 2$. Hence $z = -\frac{9}{2} + i$.

34. $-x + 3yi = -7 + 6i$ implies $-x - 7$ and $3y = 6$. Hence $z = 7 + 2i$.

35. $x^2 - y^2 + 2xyi = 0 + i$ implies $x^2 - y^2 = 0$ and $2xy = 1$. Now $y = x$ implies $2x^2 = 1$ and so $x = \pm 1/\sqrt{2}$. The choice $y = -x$ gives $-2x^2 = 1$ which has no real solution. Hence $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i$ and $z = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i$.

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36. $x^2 - y^2 - 4x + (-2xy - 4y)i = 0 + 0i$ implies $x^2 - y^2 - 4x = 0$ and $y(-2x - 4) = 0$. If $y = 0$ then $x(x - 4) = 0$ and so $z = 0$ and $z = 4$. If $-2x - 4 = 0$ or $x = -2$ then $12 - y^2 = 0$ or $y = \pm 2\sqrt{3}$. This gives $z = -2 + 2\sqrt{3}\,i$ and $z = -2 - 2\sqrt{3}\,i$.

37. $|10 + 8i| = \sqrt{164}$ and $|11 - 6i| = \sqrt{157}$. Hence $11 - 6i$ is closer to the origin.

38. $|\frac{1}{2} - \frac{1}{4}\,i| = \frac{\sqrt{5}}{4}$ and $|\frac{2}{3} + \frac{1}{6}\,i| = \frac{\sqrt{17}}{6}$. Since $\frac{\sqrt{5}}{4} < \frac{\sqrt{17}}{6}$, $\frac{1}{2} - \frac{1}{4}\,i$ is closer to the origin.

39. $|z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ which is the distance formula in the plane.

40. By the triangle inequality, $|z + 6 + 8i| \le |z| + |6 + 8i|$. On the circle, $|z| = 2$ and so $|z + 6 + 8i| \le 2 + \sqrt{100} = 12$.

Exercises 17.2

1. $2(\cos 2\pi + i \sin 2\pi)$

2. $10(\cos \pi + i \sin \pi)$

3. $3\left(\cos \dfrac{3\pi}{2} + i \sin \dfrac{3\pi}{2}\right)$

4. $6\left(\cos \dfrac{\pi}{2} + i \sin \dfrac{\pi}{2}\right)$

5. $\sqrt{2}\left(\cos \dfrac{\pi}{4} + i \sin \dfrac{\pi}{4}\right)$

6. $5\sqrt{2}\left(\cos \dfrac{7\pi}{4} + i \sin \dfrac{7\pi}{4}\right)$

7. $2\left(\cos \dfrac{5\pi}{6} + i \sin \dfrac{5\pi}{6}\right)$

8. $4\left(\cos \dfrac{4\pi}{3} + i \sin \dfrac{4\pi}{3}\right)$

9. $\dfrac{3\sqrt{2}}{2}\left(\cos \dfrac{5\pi}{4} + i \sin \dfrac{5\pi}{4}\right)$

10. $6\left[\cos\left(-\dfrac{\pi}{6}\right) + i \sin\left(-\dfrac{\pi}{6}\right)\right]$

11. $z = -\dfrac{5\sqrt{3}}{2} - \dfrac{5}{2}\,i$

12. $z = -8 + 8i$

13. $z = 5.5433 + 2.2961i$

14. $z = 8.0902 + 5.8779i$

15. $z_1 z_2 = 8\left[\cos\left(\dfrac{\pi}{8} + \dfrac{3\pi}{8}\right) + i \sin\left(\dfrac{\pi}{8} + \dfrac{3\pi}{8}\right)\right] = 8i$; $\dfrac{z_1}{z_2} = \dfrac{1}{2}\left[\cos\left(\dfrac{\pi}{8} - \dfrac{3\pi}{8}\right) + i \sin\left(\dfrac{\pi}{8} - \dfrac{3\pi}{8}\right)\right] = \dfrac{\sqrt{2}}{4} - \dfrac{\sqrt{2}}{4}\,i$

16. $z_1 z_2 = \sqrt{6}\left[\cos\left(\dfrac{\pi}{4} + \dfrac{\pi}{12}\right) + i \sin\left(\dfrac{\pi}{4} + \dfrac{\pi}{12}\right)\right] = \dfrac{\sqrt{6}}{2} + \dfrac{3\sqrt{2}}{2}\,i$

$\dfrac{z_1}{z_2} = \dfrac{\sqrt{6}}{3}\left[\cos\left(\dfrac{\pi}{4} - \dfrac{\pi}{12}\right) + i \sin\left(\dfrac{\pi}{4} - \dfrac{\pi}{12}\right)\right] = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{6}}{6}\,i$

17. $\left[3\sqrt{2}\left(\cos \dfrac{7\pi}{4} + i \sin \dfrac{7\pi}{4}\right)\right]\left[10\left(\cos \dfrac{\pi}{3} + i \sin \dfrac{\pi}{3}\right)\right] = 30\sqrt{2}\left[\cos\left(\dfrac{7\pi}{4} + \dfrac{\pi}{3}\right) + i \sin\left(\dfrac{7\pi}{4} + \dfrac{\pi}{3}\right)\right]$

$= 40.9808 + 10.9808i$

18. $\left[4\sqrt{2}\left(\cos \dfrac{\pi}{4} + i \sin \dfrac{\pi}{4}\right)\right]\left[\sqrt{2}\left(\cos \dfrac{3\pi}{4} + i \sin \dfrac{3\pi}{4}\right)\right] = 8\left[\cos\left(\dfrac{\pi}{4} + \dfrac{3\pi}{4}\right) + i \sin\left(\dfrac{\pi}{4} + \dfrac{3\pi}{4}\right)\right] = -8$

19. $\dfrac{\cos \dfrac{3\pi}{2} + i \sin \dfrac{3\pi}{2}}{2\sqrt{2}\left[\cos \dfrac{7\pi}{4} + i \sin \dfrac{7\pi}{4}\right]} = \dfrac{\sqrt{2}}{4}\left[\cos\left(\dfrac{3\pi}{2} - \dfrac{7\pi}{4}\right) + i \sin\left(\dfrac{3\pi}{2} - \dfrac{7\pi}{4}\right)\right] = \dfrac{1}{4} - \dfrac{1}{4}\,i$

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20. $\dfrac{2\sqrt{2}\left[\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right]}{2\left[\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right]}=\sqrt{2}\left[\cos\left(\dfrac{\pi}{3}-\dfrac{2\pi}{3}\right)+i\sin\left(\dfrac{\pi}{3}-\dfrac{2\pi}{3}\right)\right]=\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{6}}{2}\,i$

21. $2^9\left[\cos\dfrac{9\pi}{3}+i\sin\dfrac{9\pi}{3}\right]=-512$

22. $(2\sqrt{2}\,)^5\left[\cos\left(-\dfrac{5\pi}{4}\right)+i\sin\left(-\dfrac{5\pi}{4}\right)\right]=-128+128i$

23. $\left(\dfrac{\sqrt{2}}{2}\right)^{10}\left[\cos\dfrac{10\pi}{4}+i\sin\dfrac{10\pi}{4}\right]=\dfrac{1}{32}\,i$

24. $(2\sqrt{2}\,)^4\left[\cos\dfrac{8\pi}{3}+i\sin\dfrac{8\pi}{3}\right]=-32+32\sqrt{3}\,i$

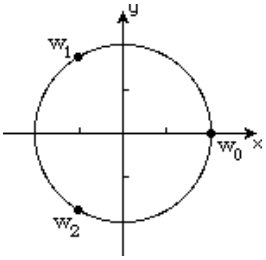
25. $\cos\dfrac{12\pi}{8}+i\sin\dfrac{12\pi}{8}=-i$

26. $(\sqrt{3}\,)^6\left[\cos\dfrac{12\pi}{9}+i\sin\dfrac{12\pi}{9}\right]=-\dfrac{27}{2}-\dfrac{27\sqrt{3}}{2}\,i$

27. $8^{1/3}=2\left[\cos\dfrac{2k\pi}{3}+i\sin\dfrac{2k\pi}{3}\right],\quad k=0,\,1,\,2$

$w_0=2[\cos 0+i\sin 0]=2;\quad w_1=2\left[\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right]=-1+\sqrt{3}\,i$

$w_2=2\left[\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}\right]=-1-\sqrt{3}\,i$



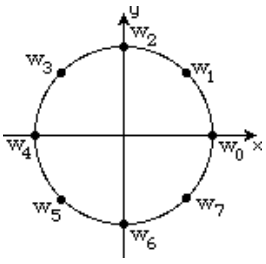
28. $(1)^{1/8}=\cos\dfrac{k\pi}{4}+i\sin\dfrac{k\pi}{4},\quad k=0,\,1,\,2,\,\ldots,\,7$

$w_0=\cos 0+i\sin 0=1;\quad w_1=\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}=\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}\,i$

$w_2=\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}=i;\quad w_3=\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}=-\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}\,i$

$w_4=\cos\pi+i\sin\pi=-1;\quad w_5=\cos\dfrac{5\pi}{4}+i\sin\dfrac{5\pi}{4}=-\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}\,i$

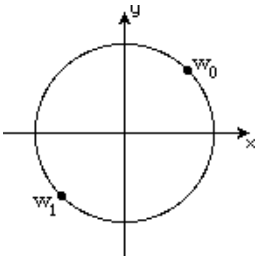
$w_6=\cos\dfrac{3\pi}{2}+i\sin\dfrac{3\pi}{2}=-i;\quad w_7=\cos\dfrac{7\pi}{4}+i\sin\dfrac{7\pi}{4}=\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}\,i$



29. $(i)^{1/2}=\cos\left(\dfrac{\pi}{4}+k\pi\right)+i\sin\left(\dfrac{\pi}{4}+k\pi\right),\quad k=0,\,1$

$w_0=\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}=\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}\,i$

$w_1=\cos\dfrac{5\pi}{4}+i\sin\dfrac{5\pi}{4}=-\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}\,i$



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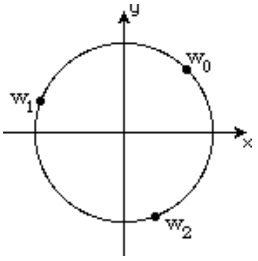
30. $(-1+i)^{1/3} = 2^{1/6}\left[\cos\left(\frac{\pi}{4}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{2k\pi}{3}\right)\right], \quad k = 0,\ 1,\ 2$

$w_0 = 2^{1/6}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right] = \frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{2}}\,i = 0.7937+0.7937i$

$w_1 = 2^{1/6}\left[\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right] = -1.0842+0.2905i$

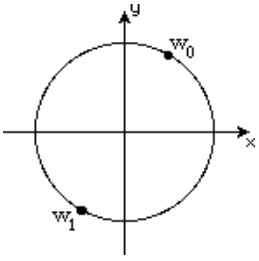
$w_2 = 2^{1/6}\left[\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right] = 0.2905-1.0842i$



31. $(-1+\sqrt{3}\,i)^{1/2} = 2^{1/2}\left[\cos\left(\frac{\pi}{3}+k\pi\right)+i\sin\left(\frac{\pi}{3}+k\pi\right)\right], \quad k = 0,\ 1$

$w_0 = 2^{1/2}\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right] = \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\,i$

$w_2 = 2^{1/2}\left[\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right] = -\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}\,i$

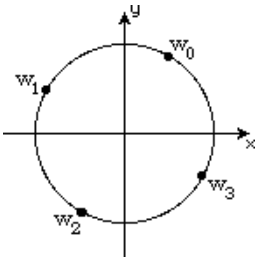


32. $(-1-\sqrt{3}\,i)^{1/4} = 2^{1/4}\left[\cos\left(\frac{\pi}{3}+\frac{k\pi}{2}\right)+i\sin\left(\frac{\pi}{3}+\frac{k\pi}{2}\right)\right], \quad k = 0,\ 1,\ 2,\ 3$

$w_0 = 2^{1/4}\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right] = 2^{1/4}\left[\frac{1}{2}+\frac{\sqrt{3}}{2}\,i\right]$

$w_1 = 2^{1/4}\left[\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right] = 2^{1/4}\left[-\frac{\sqrt{3}}{2}+\frac{1}{2}\,i\right]$

$w_2 = 2^{1/4}\left[\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right] = 2^{1/4}\left[-\frac{1}{2}-\frac{\sqrt{3}}{2}\,i\right]; \quad w_3 = 2^{1/4}\left[\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right] = 2^{1/4}\left[\frac{\sqrt{3}}{2}-\frac{1}{2}\,i\right]$



33. The solutions are the four fourth roots of -1;

$$w_k = \cos\frac{\pi+2k\pi}{4}+i\sin\frac{\pi+2k\pi}{4}, \quad k = 0,\ 1,\ 2,\ 3.$$

We have

$$w_1 = \cos\frac{\pi}{4}+i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\,i \qquad w_3 = \cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\,i$$

$$w_2 = \cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\,i \qquad w_4 = \cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\,i.$$

34. $(z^4-1)^2 = 0$ is the same as $(z-i)^2(z+i)^2(z-1)^2(z+1)^2 = 0$. Thus $z_1 = 1$, $z_2 = -1$, $z_3 = i$, and $z_4 = -i$ are roots of multiplicity two.

35. $\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right)^{12}\left[2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]^5 = 2^5\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

$$= 32\left[\cos\left(\frac{4\pi}{3}+\frac{5\pi}{6}\right)+i\sin\left(\frac{4\pi}{3}+\frac{5\pi}{6}\right)\right]$$

$$= 32\left(\cos\frac{13\pi}{6}+i\sin\frac{13\pi}{6}\right) = 32\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right) = 16\sqrt{3}+16i$$

36. $\dfrac{\left[8\left(\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}\right)\right]^3}{\left[2\left(\cos\frac{\pi}{16}+i\sin\frac{\pi}{16}\right)\right]^{10}} = \frac{2^9}{2^{10}}\left[\cos\left(\frac{9\pi}{8}-\frac{10\pi}{16}\right)+i\left(\frac{9\pi}{8}-\frac{10\pi}{16}\right)\right] = \frac{1}{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) = \frac{1}{2}\,i$

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37. We have

$$(\cos 2\theta + i \sin\theta)^2 = \cos 2\theta + i \sin 2\theta$$

Also

$$(\cos\theta + i \sin\theta)^2 = \cos^2\theta - \sin^2\theta + (2\sin\theta\cos\theta)i.$$

Equating real and imaginary parts gives

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

38. We have

$$(\cos\theta + i \sin\theta)^3 = \cos 3\theta + i \sin 3\theta.$$

Also

$$\begin{aligned}(\cos\theta + i \sin\theta)^3 &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + (3\cos^2\theta\sin\theta - \sin^3\theta)i.\end{aligned}$$

Equating real and imaginary parts gives

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta, \quad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

39. **(a)** $\text{Arg}(z_1) = \pi$, $\text{Arg}(z_2) = \dfrac{\pi}{2}$, $\text{Arg}(z_1 z_2) = -\dfrac{\pi}{2}$, $\text{Arg}(z_1) + \text{Arg}(z_2) = \dfrac{3\pi}{2} \neq \text{Arg}(z_1 z_2)$

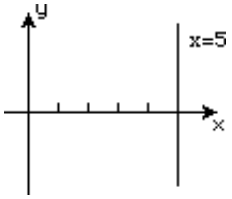
(b) $\text{Arg}(z_1/z_2) = -\dfrac{\pi}{2}$, $\text{Arg}(z_1) - \text{Arg}(z_2) = \pi - \dfrac{\pi}{2} = \dfrac{\pi}{2} \neq \text{Arg}(z_1/z_2)$

40. **(a)** If we take $\arg(z_1) = \pi$ and $\arg(z_2) = \pi/2$ then $\arg(z_1) + \arg(z_2) = 3\pi/2$ is an argument of the product $z_1 z_2 = -5i$. With these same arguments we see that $\arg(z_1) - \arg(z_2) = \pi/2$ is an argument of the quotient $z_1/z_2 = \frac{1}{5}\,i$.

(b) If we take $\arg(z_1) = \pi$ and $\arg(z_2) = -\pi/2$ then $\arg(z_1) + \arg(z_2) = \pi/2$ is an argument of the product $z_1 z_2 = 5i$. With these same arguments we see that $\arg(z_1) - \arg(z_2) = 3\pi/2$ is an argument of the quotient $z_1/z_2 = -\frac{1}{5}\,i$.

Exercises 17.3

1.



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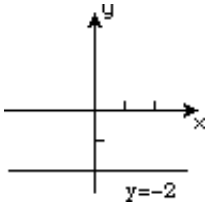

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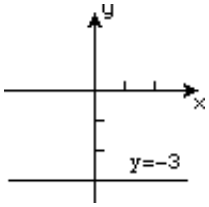

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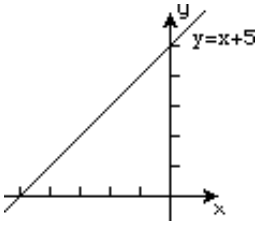

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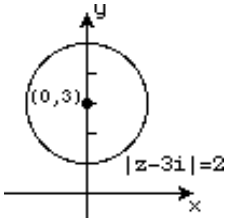

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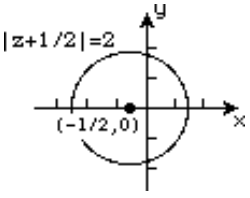

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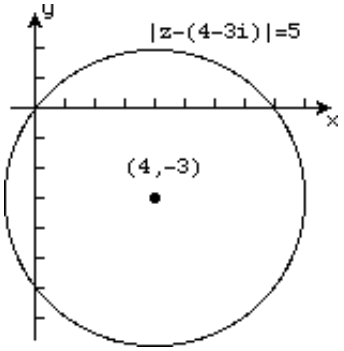

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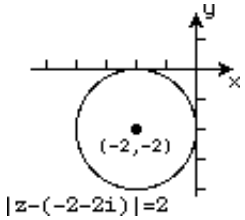

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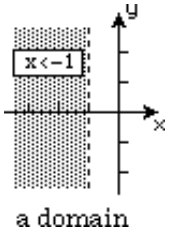

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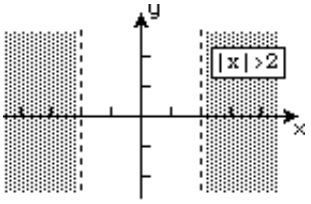

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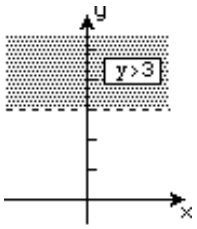

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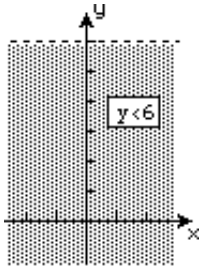

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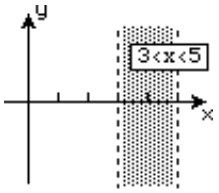

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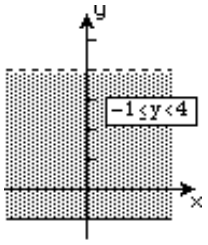

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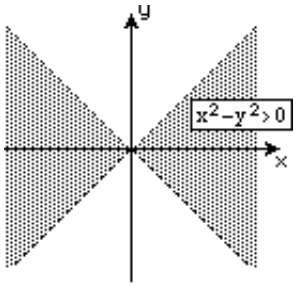

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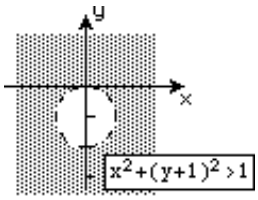

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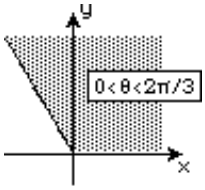

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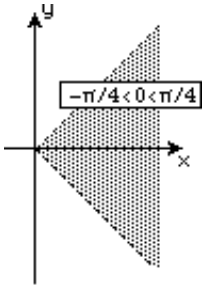

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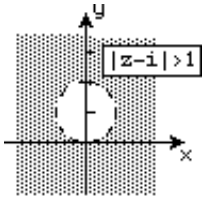

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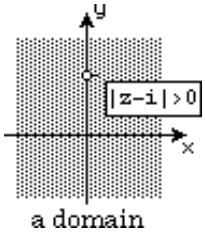

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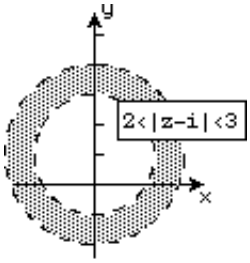

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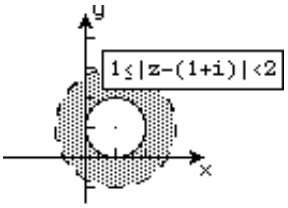

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23. The given equation is equivalent to $(x+1)^2 + y^2 = x^2 + (y-1)^2$. This simplifies to $y = -x$ which describes a straight line through the origin.

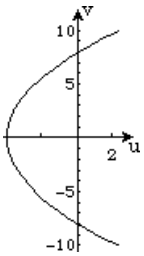
24. $|\text{Re}(z)| = |x|$ is the same as $\sqrt{x^2}$ and $|z| = \sqrt{x^2+y^2}$. Since $y^2 \geq 0$ the inequality $\sqrt{x^2} \leq \sqrt{x^2+y^2}$ is true for all complex numbers.

25. The given equation simplifies to the equation $x^2 - y^2 = 1$ which is a hyperbola with center at the origin.

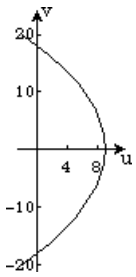
26. Since $|z-i|$ and $|z-(-i)|$ represent distances from the point (x, y) to i and $-i$, respectively, the equation is the distance formula definition of an ellipse with foci at $(0, 1)$ and $(0, -1)$.

Exercises 17.4

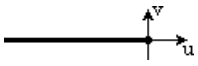
1. Substituting $y = 2$ into $u = x^2 - y^2$, $v = 2xy$ gives the parametric equations $u = x^2 - 4$, $v = 4x$. Using $x = v/4$ the first equation gives $u = v^2/16 - 4$. The graph is the parabola shown.



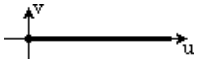
2. Substituting $x = -3$ into $u = x^2 - y^2$, $v = 2xy$ gives the parametric equations $u = 9 - y^2$, $v = -6y$. Using $y = -v/6$ the first equation gives $u = 9 - v^2/36$. The graph is the parabola shown.



3. $x = 0$ gives $u = -y^2$, $v = 0$. Since $-y^2 \leq 0$ for all real values of y, the image is the origin and the negative u-axis.



4. $y = 0$ gives $u = x^2$, $v = 0$. Since $x^2 \geq 0$ for all real values of x, the image is the origin and the positive u-axis.



5. $y = x$ gives $u = 0$, $v = 2x^2$. Since $x^2 \geq 0$ for all real values of x, the image is the origin and the positive v-axis.

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6. $y = -x$ gives $u = 0$, $v = -2x^2$. Since $-x^2 \le 0$ for all real values of x, the image is the origin and the negative v-axis.

7. $f(z) = (6x - 5) + i(6y + 9)$

8. $f(z) = (7x - 9y - 3) + i(7y - 9x + 2)$

9. $f(z) = (x^2 - y^2 - 3x) + i(2xy - 3y + 4)$

10. $f(z) = (3x^2 - 3y^2 + 2x) + i(-6xy + 2y)$

11. $f(z) = (x^3 - 3xy^2 - 4x) + i(3x^2y - y^3 - 4y)$

12. $f(z) = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)$

13. $f(z) = \left(x + \dfrac{x}{x^2 + y^2}\right) + i\left(y - \dfrac{y}{x^2 + y^2}\right)$

14. $f(z) = \dfrac{x^2 + y^2 + x}{(x + 1)^2 + y^2} + i\,\dfrac{y}{(x + 1)^2 + y^2}$

15. **(a)** $f(0 + 2i) = -4 + i$ **(b)** $f(2 - i) = 3 - 9i$ **(c)** $f(5 + 3i) = 1 + 86i$

16. **(a)** $f(1 + i) = 3 - 2i$ **(b)** $f(2 - i) = \dfrac{7}{2} + 10i$ **(c)** $f(1 + 4i) = 3 - 32i$

17. **(a)** $f(4 - 6i) = 14 - 20i$ **(b)** $f(-5 + 12i) = -13 + 43i$ **(c)** $f(2 - 7i) = 3 - 26i$

18. **(a)** $f(0 + \dfrac{\pi}{4}\,i) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\,i$ **(b)** $f(-1 - \pi i) = -e^{-1}$ **(c)** $f(3 + \dfrac{\pi}{3}\,i) = \dfrac{1}{2}e^3 + \dfrac{\sqrt{3}}{2}\,e^3 i$

19. $\lim\limits_{z \to i}(4z^3 - 5z^2 + 4z + 1 - 5i) = 6 - 5i$

20. $\lim\limits_{z \to 1-i} \dfrac{5z^2 - 2z + 2}{z + 1} = \dfrac{5(1 - i)^2 - 2(1 - i) + 2}{2 - i} = \dfrac{8}{5} - \dfrac{16}{5}\,i$

21. $\lim\limits_{z \to i} \dfrac{z^4 - 1}{z - i} = \lim\limits_{z \to i} \dfrac{(z^2 - 1)(z - i)(z + i)}{z - i} = -4i$

22. $\lim\limits_{z \to 1+i} \dfrac{z^2 - 2z + 2}{z^2 - 2i} = \lim\limits_{z \to 1+i} \dfrac{[z - (1 + i)][z - (1 - i)]}{[z - (1 + i)][z - (-1 - i)]} = \dfrac{1}{2} + \dfrac{1}{2}\,i$

23. Along the y-axis, $\lim\limits_{z \to 0} \dfrac{x + iy}{x - iy} = \lim\limits_{y \to 0} \dfrac{iy}{-iy} = -1$, whereas along the x-axis, $\lim\limits_{z \to 0} \dfrac{x + iy}{x - iy} = \lim\limits_{x \to 0} \dfrac{x}{x} = 1$.

24. Along the line $x = 1$, $\lim\limits_{z \to 1} \dfrac{x + y - 1}{z - 1} = \lim\limits_{y \to 0} \dfrac{y}{iy} = \dfrac{1}{i} = -i$, whereas along the x-axis, $\lim\limits_{z \to 1} \dfrac{x + y - 1}{z - 1} = \lim\limits_{x \to 1} \dfrac{x - 1}{x - 1} = 1$.

25. $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim\limits_{\Delta z \to 0} \dfrac{2z\Delta z + (\Delta z)^2}{\Delta z} = \lim\limits_{\Delta z \to 0}(2z + \Delta z) = 2z$

26. $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{\dfrac{1}{z + \Delta z} - \dfrac{1}{z}}{\Delta z} = \lim\limits_{\Delta z \to 0} \dfrac{-\Delta z}{(\Delta z)z(z + \Delta z)} = \lim\limits_{\Delta z \to 0} \dfrac{-1}{z(z + \Delta z)} = -\dfrac{1}{z^2}$

27. $f'(z) = 12z^2 - (6 + 2i)z - 5$

28. $f'(z) = 20z^3 - 3iz^2 + (16 - 2i)z$

29. $f'(z) = (2z + 1)(2z - 4) + 2(z^2 - 4z + 8i) = 6z^2 - 14z - 4 + 8i$

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30. $f'(z) = (z^5 + 3iz^3)(4z^2 + 3iz^2 + 4z - 6i) + (z^4 + iz^3 + 2z^2 - 6iz)(5z^4 + 9iz^2)$

31. $f'(z) = 6z(z^2 - 4i)^2$

32. $f'(z) = 6(2z - 1/z)^5(2 + 1/z^2)$

33. $f'(z) = \dfrac{(2z+i)3 - (3z - 4 + 8i)2}{(2z+1)^2} = \dfrac{8 - 13i}{(2z+i)^2}$

34. $f'(z) = \dfrac{(z^3+1)(10z-1) - (5z^2 - z)3z^2}{(z^3+1)^2} = \dfrac{-5z^4 + 2z^3 + 10z - 1}{(z^3+1)^2}$

35. $3i$ **36.** $0,\ 2 - 5i$ **37.** $-2i,\ 2i$ **38.** $3 - 4i,\ 3 + 4i$

39. We have $$\lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}.$$

If we let $\Delta z \to 0$ along a horizontal line then $\Delta z = \Delta x$, $\overline{\Delta z} = \Delta x$, and

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

If we let $\Delta z \to 0$ along a vertical line then $\Delta z = i\Delta y$, $\overline{\Delta z} = -i\Delta y$, and

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

Since these two limits are not equal, $f(z) = \overline{z}$ cannot be differentiable at any z.

40. We have $$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\overline{z} + \overline{\Delta z}) - z\overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \left(\overline{z} + z\,\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z} \right).$$

If $z = 0$, then the above limit becomes

$$f'(0) = \lim_{\Delta z \to 0} \overline{\Delta z} = 0.$$

If $z \neq 0$ then we first let $\Delta z \to 0$ along a horizontal line so that $\Delta z = \Delta x$ and $\overline{\Delta z} = \Delta x$. Thus,

$$f'(z) = \lim_{\Delta z \to 0} \left(\overline{z} + z\,\frac{\Delta x}{\Delta x} + \Delta x \right) = \overline{z} + z.$$

Next we let $\Delta z \to 0$ along a vertical line so that $\Delta z = i\Delta y$, $\overline{\Delta z} = -i\Delta y$. Thus

$$f'(z) = \lim_{\Delta y \to 0} \left(\overline{z} + z\,\frac{-i\Delta y}{i\Delta y} + i\Delta y \right) = \overline{z} - z.$$

We must have $\overline{z} + z = \overline{z} - z$ which implies $z = 0$. This is a contradiction to the assumption that $z \neq 0$. Hence $f(z) = |z|^2$ is differentiable only at $z = 0$.

41. Each linear equation in the system $\dfrac{dx}{dt} = 2x$, $\dfrac{dy}{dt} = 2y$ can be solved directly. We obtain $x(t) = c_1 e^{2t}$ and $y(t) = c_2 e^{2t}$.

42. The system $\dfrac{dx}{dt} = -y$, $\dfrac{dy}{dt} = x$ can be solved as in Section 3.11. We obtain $x(t) = c_1 \cos t + c_2 \sin t$, $y(t) = c_1 \sin t - c_2 \cos t$.

43. The equations in the system $\dfrac{dx}{dt} = \dfrac{x}{x^2 + y^2}$, $\dfrac{dy}{dt} = \dfrac{y}{x^2 + y^2}$ can be divided to give $\dfrac{dy}{dx} = \dfrac{y}{x}$. By separation of variables we obtain $y = cx$.

44. Each equation in the system $\dfrac{dx}{dt} = x^2$, $\dfrac{dy}{dt} = -y^2$ can be solved directly by separation of variables. We obtain $x(t) = \dfrac{-1}{t + c_1}$, $y = \dfrac{1}{t + c_2}$.

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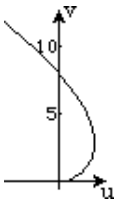
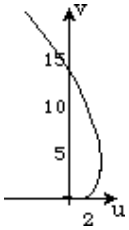
45. If $y = \frac{1}{2}x^2$ the equations $u = x^2 - y^2$, $v = 2xy$ give $u = x^2 - \frac{1}{4}x^4$, $v = x^3$. With the aid of a computer, the graph of these parametric equations is shown.



46. If $y = (x-1)^2$ the equations $u = x^2 - y^2$, $v = 2xy$ give $u = x^2 - (x-1)^4$, $v = 2x(x-1)^2$. With the aid of a computer the graph of these parametric equations is shown.



Exercises 17.5

1. $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$; $\dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -6xy = -\dfrac{\partial v}{\partial x}$

2. $u = 3x^2 - 3y^2 + 5x$, $v = 6xy + 5y - 6$; $\dfrac{\partial u}{\partial x} = 6x + 5 = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -6y = -\dfrac{\partial v}{\partial x}$

3. $u = x$, $v = 0$; $\dfrac{\partial u}{\partial x} = 1$, $\dfrac{\partial v}{\partial y} = 0$. Since $1 \neq 0$, f is not analytic at any point.

4. $u = y$, $v = x$; $\dfrac{\partial u}{\partial x} = 0 = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = 1$, $-\dfrac{\partial v}{\partial x} = -1$. Since $1 \neq -1$, f is not analytic at any point.

5. $u = -2x + 3$, $v = 10y$; $\dfrac{\partial u}{\partial x} = -2$, $\dfrac{\partial v}{\partial y} = 10$. Since $-2 \neq 10$, f is not analytic at any point.

6. $u = x^2 - y^2$, $v = -2xy$; $\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial v}{\partial y} = -2x$; $\dfrac{\partial u}{\partial y} = -2y$, $-\dfrac{\partial v}{\partial x} = 2y$

The Cauchy-Riemann equations hold only at $(0, 0)$. Since there is no neighborhood about $z = 0$ within which f is differentiable we conclude f is nowhere analytic.

7. $u = x^2 + y^2$, $v = 0$; $\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial v}{\partial y} = 0$; $\dfrac{\partial u}{\partial y} = 2y$, $-\dfrac{\partial v}{\partial x} = 0$

The Cauchy-Riemann equations hold only at $(0, 0)$. Since there is no neighborhood about $z = 0$ within which f is differentiable we conclude f is nowhere analytic.

8. $u = \dfrac{x}{x^2 + y^2}$, $v = \dfrac{y}{x^2 + y^2}$; $\dfrac{\partial u}{\partial x} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$, $\dfrac{\partial v}{\partial y} = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$; $\dfrac{\partial u}{\partial y} = -\dfrac{2xy}{(x^2 + y^2)^2} = \dfrac{\partial v}{\partial x}$

The Cauchy-Riemann equations hold only at $(0, 0)$. Since there is no neighborhood about $z = 0$ within which f is differentiable, we conclude f is nowhere analytic.

9. $u = e^x \cos y$, $v = e^x \sin y$; $\dfrac{\partial u}{\partial x} = e^x \cos y = \dfrac{\partial v}{\partial y}$; $\dfrac{\partial u}{\partial y} = -e^x \sin y = -\dfrac{\partial v}{\partial x}$. f is analytic for all z.

10. $u = x + \sin x \cosh y$, $v = y + \cos x \sinh y$; $\dfrac{\partial u}{\partial x} = 1 + \cos x \cosh y = \dfrac{\partial v}{\partial y}$; $\dfrac{\partial u}{\partial y} = \sin x \sinh y = -\dfrac{\partial v}{\partial x}$.

f is analytic for all z.

11. $u = e^{x^2 - y^2} \cos 2xy$, $v = e^{x^2 - y^2} \sin 2xy$; $\dfrac{\partial u}{\partial x} = -2ye^{x^2 - y^2} \sin 2xy + 2xe^{x^2 - y^2} \cos 2xy = \dfrac{\partial v}{\partial y}$;

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$\dfrac{\partial u}{\partial y} = -2xe^{x^2-y^2}\sin 2xy - 2ye^{x^2-y^2}\cos 2xy = -\dfrac{\partial v}{\partial x}$. f is analytic for all z.

12. $u = 4x^2 + 5x - 4y^2 + 9$, $v = 8xy + 5y - 1$; $\dfrac{\partial u}{\partial x} = 8x + 5 = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -8y = -\dfrac{\partial v}{\partial x}$. f is analytic for all z.

13. $u = \dfrac{x-1}{(x-1)^2+y^2}$, $v = -\dfrac{y}{(x-1)^2+y^2}$; $\dfrac{\partial u}{\partial x} = \dfrac{y^2-(x-1)^2}{[(x-1)^2+y^2]} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{2y(x-1)}{[(x-1)^2+y^2]^2} = -\dfrac{\partial v}{\partial x}$

f is analytic in any domain not containing $z = 1$.

14. $u = \dfrac{x^3+xy^2+x}{x^2+y^2}$, $v = \dfrac{x^2y+y^3-y}{x^2+y^2}$; $\dfrac{\partial u}{\partial x} = \dfrac{x^4+2x^2y^2-x^2+y^2+y^4}{(x^2+y^2)^2} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = \dfrac{-2xy}{(x^2+y^2)^2} = -\dfrac{\partial v}{\partial x}$

f is analytic in any domain not containing $z = 0$.

15. $\dfrac{\partial u}{\partial x} = 3 = b = \dfrac{\partial v}{\partial y}$; $\dfrac{\partial u}{\partial y} = -1 = -a = -\dfrac{\partial v}{\partial x}$. f is analytic for all z when $b = 3$, $a = 1$.

16. The Cauchy-Riemann equations yield the system

$$\begin{aligned} 2x + ay &= dx + 2y \\ ax + 2by &= -2cx - dy \end{aligned} \quad \text{or} \quad \begin{aligned} (2-d)x + (2-a)y &= 0 \\ (a+2c)x + (2b+d)y &= 0. \end{aligned}$$

The system holds for $z = x + iy$ whenever $2 - d = 0$, $2 - a = 0$, $a + 2c = 0$, and $2b + d = 0$. That is, f is analytic for all z when $a = 2$, $b = -1$, $c = -1$, and $d = 2$.

17. $u = x^2 + y^2$, $v = 2xy$; $\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial v}{\partial y} = 2x$; $\dfrac{\partial u}{\partial y} = 2y$, $-\dfrac{\partial v}{\partial x} = -2y$

u and v are continuous and have continuous first partial derivatives. The Cauchy-Riemann equations are satisfied for any x and for $y = 0$, that is, for points on the real axis. The function f is differentiable but not analytic along this axis; there is no neighborhood about any point $z = x$ within which f is differentiable.z

18. $u = 3x^2y^2$, $v = -6x^2y^2$; $\dfrac{\partial u}{\partial x} = 6xy^2$, $\dfrac{\partial v}{\partial y} = -12x^2y$; $\dfrac{\partial u}{\partial y} = 6x^2y$, $-\dfrac{\partial v}{\partial x} = 12xy^2$

u and v are continuous and have continuous first partial derivatives. The Cauchy-Riemann equations are satisfied whenever $6xy(y + 2x) = 0$ and $6xy(x - 2y) = 0$. The point satisfying $y + 2x = 0$ and $x - 2y = 0$ is $z = 0$. The points that satisfy $6xy = 0$ are the points along the y-axis ($x = 0$) or along the x-axis ($y = 0$). The function f is differentiable but not analytic on either axis; there is no neighborhood about any point $z = x$ or $z = iy$ within which f is differentiable.

19. $u = x^3 + 3xy^2 - x$, $v = y^3 + 3x^2y - y$; $\dfrac{\partial u}{\partial x} = 3x^2 + 3y^2 - 1$, $\dfrac{\partial v}{\partial y} = 3y^2 + 3x^2 - 1$; $\dfrac{\partial u}{\partial y} = 6xy$, $-\dfrac{\partial v}{\partial x} = -6xy$.

u and v are continuous and have continuous first partial derivatives. The Cauchy-Riemann equations are satisfied whenever $6xy = -6xy$ or $12xy = 0$. The points satisfying $12xy = 0$ are the points along the y-axis ($x = 0$) or along the x-axis ($y = 0$). The function f is differentiable but not analytic on either axis; there is no neighborhood about any point $z = x$ or $z = iy$ within which f is differentiable.

20. $u = x^2 - x + y$, $v = y^2 - 5y - x$; $\dfrac{\partial u}{\partial x} = 2x - 1$, $\dfrac{\partial v}{\partial y} = 2y - 5$; $\dfrac{\partial u}{\partial y} = 1$, $-\dfrac{\partial v}{\partial x} = 1$

u and v are continuous and have continuous first partial derivatives. The Cauchy-Riemann equations are satisfied whenever $2x - 1 = 2y - 5$ or for points on the line $y = x + 2$. The function f is differentiable but not analytic on this line; there is no neighborhood about any point $z = x + (x + 2)i$ within which f is differentiable.

21. Since f is entire,

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = e^x\cos y + ie^x\sin y = f(z).$$

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22. Since f is entire,

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = -2ye^{x^2-y^2}\sin 2xy + 2xe^{x^2-y^2}\cos 2xy + i(2ye^{x^2-y^2}\cos 2xy + 2xe^{x^2-y^2}\sin 2xy).$$

23. $\dfrac{\partial^2 u}{\partial x^2} = 0,\ \dfrac{\partial^2 u}{\partial y^2} = 0$ gives $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Thus u is harmonic. Now $\dfrac{\partial u}{\partial x} = 1 = \dfrac{\partial v}{\partial y}$ implies $v = y + h(x)$, $\dfrac{\partial u}{\partial y} = 0 = -\dfrac{\partial v}{\partial x}$ implies $0 = -h'(x)$, and so $h(x) = C$ (a constant.) Therefore $f(z) = x + i(y + C)$.

24. $\dfrac{\partial^2 u}{\partial x^2} = 0,\ \dfrac{\partial^2 u}{\partial y^2} = 0$ gives $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Thus u is harmonic. Now $\dfrac{\partial u}{\partial x} = 2 - 2y = \dfrac{\partial v}{\partial y}$ implies $v = 2y - y^2 + h(x)$, $\dfrac{\partial u}{\partial y} = -2x = -\dfrac{\partial v}{\partial x} = -h'(x)$ implies $h'(x) = 2x$ or $h(x) = x^2 + C$. Therefore $f(z) = 2x - 2xy + i(2y - y^2 + x^2 + C)$.

25. $\dfrac{\partial^2 u}{\partial x^2} = 2,\ \dfrac{\partial^2 u}{\partial y^2} = -2$ gives $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Thus u is harmonic. Now $\dfrac{\partial u}{\partial x} = 2x = \dfrac{\partial v}{\partial y}$ implies $v = 2xy + h(x)$, $\dfrac{\partial u}{\partial y} = -2y = -\dfrac{\partial v}{\partial x} = -2y - h'(x)$ implies $h'(x) = 0$ or $h(x) = C$. Therefore $f(z) = x^2 - y^2 + i(2xy + C)$.

26. $\dfrac{\partial^2 u}{\partial x^2} = -24xy,\ \dfrac{\partial^2 u}{\partial y^2} = 24xy$ gives $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Thus u is harmonic. Now $\dfrac{\partial u}{\partial x} = 4y^3 - 12x^2y + 1 = \dfrac{\partial v}{\partial y}$ implies $v = y^4 - 6x^2y^2 + y + h(x)$, $\dfrac{\partial u}{\partial y} = 12xy^2 - 4x^3 = -\dfrac{\partial v}{\partial x} = 12xy^2 - h'(x)$ implies $h'(x) = 4x^3$ or $h(x) = x^4 + C$. Therefore $f(z) = 4xy^3 - 4x^3y + x + i(y^4 - 6x^2y^2 + y + x^4 + C)$.

27. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{2y^2 - 2x^2}{(x^2+y^2)^2},\ \dfrac{\partial^2 u}{\partial y^2} = \dfrac{2x^2 - 2y^2}{(x^2+y^2)^2}$ gives $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Thus u is harmonic. Now $\dfrac{\partial u}{\partial x} = \dfrac{2x}{x^2+y^2} = \dfrac{\partial v}{\partial y}$ implies $v = 2\tan^{-1}\dfrac{y}{x} + h(x)$, $\dfrac{\partial u}{\partial y} = \dfrac{2y}{x^2+y^2} = -\dfrac{\partial v}{\partial x} = \dfrac{2y}{x^2+y^2} - h'(x)$ implies $h'(x) = 0$ or $h(x) = C$. Therefore $f(z) = \log_e(x^2+y^2) + i\left(\tan^{-1}\dfrac{y}{x} + C\right),\ z \neq 0$.

28. $\dfrac{\partial^2 u}{\partial x^2} = 2e^x\cos y + e^x(x\cos y - y\sin y),\ \dfrac{\partial^2 u}{\partial y^2} = e^x(-x\cos y + y\sin y - 2\cos y)$ gives $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Thus u is harmonic. Now $\dfrac{\partial u}{\partial x} = e^x\cos y + e^x(x\cos y - y\sin y) = \dfrac{\partial v}{\partial y}$. Integrating by parts with respect to y implies

$$v = e^x\sin y + e^x(x\sin y + y\cos y - \sin y) + h(x) = xe^x\sin y + ye^x\cos y + h(x),$$

and

$$\frac{\partial u}{\partial y} = -xe^x\sin y - ye^x\cos y - e^x\sin y = -\frac{\partial v}{\partial x} = -xe^x\sin y - e^x\sin y - ye^x\cos y + h'(x)$$

implies $h'(x) = 0$ or $h(x) = C$. Therefore

$$f(z) = e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y + C).$$

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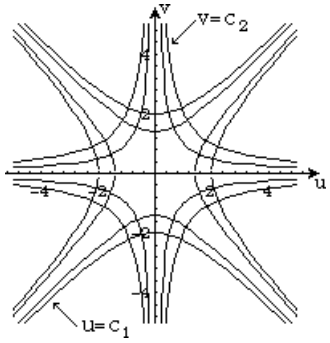
29. The level curves $u(x,y)=c_1$ and $v(x,y)=c_2$ are the families of hyperbolas $x^2-y^2=c_1$ and $2xy=c_2$, respectively. The graphs of these families are displayed on the same axes in the figure.



30. $f(x)=\dfrac{x}{x^2+y^2}-i\dfrac{y}{x^2+y^2}$. The level curves $u(x,y)=c_1$ and $v(x,y)=c_2$ are the family of circles $x=c_1(x^2+y^2)$ and $-y=c_2(x^2+y^2)$, with the exception that $(0,0)$ is not on the circumference of any circle.

31. $f(z)=x+\dfrac{x}{x^2+y^2}+i\left(y-\dfrac{y}{x^2+y^2}\right)$. The level curve $v(x,y)=0$ is described by $y-\dfrac{y}{x^2+y^2}=0$ or $y(x^2+y^2-1)=0$. We see that either $y=0$ or $x^2+y^2=1$. Thus $v(x,y)=0$ gives either the x-axis (without the origin $(0,0)$) or the unit circle $x^2+y^2=1$.

32. If $\nabla u=\dfrac{\partial u}{\partial x}\,\mathbf{i}+\dfrac{\partial u}{\partial y}\,\mathbf{j}$ and $\nabla v=\dfrac{\partial v}{\partial x}\,\mathbf{i}+\dfrac{\partial v}{\partial y}\,\mathbf{j}$, then $\nabla u\cdot\nabla v=\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial y}$. By the Cauchy-Riemann equations this becomes

$$\nabla u\cdot\nabla v=\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}+\left(-\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial y}=0.$$

Since the gradients of u and v are orthogonal vectors, the level curves $u(x,y)=c_1$ and $u(x,y)=c_2$ are orthogonal families.

Exercises 17.6

1. $e^{\frac{\pi}{6}i}=\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}=\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\,i$

2. $e^{-\frac{\pi}{3}i}=\cos\dfrac{\pi}{3}-i\sin\dfrac{\pi}{3}=\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}\,i$

3. $e^{-1+\frac{\pi}{4}i}=e^{-1}\cos\dfrac{\pi}{4}+ie^{-1}\sin\dfrac{\pi}{4}=e^{-1}\left(\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}\,i\right)$

4. $e^{2-\frac{\pi}{2}i}=e^2\cos\left(-\dfrac{\pi}{2}\right)+ie^2\sin\left(-\dfrac{\pi}{2}\right)=-e^2i$

5. $e^{\pi+\pi i}=e^{\pi}\cos\pi+ie^{\pi}\sin\pi=-e^{\pi}$

6. $e^{-\pi+\frac{3\pi}{2}i}=e^{-\pi}\cos\dfrac{3\pi}{2}+ie^{-\pi}\sin\dfrac{3\pi}{2}=-e^{-\pi}i$

7. $e^{1.5+2i}=e^{1.5}\cos 2+ie^{1.5}\sin 2=-1.8650+4.0752i$

8. $e^{-0.3+0.5i}=e^{-0.3}\cos 0.5+ie^{-0.3}\sin 0.5=0.6501+0.3552i$

9. $e^{5i}=\cos 5+i\sin 5=0.2837-0.9589i$

10. $e^{-0.23-i}=e^{-0.23}\cos(-1)+ie^{-0.23}\sin(-1)=0.4293-0.6686i$

11. $e^{\frac{11\pi}{12}i}=\cos\dfrac{11\pi}{12}+i\sin\dfrac{11\pi}{12}=-0.9659+0.2588i$

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12. $e^{5+\frac{5\pi}{2}i} = e^5\cos\dfrac{5\pi}{2} + ie^5\sin\dfrac{5\pi}{2} = e^5 i$

13. $e^{-iz} = e^{y-xi} = e^y\cos x - ie^y\sin x$

14. $e^{2\bar{z}} = e^{2x-2yi} = e^{2x}\cos 2y - ie^{2x}\sin 2y$

15. $e^{z^2} = e^{x^2-y^2+2xyi} = e^{x^2-y^2}\cos 2xy + ie^{x^2-y^2}\sin 2xy$

16. $e^{1/z} = e^{x/(x^2+y^2)-iy/(x^2+y^2)} = e^{x/(x^2+y^2)}\cos\dfrac{y}{x^2+y^2} - ie^{x/(x^2+y^2)}\sin\dfrac{y}{x^2+y^2}$

17. $|e^z|^2 = e^{2x}\cos^2 y + e^{2x}\sin^2 y = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x}$ implies $|e^z| = e^x$.

18. $\dfrac{e^{z_1}}{e^{z_2}} = \dfrac{e^{x_1}\cos y_1 + ie^{x_1}\sin y_1}{e^{x_2}\cos y_2 + ie^{x_2}\sin y_2} = \dfrac{(e^{x_1}\cos y_1 + ie^{x_1}\sin y_1)(e^{x_2}\cos y_2 - ie^{x_2}\sin y_2)}{e^{2x_2}}$

$= e^{x_1-x_2}[(\cos y_1\cos y_2 + \sin y_1\sin y_2) + i(\sin y_1\cos y_2 - \cos y_1\sin y_2)]$

$= e^{x_1-x_2}[\cos(y_1-y_2) + i\sin(y_1-y_2)] = e^{x_1-x_2+i(y_1-y_2)} = e^{(x_1+iy_1)-(x_2+iy_2)} = e^{z_1-z_2}$

19. $e^{z+\pi i} = e^{x+(y+\pi)i} = e^x[\cos(y+\pi) + i\sin(y+\pi)] = e^x[\cos(y-\pi) + i\sin(y-\pi)] = e^{x+(y-\pi)i} = e^{z-\pi i}$

20. $(e^z)^n = (e^x[\cos y + i\sin y])^n = e^{nx}[\cos y + i\sin y]^n = e^{nx}[\cos ny + i\sin ny] = e^{nz}$, n an integer

21. $u = e^x\cos y$, $v = -e^x\sin y$; $\dfrac{\partial u}{\partial x} = e^x\cos y$, $\dfrac{\partial v}{\partial y} = -e^x\cos y$; $\dfrac{\partial u}{\partial y} = -e^x\sin y$, $-\dfrac{\partial v}{\partial x} = e^x\sin y$

Since the Cauchy-Riemann equations are not satisfied at any point, f is nowhere analytic.

22. **(a)** $u = e^{x^2-y^2}\cos 2xy$, $v = e^{x^2-y^2}\sin 2xy$; $\dfrac{\partial u}{\partial x} = -2ye^{x^2-y^2}\sin 2xy + 2xe^{x^2-y^2}\cos 2xy = \dfrac{\partial v}{\partial y}$;

$\dfrac{\partial u}{\partial y} = -2xe^{x^2-y^2}\sin 2xy - 2ye^{x^2-y^2}\cos 2xy = -\dfrac{\partial v}{\partial x}$

Since u, v, and their first partial derivatives are continuous, and u and v satisfy the Cauchy-Riemann equations everywhere, the function f is differentiable everywhere. Hence f is entire.

(b) $\dfrac{\partial^2 u}{\partial x^2} = -4y^2e^{x^2-y^2}\cos 2xy - 4xye^{x^2-y^2}\sin 2xy - 4xye^{x^2-y^2}\sin 2xy + \cos 2xy[4x^2e^{x^2-y^2} + 2e^{x^2-y^2}]$;

$\dfrac{\partial^2 u}{\partial y^2} = -4x^2e^{x^2-y^2}\cos 2xy + 4xye^{x^2-y^2}\sin 2xy + 4xye^{x^2-y^2}\sin 2xy + \cos 2xy[4y^2e^{x^2-y^2} - 2e^{x^2-y^2}]$

Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ the function u is harmonic throughout the z-plane.

23. $\ln(-5) = \log_e 5 + i(\pi + 2n\pi) = 1.16094 + (\pi + 2n\pi)i$

24. $\ln(-ei) = \log_e e + i\left(-\dfrac{\pi}{2} + 2n\pi\right) = 1 + \left(-\dfrac{\pi}{2} + 2n\pi\right)i$

25. $\ln(-2+2i) = \log_e 2\sqrt{2} + i\left(\dfrac{3\pi}{4} + 2n\pi\right) = 1.0397 + \left(\dfrac{3\pi}{4} + 2n\pi\right)i$

26. $\ln(1+i) = \log_e\sqrt{2} + i\left(\dfrac{\pi}{4} + 2n\pi\right) = 0.3466 + \left(\dfrac{\pi}{4} + 2n\pi\right)i$

27. $\ln(\sqrt{2} + \sqrt{6}\,i) = \log_e 2\sqrt{2} + i\left(\dfrac{\pi}{3} + 2n\pi\right) = 1.0397 + \left(\dfrac{\pi}{3} + 2n\pi\right)i$

28. $\ln(-\sqrt{3} + i) = \log_e 2 + i\left(\dfrac{5\pi}{6} + 2n\pi\right) = 0.6932 + \left(\dfrac{5\pi}{6} + 2n\pi\right)i$

29. $\mathrm{Ln}(6-6i) = \log_e 6\sqrt{2} + i\left(-\dfrac{\pi}{4}\right) = 2.1383 - \dfrac{\pi}{4}i$

30. $\mathrm{Ln}(-e^3) = \log_e e^3 + \pi i = 3 + \pi i$

31. $\mathrm{Ln}(-12+5i) = \log_e 13 + i\left(\tan^{-1}\left(-\dfrac{5}{12}\right) + \pi\right) = 2.5649 + 2.7468i$

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32. $\text{Ln}(3-4i) = \log_e 5 + i\tan^{-1}\left(-\dfrac{4}{3}\right) = 1.6094 - 0.9273i$

33. $\text{Ln}(1+\sqrt{3}\,i)^5 = \text{Ln}(16-16\sqrt{3}\,i) = \log_e 32 - \dfrac{\pi}{3}\,i = 3.4657 - \dfrac{\pi}{3}\,i$

34. $\text{Ln}(1+i)^4 = \text{Ln}(-4) = \log_e 4 + \pi i = 1.3863 + \pi i$

35. $z = \ln(4i) = \log_e 4 + i\left(\dfrac{\pi}{2} + 2n\pi\right) = 1.3863 + \left(\dfrac{\pi}{2} + 2n\pi\right)i$

36. $\dfrac{1}{z} = \ln(-1) = \log_e 1 + i(\pi + 2n\pi) = (2n+1)\pi i$ and so $z = -\dfrac{i}{(2n+1)\pi}$.

37. $z - 1 = \ln(-ie^2) = \log_e e^2 + i\left(\dfrac{3\pi}{2} + 2n\pi\right) = 2 + \left(\dfrac{3\pi}{2} + 2n\pi\right)i$ and so $z = 3\left(\dfrac{3\pi}{2} + 2n\pi\right)i$.

38. By the quadratic formula, $e^z = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\,i$ or $e^z = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\,i$. Hence

$$z = \ln\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right) = \left(\frac{2\pi}{3} + 2n\pi\right)i \quad \text{or} \quad z = \ln\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right) = \left(\frac{4\pi}{3} + 2n\pi\right)i.$$

39. $(-i)^{4i} = e^{4i\ln(-i)} = e^{4i[\log_e 1 + i(-\frac{\pi}{2} + 2n\pi)]} = e^{(2-8n)\pi}$

40. $3^{1/\pi} = e^{\frac{i}{\pi}\ln 3} = e^{\frac{i}{\pi}[\log_e 3 + 2n\pi i]} = e^{-2n}\left[\cos\left(\dfrac{1}{\pi}\log_e 3\right) + i\sin\left(\dfrac{1}{\pi}\log_e 3\right)\right] = e^{-2n}[0.9395 + 0.3426i]$

41. $(1+i)^{(1+i)} = e^{(1+i)\ln(1+i)} = e^{(1+i)[\log_e\sqrt{2} + i(\frac{\pi}{4} + 2n\pi)]}$

$$= e^{\log_e\sqrt{2} - (\frac{\pi}{4} + 2n\pi)}\left[\cos\left(\frac{\pi}{4} + \log_e\sqrt{2}\right) + i\sin\left(\frac{\pi}{4} + \log_e\sqrt{2}\right)\right] = e^{-2n\pi}[0.2740 + 0.5837i]$$

42. $(1-i)^{2i} = e^{2i\ln(1-i)} = e^{2i[\log_e\sqrt{2} + i(-\frac{\pi}{4} + 2n\pi)]} = e^{\frac{\pi}{2} - 4n\pi}[\cos(\log_e 2) + i\sin(\log_e 2)] = e^{-4n\pi}[3.7004 + 3.0737i]$

43. $(-1)^{-\frac{2i}{\pi}} = e^{-\frac{2i}{\pi}\text{Ln}(-1)} = e^{-\frac{2i}{\pi}(\pi i)} = e^2 = 7.3891$

44. $(1-i)^{2i} = e^{2i\text{Ln}(1-i)} = e^{2i[\log_e\sqrt{2} - \frac{\pi}{4}i]} = e^{\frac{\pi}{2}}[\cos(\log_e 2) + i\sin(\log_e 2)] = 3.7004 + 3.0737i$

45. If $z_1 = i$ and $z_2 = -1 + i$ then

$$\text{Ln}(z_1 z_2) = \text{Ln}(-1-i) = \log_e\sqrt{2} - \frac{3\pi}{4}\,i,$$

whereas

$$\text{Ln}z_1 + \text{Ln}z_2 = \frac{\pi}{2}\,i + \left(\log_e\sqrt{2} + \frac{3\pi}{4}\,i\right) = \log_e\sqrt{2} + \frac{5\pi}{4}\,i.$$

46. If $z_1 = -i$ and $z_2 = i$ then

$$\text{Ln}(z_1/z_2) = \text{Ln}(-1) = \pi i, \quad \text{whereas} \quad \text{Ln}z_1 - \text{Ln}z_2 = -\frac{\pi}{2}\,i - \frac{\pi}{2}\,i = -\pi i.$$

47. **(a)** The statement is false.

$$\text{Ln}(-1+i)^2 = \text{Ln}(-2i) = \log_e 2 - \frac{\pi}{2}\,i, \quad \text{whereas} \quad 2\text{Ln}(-1+i) = 2\left(\log_e\sqrt{2} + \frac{3\pi}{4}\,i\right) = \log_e 2 + \frac{3\pi}{2}\,i.$$

(b) The statement is false.

$$\text{Ln}i^3 = \text{Ln}(-i) = -\frac{\pi}{2}\,i, \quad \text{whereas} \quad 3\text{Ln}i = \frac{3\pi}{2}\,i.$$

(c) The statement is true. If we take $\arg(-i) = \dfrac{3\pi}{2}$ then $\ln i^3 = \ln(-i) = \dfrac{3\pi}{2}\,i$ for $n = 0$. Also, $3\ln i = 3\left(\dfrac{\pi}{2}\,i\right)$.

48. **(a)** $(i^i)^2 = (e^{i\ln i})^2 = [e^{-(\frac{\pi}{2} + 2n\pi)}]^2 = e^{-(\pi + 4n\pi)}$ and $i^{2i} = e^{2i\ln i} = e^{-(\pi + 4n\pi)}$

(b) $(i^2)^i = (-1)^i = e^{i\ln(-1)} = e^{-(\pi + 2n\pi)}$, whereas $i^{2i} = e^{-(\pi + 4n\pi)}$

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49. Since $|z| = \sqrt{x^2+y^2}$ and $\text{Arg}z = \tan^{-1}\dfrac{y}{x}$ for $x > 0$ we have

$$\text{Ln}z = \log_e|z| + i\text{Arg}z = \log_e(x^2+y^2)^{1/2} + i\tan^{-1}\frac{y}{x} = \frac{1}{2}\log_e(x^2+y^2) + i\tan^{-1}\frac{y}{x}.$$

50. **(a)** $u = \log_e(x^2+y^2)$; $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{2(y^2-x^2)}{(x^2+y^2)^2}$, $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{2(x^2-y^2)}{(x^2+y^2)^2}$

Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ the function u is harmonic in any domain not containing the point $(0,0)$.

(b) $v = \tan^{-1}\dfrac{y}{x}$; $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{2xy}{(x^2+y^2)^2}$, $\dfrac{\partial^2 v}{\partial y^2} = \dfrac{-2xy}{(x^2+y^2)}$

Since $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ the function v is harmonic in any domain not containing the point $(0,0)$.

Exercises 17.7

1. $\cos(3i) = \cosh 3 = 10.0677$

2. $\sin(-2i) = i\sinh(-2) = -3.6269i$

3. $\sin\left(\dfrac{\pi}{4}+i\right) = \sin\dfrac{\pi}{4}\cosh(1) + i\cos\dfrac{\pi}{4}\sinh(1) = 1.0911 + 0.8310i$

4. $\cos(2-4i) = \cos(2)\cosh(-4) - \sin(2)\sinh(-4) = -11.3642 - 24.8147i$

5. $\tan(i) = \dfrac{\sin(i)}{\cos(i)} = \dfrac{i\sinh(1)}{\cosh(1)} = 0.7616i$

6. $\cot\left(\dfrac{\pi}{2}+3i\right) = \dfrac{\cos(\frac{\pi}{2}+3i)}{\sin(\frac{\pi}{2}+3i)} = \dfrac{-i\sinh(3)}{\cosh(3)} = -0.9951i$

7. $\sec(\pi+i) = \dfrac{1}{\cos(\pi+i)} = \dfrac{1}{-\cosh(1)} = -0.6481$

8. $\csc(1+i) = \dfrac{1}{\sin(1+i)} = \dfrac{1}{\sin(1)\cosh(1) + i\cos(1)\sinh(1)} = 0.6215 - 0.3039i$

9. $\cosh(\pi i) = \cos(i(\pi i)) = \cos(-\pi) = \cos\pi = -1$

10. $\sinh\left(\dfrac{3\pi}{2}i\right) = -i\sin\left(i\left(\dfrac{3\pi}{2}i\right)\right) = -i\sin\left(-\dfrac{3\pi}{2}\right) = i\sin\dfrac{3\pi}{2} = -i$

11. $\sinh\left(1+\dfrac{\pi}{3}i\right) = \sinh(1)\cos\dfrac{\pi}{3} + i\cosh(1)\sin\dfrac{\pi}{3} = 0.5876 + 1.3363i$

12. $\cosh(2+3i) = \cosh(2)\cos(3) + i\sinh(2)\sin(3) = -3.7245 + 0.5118i$

13. $\sin\left(\dfrac{\pi}{2}+i\ln 2\right) = \sin\dfrac{\pi}{2}\cosh(\ln 2) + i\cos\dfrac{\pi}{2}\sinh(\ln 2) = \dfrac{e^{\ln 2} + e^{\ln 2^{-1}}}{2} = \dfrac{2+\frac{1}{2}}{2} = \dfrac{5}{4}$

14. $\cos\left(\dfrac{\pi}{2}+i\ln 2\right) = \cos\dfrac{\pi}{2}\cosh(\ln 2) - i\sin\dfrac{\pi}{2}\sinh(\ln 2) = -i\cdot\dfrac{e^{\ln 2} - e^{\ln 2^{-1}}}{2} = -i\cdot\dfrac{2-\frac{1}{2}}{2} = -\dfrac{3}{4}i$

15. $\dfrac{e^{iz}-e^{-iz}}{2i} = 2$ gives $e^{2(iz)} - 4ie^{iz} - 1 = 0$. By the quadratic formula, $e^{iz} = 2i \pm \sqrt{3}\,i$ and so

$$iz = \ln[(2\pm\sqrt{3}\,)i]$$

$$z = -i\left[\log_e(2\pm\sqrt{3}\,) + \left(\frac{\pi}{2}+2n\pi\right)i\right] = \frac{\pi}{2} + 2n\pi - i\log_e(2\pm\sqrt{3}\,), \quad n = 0,\ \pm 1.,\ \pm 2,\ \ldots.$$

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16. $\dfrac{e^{iz}+e^{-iz}}{2} = -3i$ gives $e^{2(iz)} + 6ie^{iz} + 1 = 0$. By the quadratic formula, $e^{iz} = -3i \pm \sqrt{10}\,i$ and so $iz = \ln[-3 \pm \sqrt{10}\,)i]$. Hence

$$z = -i\left[\log_e(\sqrt{10}-3) + \left(\frac{\pi}{2}+2n\pi\right)i\right] \qquad \text{or} \qquad z = -i\left[\log_e(\sqrt{10}+3) + \left(\frac{3\pi}{2}+2n\pi\right)i\right]$$

$$z = \frac{\pi}{2} + 2n\pi - i\log_e(\sqrt{10}-3) \qquad \text{or} \qquad z = \frac{3\pi}{2} + 2n\pi - i\log_e(\sqrt{10}+3)$$

$n = 0, \pm1, \pm2, \ldots$.

17. $\dfrac{e^z - e^{-z}}{2} = i$ gives $e^{2z} - 2ie^z - 1 = 0$. By the quadratic formula, $e^z = -i$ and so

$$z = \ln(-i) = \log_e 1 + \left(-\frac{\pi}{2} + 2n\pi\right)i = \left(-\frac{\pi}{2} + 2n\pi\right)i, \quad n = 0,\ \pm1,\ \pm2,\ \ldots\ .$$

18. $\dfrac{e^z - e^{-z}}{2} = -1$ gives $e^{2z} + 2e^z - 1 = 0$. By the quadratic formula, $e^z = -1 \pm \sqrt{2}$, and so

$$z = \ln(-1 \pm \sqrt{2}\,)$$

$$z = \log_e(\sqrt{2}-1) + 2n\pi i \quad \text{or} \quad z = \log_e(\sqrt{2}+1) + (\pi + 2n\pi)i,$$

$n = 0, \pm1, \pm2, \ldots$.

19. $\cos z = \sin z$ gives $\tan z = 1$. One solution is $z = \dfrac{\pi}{4}$. Since $\tan z$ is π-periodic, $z = \dfrac{\pi}{4} + n\pi$, $n = 0, \pm1, \pm2, \ldots$ are also solutions. That these are the only solutions can be proved by solving

$$\frac{e^{iz}+e^{-iz}}{2} = \frac{e^{iz}-e^{-iz}}{2i}$$

by the method illustrated in Problems 15-18.

20. $\cos z = i\sin z$ gives $e^{iz} + e^{-iz} = e^{iz} - e^{-iz}$ or $e^{-iz} = 0$. Since this last equation has no solutions, the original equation has no solutions.

21. $\cos z = \cosh 2$ implies $\cos x\cosh y - i\sin x\sinh y = \cosh 2 + 0i$ and so we must have $\cos x\cosh y = \cosh 2$ and $\sin x\sinh y = 0$. The last equation has solutions $x = n\pi$, $n = 0, \pm1, \pm2, \ldots$, or $y = 0$. For $y = 0$ the first equation becomes $\cos x = \cosh 2$. Since $\cosh 2 > 1$ this equation has no solutions. For $x = n\pi$ the first equation becomes $(-1)^n\cosh y = \cosh 2$. Since $\cosh y > 0$ we see n must be even, say, $n = 2k$, $k = 0, \pm1, \pm2, \ldots$. Now $\cosh y = \cosh 2$ implies $y = \pm 2$. Solutions of the original equation are then

$$z = 2k\pi \pm 2i, \quad k = 0,\ \pm1,\ \pm2,\ \ldots\ .$$

22. $\sin z = i\sinh 2$ implies $\sin x\cosh y + i\cos x\sinh y = 0 + i\sinh 2$ and so we must have $\sin x\cosh y = 0$ and $\cos x\sinh y = \sinh 2$. Since $\cosh y > 0$ for all real numbers, the first equation has only the solutions $x = n\pi$, $n = 0, \pm1, \pm2, \ldots$. For $x = n\pi$ the second equation becomes $(-1)^n\sinh y = \sinh 2$. If n is even, $\sinh y = \sinh 2$ implies $y = 2$ ($\sinh y$ is one-to-one.) If n is odd, $-\sinh y = \sinh 2$ implies $\sinh y = -\sinh(-2)$ and so $y = -2$. Solutions of the original equation are then

$$z = 2k\pi + 2i, \quad z = (2k+1)\pi - 2i, \quad k = 0,\ \pm1,\ \pm2,\ \ldots\ .$$

23. $\cos z = \dfrac{e^{i(x+iy)} + e^{-i(x+i)}}{2} = \dfrac{1}{2}(e^{-y}e^{ix} + e^{y}e^{-ix}) = \dfrac{1}{2}[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)]$

$$= \cos x\left(\frac{e^y + e^{-y}}{2}\right) - i\sin x\left(\frac{e^y - e^{-y}}{2}\right) = \cos x\cosh y - i\sin x\sinh y$$

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24. $\sinh z = \dfrac{e^{x+iy} - e^{-x-iy}}{2} = \dfrac{1}{2}(e^x e^{iy} - e^{-x}e^{-iy}) = \dfrac{1}{2}[e^x(\cos y + i\sin y) - e^{-x}(\cos y - i\sin y)]$

$= \left(\dfrac{e^x - e^{-x}}{2}\right)\cos y + i\left(\dfrac{e^x + e^{-x}}{2}\right)\sin y = \sinh x\cos y + i\cosh x\sin y$

25. $\cosh z = \dfrac{e^{x+iy} + e^{-x-iy}}{2} = \dfrac{1}{2}(e^x e^{iy} + e^{-x}e^{-iy}) = \dfrac{1}{2}[e^x(\cos y + i\sin y) + e^{-x}(\cos y - i\sin y)]$

$= \left(\dfrac{e^x + e^{-x}}{2}\right)\cos y + i\left(\dfrac{e^x - e^{-x}}{2}\right)\sin y = \cosh x\cos y + i\sinh x\sin y$

26. $|\sinh z|^2 = \sinh^2 x\cos^2 y + \cosh^2 x\sin^2 y = \sinh^2 x\cos^2 y + (1+\sinh^2 x)\sin^2 y$

$= \sinh^2 x(\cos^2 y + \sin^2 y) + \sin^2 y = \sinh^2 x + \sin^2 y$

27. $|\cosh z|^2 = \cosh^2 x\cos^2 y + \sinh^2 x\sin^2 y = (1+\sinh^2 x)\cos^2 y + \sinh^2 x\sin^2 y$

$= \cos^2 y + \sinh^2 x(\cos^2 y + \sin^2 y) = \cos^2 y + \sinh^2 x$

28. $\cos^2 z + \sin^2 z = \left(\dfrac{e^{1z} + e^{-iz}}{2}\right)^2 + \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2 = \dfrac{1}{4}[e^{2iz} + 2 + e^{-2iz} - (e^{2iz} - 2 + e^{-2iz})] = \dfrac{4}{4} = 1$

29. $\cosh^2 z - \sinh^2 z = \left(\dfrac{e^z + e^{-z}}{2}\right) - \left(\dfrac{e^z - e^{-z}}{2}\right)^2 = \dfrac{1}{4}[e^{2z} + 2 + e^{-2z} - (e^{2z} - 2 + e^{-2z})] = \dfrac{4}{4} = 1$

30. $\tan z = \dfrac{\sin z}{\cos z} = \dfrac{\sin z\cos z}{|\cos z|^2} = \dfrac{[\sin x\cosh y + i\cos x\sinh y][\cos x\cosh y + i\sin x\sinh y]}{\cos^2 x + \sinh^2 y}$

$= \dfrac{(\sin x\cos x\cosh^2 y - \sin x\cos x\sinh^2 y)}{\cos^2 x + \sinh^2 y} + i\,\dfrac{\cos^2 x\sinh y\cosh y + \sin^2 x\sinh y\cosh y}{\cos^2 x + \sinh^2 y}$

$= \dfrac{\sin x\cos x(\cosh^2 y - \sinh^2 y)}{\cos^2 x + \sinh^2 y} + i\,\dfrac{\sin y\cosh y(\cos^2 x + \sin^2 x)}{\cos^2 x + \sinh^2 y}$

$= \dfrac{\sin x\cos x}{\cos^2 x + \sinh^2 y} + i\,\dfrac{\sinh y\cosh y}{\cos^2 x + \sinh^2 y} = \dfrac{\sin 2x}{2(\cos^2 x + \sinh^2 y)} + i\,\dfrac{\sinh 2y}{2(\cos^2 x + \sinh^2 y)}$

But

$$2\cos^2 x + 2\sinh^2 y = (2\cos^2 x - 1) + (2\sinh^2 y + 1) = \cos 2x + \cosh 2y.$$

Therefore $\tan z = u + iz$ where

$$u = \frac{\sin 2x}{\cos 2x + \cosh 2y}, \qquad v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}.$$

31. $\tanh(z + \pi i) = \dfrac{\sinh(x + (y+\pi)i)}{\cosh(x + (y+\pi)i)} = \dfrac{\sinh x\cos(y+\pi) + i\cosh x\sin(y+\pi)}{\cosh x\cos(y+\pi) + i\sinh x\sin(y+\pi)}$

$= \dfrac{-[\sinh x\cos y + i\cosh x\sinh y]}{-[\cosh x\cos y + i\sinh x\sin y]} = \dfrac{-\sinh z}{-\cosh z} = \tanh z$

32. **(a)** $\overline{\sin z} = \sin x\cosh y - i\cos x\sinh y = \sin x\cosh(-y) + i\cos x\sinh(-y) = \sin(x - iy) = \sin\bar{z}$

(b) $\overline{\cos z} = \cos x\cosh y + i\sin x\sinh y = \cos x\cosh(-y) - i\sin x\sinh(-y) = \cos(x - iy) = \cos\bar{z}$

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Exercises 17.8

1. $\sin^{-1}(-i) = -i\ln(1 \pm \sqrt{2}\,) = \begin{cases} 2n\pi - i\log_e(1+\sqrt{2}\,) \\ (2n+1)\pi - i\log_e(\sqrt{2}-1) \end{cases}$

Since $\sqrt{2}-1 = 1/(\sqrt{2}+1)$ we can have $\sin^{-1}(-i) = \begin{cases} 2n\pi - i\log_e(1+\sqrt{2}\,) \\ (2n+1)\pi + i\log_e(1+\sqrt{2}). \end{cases}$

This can be written compactly as

$$\sin^{-1}(-i) = n\pi + (-1)^{n+1} i\log_e(1+\sqrt{2}\,), \quad k = 0,\ \pm1,\ \pm2,\ \ldots\ .$$

2. $\sin^{-1}\sqrt{2} = -i\ln[\sqrt{2} \pm 1)i] = 2n\pi + \frac{\pi}{2} - i\log_e(\sqrt{2}\pm 1) = 2n\pi + \frac{\pi}{2} \pm i\log_e(1+\sqrt{2}\,),\ \ n = 0, \pm1, \pm2, \ldots$

3. $\sin^{-1} 0 = -i\ln(\pm 1) = \begin{cases} 2n\pi + i\log_e 1 \\ (2n+1)\pi + i\log_e 1 \end{cases} = \begin{cases} 2n\pi \\ (2n+1)\pi \end{cases} = n\pi, \quad n = 0, \pm1, \pm2, \ldots$

4. $\sin^{-1}\dfrac{13}{5} = -i\ln\left[\left(\dfrac{13}{5} \pm \dfrac{12}{5}\right) i\right] = \begin{cases} 2n\pi + \frac{\pi}{2} - i\log_e 5 \\ 2n\pi + \frac{\pi}{2} - i\log_e \frac{1}{5} \end{cases} = 2n\pi + \dfrac{\pi}{2} \pm i\log_e 5, \quad n = 0, \pm1, \pm2, \ldots$

5. $\cos^{-1} 2 = -i\ln(2 \pm \sqrt{3}\,) = \begin{cases} 2n\pi - i\log_e(2+\sqrt{3}\,) \\ 2n\pi - i\log_e(2-\sqrt{3}\,) \end{cases}$

Since $2 - \sqrt{3} = 1/(2+\sqrt{3}\,)$ this can be written compactly as

$$\cos^{-1} 2 = 2n\pi \pm i\log_e(2+\sqrt{3}\,), \quad k = 0,\ \pm1,\ \pm2,\ \ldots\ .$$

6. $\cos^{-1} 2i = -i\ln[(2 \pm \sqrt{5}\,)i] = \begin{cases} 2n\pi - \frac{\pi}{2} + i\log_e(2+\sqrt{5}\,) \\ 2n\pi + \frac{\pi}{2} - i\log_e(2+\sqrt{5}\,) \end{cases}, \quad n = 0, \pm1, \pm2, \ldots$

7. $\cos^{-1}\dfrac{1}{2} = -i\ln\left(\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}\, i\right) = \begin{cases} 2n\pi + \frac{\pi}{3} - i\log_e 1 \\ 2n\pi + \frac{\pi}{3} - i\log_e 1 \end{cases} = 2n\pi \pm \dfrac{\pi}{2}, \quad n = 0, \pm1, \pm2, \ldots$

8. $\cos^{-1}\dfrac{5}{3} = -i\ln\left(\dfrac{5}{3} \pm \dfrac{4}{3}\right) = \begin{cases} 2n\pi - i\log_e 3 \\ 2n\pi - i\log_e \frac{1}{3} \end{cases} = 2n\pi \pm i\log_e 3, \quad n = 0, \pm1, \pm2, \ldots$

9. $\tan^{-1} 1 = \dfrac{i}{2}\ln\dfrac{i+1}{i-1} = \dfrac{i}{2}\ln(-i) = -n\pi + \dfrac{\pi}{4} + \dfrac{i}{2}\log_e 1 = \dfrac{\pi}{4} - n\pi, \quad n = 0, \pm1, \pm2, \ldots$

Note that this can also be written as $\tan^{-1} 1 = \frac{\pi}{4} + n\pi, \quad n = 0, \pm1, \pm2, \ldots\ .$

10. $\tan^{-1} 3i = \dfrac{i}{2}\ln\left(\dfrac{4i}{-2i}\right) = \dfrac{i}{2}\ln(-2) = -\dfrac{\pi}{2} - n\pi + i\log_e\sqrt{2}, \quad n = 0, \pm1, \pm2, \ldots$

11. $\sinh^{-1}\dfrac{4}{3} = \ln\left(\dfrac{4}{3} \pm \dfrac{5}{3}\right) = \begin{cases} \log_e 3 + 2n\pi i \\ \log_e \frac{1}{3} + (2n+1)\pi i \end{cases} = (-1)^n \log_e 3 + n\pi i, \quad n = 0, \pm1, \pm2, \ldots$

12. $\cosh^{-1} i = \ln[(1 + \pm\sqrt{2}\,)i] = \begin{cases} \log_e(1+\sqrt{2}\,) + (\frac{\pi}{2} + 2n\pi)i \\ \log_e(\sqrt{2}-1) + (-\frac{\pi}{2} + 2n\pi)i \end{cases}, \quad n = 0, \pm1, \pm2, \ldots$

13. $\tanh^{-1}(1+2i) = \dfrac{1}{2}\ln\dfrac{2+2i}{-2i} = \dfrac{1}{2}\ln(-1+i) = \dfrac{1}{2}\left[\log_e\sqrt{2} + \left(\dfrac{3\pi}{4} + 2n\pi\right) i\right] = \dfrac{1}{4}\log_e 2 + \left(\dfrac{3\pi}{8} + n\pi\right) i$

14. $\tanh^{-1}(-\sqrt{3}\,i) = \dfrac{1}{2}\ln\dfrac{1-\sqrt{3}\,i}{1+\sqrt{3}\,i} = \dfrac{1}{2}\ln\left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\, i\right) = \dfrac{1}{2}\left[\log_e 1 + \left(\dfrac{4\pi}{3} + 2n\pi\right) i\right] = \left(\dfrac{2\pi}{3} + n\pi\right) i,$

$n = 0, \pm1, \pm2, \ldots$

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Chapter 17 Review Exercises

1. 0; 32 **2.** third **3.** $-7/25$ **4.** $-8i$ **5.** $4/5$

6. The closed annular region between the circles $|z+2|=1$ and $|z+2|=3$. These circles have center at $z=-2$.

7. False. $\operatorname{Arg}[(-1+i)+(-1-i)] = \operatorname{Arg}(-2) = \pi$

8. $-5\pi/6$

9. $z = \ln(2i) = \log_e 2 + i\left(\dfrac{\pi}{2} + 2n\pi\right),\ n = 0, \pm 1, \pm 2, \ldots$

10. True

11. $(1+i)^{2+i} = e^{(2+i)[\log_e 2 + \frac{\pi}{4}i]} = e^{(\log_e 2 - \frac{\pi}{4}) + i(\log_e \sqrt{2} + \frac{\pi}{2})} = e^{\log_e 2 - \frac{\pi}{4}}\left[\cos\left(\log_e \sqrt{2} + \dfrac{\pi}{2}\right) + i\sin\left(\log_e \sqrt{2} + \dfrac{\pi}{2}\right)\right]$

$= -0.3097 + 0.8577i$

12. $f(-1+i) = -33 + 26i$

13. False

14. $2\pi i$

15. $\operatorname{Ln}(-ie^3) = \log_e e^3 + \left(-\dfrac{\pi}{2}\right)i = 3 - \dfrac{\pi}{2}\,i$

16. True

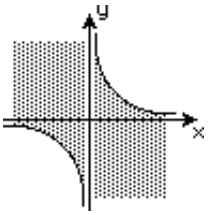
17. $58 - 4i$

18. $-\dfrac{1}{13} - \dfrac{17}{13}\,i$

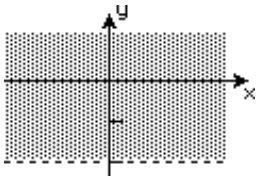
19. $-8 + 8i$

20. $4e^{\pi i/12} = 4\left(\cos\dfrac{\pi}{12} + i\,\sin\dfrac{\pi}{12}\right) = 3.8637 + 1.0353i$

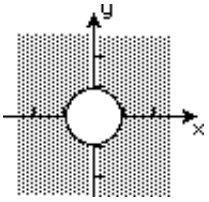
21. The region satisfying $xy \le 1$ is shown in the figure.



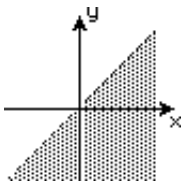
22. The region satisfying $y + 5 > 3$ or $y > -2$ is shown in the figure.



23. The region satisfying $|z| \ge 1$ is shown in the figure.



24. The region satisfying $y < x$ is shown in the figure.



25. Ellipse with foci $(0, -2)$ and $(0, 2)$

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26. $\left|\dfrac{z-w}{1-z\bar{w}}\right|^2 = \dfrac{z-w}{1-z\bar{w}}\cdot\dfrac{\bar{z}-\bar{w}}{1-\bar{z}w} = \dfrac{z\bar{z}-z\bar{w}-w\bar{z}+w\bar{w}}{1-\bar{z}w-z\bar{w}+z\bar{z}w\bar{w}} = \dfrac{1-z\bar{w}-w\bar{z}+|w|^2}{1-\bar{z}w-z\bar{w}+|w|^2} = 1,$

since $|z|^2 = z\bar{z} = 1$ and $|w| \neq 1$.

27. The four fourth roots of $1-i$ are given by

$$w_R = 2^{1/8}\left[\cos\left(-\frac{\pi}{16}+\frac{k\pi}{2}\right)+i\sin\left(-\frac{\pi}{16}+\frac{k\pi}{2}\right)\right], \quad n = 0,\ 1,\ 2,\ 3$$

$$w_0 = 2^{1/8}\left[\cos\left(-\frac{\pi}{16}\right)+i\sin\left(-\frac{\pi}{16}\right)\right] = 1.0696 - 0.2127i$$

$$w_1 = 2^{1/8}\left[\cos\frac{7\pi}{16}+i\sin\frac{7\pi}{16}\right] = 0.2127 + 1.0696i$$

$$w_2 = 2^{1/8}\left[\cos\frac{15\pi}{16}+i\sin\frac{15\pi}{16}\right] = -1.0696 + 0.2127i$$

$$w_3 = 2^{1/8}\left[\cos\frac{23\pi}{16}+i\sin\frac{23\pi}{16}\right] = -0.2127 - 1.0696i$$

28. $z^{3/2} = \frac{2}{5}+\frac{1}{5}i$ implies $z^3 = \dfrac{3}{25}+\dfrac{4}{25}i$. The three cube roots of $\dfrac{3}{25}+\dfrac{4}{25}i$ are

$$w_k = \left(\frac{1}{5}\right)^{1/3}\left[\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{4}{3}\right)+\frac{2k\pi}{3}\right)+i\sin\left(\frac{1}{3}\tan^{-1}\left(\frac{4}{3}\right)+\frac{2k\pi}{3}\right)\right], \quad k = 0,\ 1,\ 2$$

$$w_0 = \left(\frac{1}{5}\right)^{1/3}[\cos(0.3091)+i\sin(0.3091)] = 0.5571 + 0.1779i$$

$$w_1 = \left(\frac{1}{5}\right)^{1/3}[\cos(2.4035)+i\sin(2.4035)] = -0.4326 + 0.3935i$$

$$w_2 = \left(\frac{1}{5}\right)^{1/3}[\cos(4.4979)+i\sin(4.4979)] = -0.1245 - 0.5714i.$$

29. Write $\dfrac{1+i}{\sqrt{2}} = e^{\pi i/4}$ so that

$$z^{24} = e^{6\pi i} = 1, \quad z^{20} = e^{5\pi i} = -1, \quad z^{12} = e^{3\pi i} = -1, \quad z^6 = e^{3\pi i/2} = -i.$$

Therefore

$$f\left(\frac{1+i}{\sqrt{2}}\right) = 1 - 3(-1) + 4(-1) - 5(-i) = 5i.$$

30. $\text{Im}(z-3\bar{z}) = 4y$, $z\text{Re}(z^2) = (x^3-xy^2)+i(x^2y-y^3)$. Thus,

$$f(z) = (4y+x^3-xy^2-5x)+i(x^2y-y^3-5y).$$

31. $u = x^2-y$, $v = y^2-x$. When $x = 1$ we get the parametric equations $u = 1-y$, $v = y^2-1$. Eliminating y then gives $v = (1-u)^2-1 = u^2-2u$. This is an equation of a parabola.

32. $u = x/(x^2+y^2)$, $v = -y/(x^2+y^2)$. When $x = 1$ we get the parametric equations $u = 1/(1+y^2)$, $v = -y/(1+y^2)$. From this we find $u^2+v^2-u = 0$. This describes a circle with the exception that $(0,0)$ is not on its circumference.

33. $z = z^{-1}$ gives $z^2 = 1$ or $(z-1)(z+1) = 0$. Thus $z = \pm 1$.

34. $\bar{z} = 1/z$ gives $z\bar{z} = 1$ or $|z|^2 = 1$. All points on the circle $|z| = 1$ satisfy the equation.

35. $\bar{z} = -z$ gives $x = -x$ or $x = 0$. All complex numbers of the form $z = 0+iy$ (pure imaginaries) satisfy the equation.

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36. $z^2 = \bar{z}^2$ gives $xy = -xy$ or $xy = 0$. This implies $x = 0$ or $y = 0$. All real numbers ($y = 0$) and all pure imaginary numbers ($x = 0$) satisfy the equation.

37. $u = -2xy - 5x$, $v = x^2 - 5y - y^2$; $\dfrac{\partial u}{\partial x} = -2y - 5 = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -2x = -\dfrac{\partial v}{\partial x}$; $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = -2y - 5 + 2xi$

38. $u = x^3 + xy^2 - 4x$, $v = 4y - y^3 - x^2y$; $\dfrac{\partial u}{\partial x} = 3x^2 + y^2 - 4$, $\dfrac{\partial v}{\partial y} = 4 - 3y^2 - x^2$, $\dfrac{\partial u}{\partial y} = 2xy = -\dfrac{\partial v}{\partial x}$

The Cauchy-Riemann equations are satisfied at all points on the circle $x^2 + y^2 = 2$. Continuity of u, v, and the first partial derivatives guarantee f is differentiable on the circle. However, f is nowhere analytic.

39. $\text{Ln}(1+i)(1-i) = \text{Ln}(2) = \log_e 2$; $\text{Ln}(1+i) = \log_e \sqrt{2} + \dfrac{\pi}{4}\,i$; $\text{Ln}(1-i) = \log_e \sqrt{2} - \dfrac{\pi}{4}\,i$. Therefore

$$\text{Ln}(1+i) + \text{Ln}(1-i) = 2\log_e \sqrt{2} = \log_e 2 = \text{Ln}(1+i)(1-i).$$

40. $\text{Ln}\dfrac{1+i}{1-i} = \text{Ln}\ i = \log_e 1 + \dfrac{\pi}{2}\,i = \dfrac{\pi}{2}\,i$; $\text{Ln}(1+i) = \log_e \sqrt{2} + \dfrac{\pi}{4}\,i$; $\text{Ln}(1-i) = \log_e \sqrt{2} - \dfrac{\pi}{4}\,i$. Therefore

$$\text{Ln}(1+i) - \text{Ln}(1-i) = \frac{\pi}{4}\,i - \left(-\frac{\pi}{4}\,i\right) = \frac{\pi}{2}\,i = \text{Ln}\frac{1+i}{1-i}\,.$$

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18 Integration in the Complex Plane

Exercises 18.1

1. $\displaystyle\int_C (z+3)\,dz = (2+4i)\left[\int_1^3 (2t+3)\,dt + i\int_1^3 (4t-1)\,dt\right] = (2+4i)[14+14i] = -28+84i$

2. $\displaystyle\int_C (2\bar{z}-z)\,dz = \int_0^2 [-t-3(t^2+2)i](-1+2ti)\,dt = \int_0^2 (6t^3+13t)\,dt + i\int_0^2 (t^2+2)\,dt = 50+\frac{20}{3}\,i$

3. $\displaystyle\int_C z^2\,dz = (3+2i)^3\int_{-2}^2 t^2\,dt = \frac{16}{3}(3+2i)^3 = -48+\frac{736}{3}\,i$

4. $\displaystyle\int_C (3z^2-2z)\,dz = \int_0^1 (-15t^4+4t^3+3t^2-2t)\,dt + i\int_0^1 (-6t^5+12t^3-6t^2)\,dt = -2+0i = -2$

5. Using $z = e^{it}$, $-\pi/2 \le t \le \pi/2$, and $dz = ie^{it}\,dt$, $\displaystyle\int_C \frac{1+z}{z}\,dz = -\int_{-\pi/2}^{\pi/2}(1+e^{it})\,dt = (2+\pi)i.$

6. $\displaystyle\int_C |z|^2\,dz = \int_1^2 \left(2t^5+\frac{2}{t}\right)dt - i\int_1^2 \left(t^2+\frac{1}{t^4}\right)dt = 21+\ln 4-\frac{21}{8}\,i$

7. Using $z = e^{it} = \cos t + i\sin t$, $dz = (-\sin t + i\cos t)\,dt$ and $x = \cos t$,

$$\oint_C \operatorname{Re}(z)\,dz = \int_0^{2\pi} \cos t(-\sin t + i\cos t)\,dt = -\int_0^{2\pi} \sin t\cos t\,dt + i\int_0^{2\pi} \cos^2 t\,dt$$

$$= -\frac{1}{2}\int_0^{2\pi} \sin 2t\,dt + \frac{1}{2}\,i\int_0^{2\pi}(1+\cos 2t)\,dt = \pi i.$$

8. Using $z+i = e^{it}$, $0 \le t \le 2\pi$, and $dz = ie^{it}\,dt$,

$$\oint_C \left[\frac{1}{(z+i)^3} - \frac{5}{z+i} + 8\right] dz = i\int_0^{2\pi} [e^{-2it} - 5 + 8e^{it}]\,dt = -10\pi i.$$

9. Using $y = -x+1$, $0 \le x \le 1$, $z = x + (-x+1)i$, $dz = (1-i)\,dx$,

$$\int_C (x^2+iy^3)\,dz = (1-i)\int_1^0 [x^2 + (1-x)^3 i]\,dx = -\frac{7}{12}+\frac{1}{12}\,i.$$

10. Using $z = e^{it}$, $\pi \le t \le 2\pi$, $dz = ie^{it}\,dt$, $x = \cos t = (e^{it}+e^{-it})/2$, $y = \sin t = (e^{it}-e^{-it})/2i$,

$$\int_C (x^3 - iy^3)\,dz = \frac{1}{8}\,i\int_\pi^{2\pi} (e^{3it}+3e^{it}+3e^{-it}+e^{-3it})e^{it}\,dt + \frac{1}{8}\,i\int_\pi^{2\pi} (e^{3it}-3e^{it}+3e^{-it}-e^{-3it})e^{it}\,dt$$

$$= \frac{1}{8}\,i\int_\pi^{2\pi} (2e^{4it}+6)\,dt = \frac{3\pi}{4}\,i.$$

11. $\displaystyle\int_C e^z\,dz = \int_{C_1} e^z\,dz + \int_{C_2} e^z\,dz$ where C_1 and C_2 are the line segments $y = 0$, $0 \le x \le 2$ and $y = -\pi x + 2\pi$, $1 \le x \le 2$, respectively. Now

$$\int_{C_1} e^z\,dz = \int_0^2 e^x\,dx = e^2 - 1$$

$$\int_{C_2} e^z\,dz = (1-\pi i)\int_2^1 e^{x+(-\pi x+2\pi)i}dx = (1-\pi i)\int_2^1 e^{(1-\pi i)x}dx = e^{1-\pi i} - e^{2(1-\pi i)} = -e - e^2.$$

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In the second integral we have used the fact that e^z has period $2\pi i$. Thus

$$\int_C e^z\,dz = (e^2-1)+(-e-e^2) = -1-e.$$

12. $\displaystyle\int_C \sin z\,dz = \int_{C_1} \sin z\,dz + \int_{C_2} \sin z\,dz$ where C_1 and C_2 are the line segments $y=0$, $0\le x\le 1$, and $x=1$, $0\le y\le 1$, respectively. Now

$$\int_{C_1} \sin z\,dz = \int_0^1 \sin x\,dx = 1-\cos 1$$

$$\int_{C_2} \sin z\,dz = i\int_0^1 \sin(1+iy)\,dy = \cos 1 - \cos(1+i).$$

Thus

$$\int_C \sin z\,dz = (1-\cos 1)+(\cos 1-\cos(1+i)) = 1-\cos(1+i) = (1-\cos 1\cosh 1)+i\sin 1\sinh 1 = 0.1663+0.9889i.$$

13. We have $$\int_C \operatorname{Im}(z-i)\,dz = \int_{C_1}(y-1)\,dz + \int_{C_2}(y-1)\,dz$$

On C_1, $z=e^{it}$, $0\le t\le \pi/2$, $dz = ie^{it}\,dt$, $y=\sin t = (e^{it}-e^{-it})/2i$,

$$\int_{C_1} = (y-1)\,dz = \frac{1}{2}\int_0^{\pi/2}[e^{it}-e^{-it}-2i]e^{it}\,dt = \frac{1}{2}\int_0^{\pi/2}[e^{2it}-1+2ie^{it}]\,dt = 1-\frac{\pi}{4}-\frac{1}{2}\,i.$$

On C_2, $y=x+1$, $-1\le x\le 0$, $z=x+(x+1)i$, $dz=(1+i)\,dx$,

$$\int_{C_2}(y-1)\,dz = (1+i)\int_0^{-1} x\,dx = \frac{1}{2}+\frac{1}{2}\,i.$$

Thus $$\int_C \operatorname{Im}(z-i)\,dz = \left(1-\frac{\pi}{4}-\frac{1}{2}\,i\right)+\left(\frac{1}{2}+\frac{1}{2}\,i\right) = \frac{3}{2}-\frac{\pi}{4}\,.$$

14. Using $x=6\cos t$, $y=2\sin t$, $\pi/2\le t\le 3\pi/2$, $z=6\cos t+2i\sin t$, $dz=(-6\sin t+2i\cos t)\,dt$,

$$\int_C dz = -6\int_{\pi/2}^{3\pi/2}\sin t\,dt + 2i\int_{\pi/2}^{3\pi/2}\cos t\,dt = 2i(-2) = -4i.$$

15. We have $$\oint_C ze^z\,dz = \int_{C_1} ze^z\,dz + \int_{C_2} ze^z\,dz + \int_{C_3} ze^z\,dz + \int_{C_4} ze^z\,dz$$

On C_1, $y=0$, $0\le x\le 1$, $z=x$, $dz=dx$,

$$\int_{C_1} ze^z\,dz = \int_0^1 xe^x\,dx = xe^x - e^x\,\Big|_0^1 = 1.$$

On C_2, $x=1$, $0\le y\le 1$, $z=1+iy$, $dz=i\,dy$,

$$\int_{C_2} ze^z\,dz = i\int_0^1 (1+iy)e^{1+iy}\,dy = ie^{i+1}.$$

On C_3, $y=1$, $0\le x\le 1$, $z=x+i$, $dz=dx$,

$$\int_{C_3} ze^z\,dz = \int_1^0 (x+i)e^{x+i}dx = (i-1)e^i - ie^{1+i}.$$

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On C_4, $x = 0$, $0 \le y \le 1$, $z = iy$, $dz = i\,dy$,

$$\int_{C_4} ze^z\,dz = -\int_1^0 ye^{iy}\,dy = (1-i)e^i - 1.$$

Thus $$\oint_C ze^z\,dz = 1 + ie^{i+1} + (i-1)e^i - ie^{1+i} + (1-i)e^i - 1 = 0.$$

16. We have $$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

On C_1, $y = x^2$, $-1 \le x \le 0$, $z = x + ix^2$, $dz = (1 + 2xi)\,dx$,

$$\int_{C_1} f(z)\,dz = \int_{-1}^0 2(1+2xi)\,dx = 2 - 2i.$$

On C_2, $y = x^2$, $0 \le x \le 1$, $z = x + ix^2$, $dz = (1 + 2xi)\,dx$,

$$\int_{C_2} f(z)\,dz = \int_0^1 6x(1+2xi)\,dx = 3 + 4i.$$

Thus $$\int_C f(z)\,dz = 2 - 2i + 3 + 4i = 5 + 2i.$$

17. We have $$\oint_C x\,dz = \int_{C_1} x\,dz + \int_{C_2} x\,dz + \int_{C_3} x\,dz$$

On C_1, $y = 0$, $0 \le x \le 1$, $z = x$, $dz = dx$,

$$\int_{C_1} x\,dz = \int_0^1 x\,dx = \frac{1}{2}.$$

On C_2, $x = 1$, $0 \le y \le 1$, $z = 1 + iy$, $dz = i\,dy$,

$$\int_{C_2} x\,dz = i\int_0^1 dy = i.$$

On C_3, $y = x$, $0 \le x \le 1$, $z = x + ix$, $dz = (1 + i)\,dx$,

$$\int_{C_3} x\,dz = (1+i)\int_1^0 x\,dx = -\frac{1}{2} - \frac{1}{2}\,i.$$

Thus $$\oint_C x\,dz = \frac{1}{2} + i - \frac{1}{2} - \frac{1}{2}\,i = \frac{1}{2}\,i.$$

18. We have $$\oint_C (2z-1)\,dz = \int_{C_1} (2z-1)\,dz + \int_{C_2} (2z-1)\,dz + \int_{C_3} (2z-1)\,dz$$

On C_1, $y = 0$, $0 \le x \le 1$, $z = x$, $dz = dx$,

$$\int_{C_1} (2z-1)\,dz = \int_0^1 (2x-1)\,dx = 0.$$

On C_2, $x = 1$, $0 \le y \le 1$, $z = 1 + iy$, $dz = i\,dy$,

$$\int_{C_2} (2z-1)\,dz = -2\int_0^1 y\,dy + i\int_0^1 dy = -1 + i.$$

On C_3, $y = x$, $z = x + ix$, $dz = (1 + i)\,dx$,

$$\int_{C_3} (2z-1)\,dz = (1+i)\int_1^0 (2x - 1 + 2ix)\,dx = 1 - i.$$

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Thus $$\oint_C (2z-1)\,dz = 0 - 1 + i + 1 - i = 0.$$

19. We have $$\oint_C z^2\,dz = \int_{C_1} z^2\,dz + \int_{C_2} z^2\,dz + \int_{C_3} z^2\,dz$$

On C_1 $y = 0$, $0 \le x \le 1$, $z = x$, $dz = dx$,

$$\int_{C_1} z^2\,dz = \int_0^1 x^2\,dx = \frac{1}{3}.$$

On C_2, $x = 1$, $0 \le y \le 1$, $z = 1 + iy$, $dz = i\,dy$,

$$\int_{C_2} z^2\,dz = \int_0^1 (1+iy)^2 i\,dy = -1 + \frac{2}{3}\,i.$$

On C_3, $y = x$, $0 \le x \le 1$, $z = x + ix$, $dz = (1+i)\,dx$,

$$\int_{C_3} z^2\,dz = (1+i)^3 \int_1^0 x^2\,dx = \frac{2}{3} - \frac{2}{3}\,i.$$

Thus $$\oint_C z^2\,dz = \frac{1}{3} - 1 + \frac{2}{3}\,i + \frac{2}{3} - \frac{2}{3}\,i = 0.$$

20. We have $$\oint_C \bar{z}^2\,dz = \int_{C_1} \bar{z}^2\,dz + \int_{C_2} \bar{z}^2\,dz + \int_{C_3} \bar{z}^2\,dz$$

On C_1, $y = 0$, $0 \le x \le 1$, $z = x$, $dz = dx$,

$$\int \bar{z}^2\,dz = \int_0^1 x^2\,dx = \frac{1}{3}.$$

On C_2, $x = 1$, $0 \le y \le 1$, $z = 1 + iy$, $dz = i\,dy$,

$$\int_{C_2} \bar{z}^2\,dz = -\int_0^1 (1-iy)^2(-i\,dy) = 1 + \frac{2}{3}\,i.$$

On C_3, $y = x$, $0 \le x \le 1$, $z = x + ix$, $dz = (1+i)\,dx$,

$$\int_{C_3} \bar{z}^2\,dz = (1-i)^2(1+i) \int_1^0 x^2\,dx = -\frac{2}{3} + \frac{2}{3}\,i.$$

Thus $$\oint_C \bar{z}^2\,dz = \frac{1}{3} + 1 + \frac{2}{3}\,i - \frac{2}{3} + \frac{2}{3}\,i = \frac{2}{3} + \frac{4}{3}\,i.$$

21. On C, $y = -x + 1$, $0 \le x \le 1$, $z = x + (-x+1)i$, $dz = (1-i)\,dx$,

$$\int_C (z^2 - z + 2)\,dz = (1-i)\int_0^1 [x^2 - (1-x)^2 - x + 2 + (3x - 2x^2 - 1)i]\,dx = \frac{4}{3} - \frac{5}{3}\,i.$$

22. We have $$\int_C (z^2 - z + 2)\,dz = \int_{C_1} (z^2 - z + 2)\,dz + \int_{C_2} (z^2 - z + 2)\,dz$$

On C_1, $y = 1$, $0 \le x \le 1$, $z = x + i$, $dz = dx$,

$$\int_{C_1} (z^2 - z + 2)\,dz = \int_0^1 [(x+i)^2 - x + 2 - i]\,dx = \frac{5}{6}.$$

On C_2, $x = 1$, $0 \le y \le 1$, $z = 1 + iy$, $dz = i\,dy$,

$$\int_{C_2} (z^2 - z + 2)\,dz = i\int_1^0 [(1+iy)^2 + 1 - iy]\,dy = \frac{1}{2} - \frac{5}{3}\,i.$$

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Thus $$\int_C (z^2 - z + 2)\,dz = \frac{1}{2} - \frac{5}{3}\,i + \frac{5}{6} = \frac{4}{3} - \frac{5}{3}\,i.$$

23. On C, $y = 1 - x^2$, $0 \le x \le 1$, $z = x + i(1 - x^2)$, $dz = (1 - 2xi)\,dx$,

$$\int_C (z^2 - z + 2)\,dz = \int_0^1 (-5x^4 + 2x^3 + 7x^2 - 3x + 1)\,dx + i\int_0^1 (2x^5 - 8x^3 + 3x^2 - 1)\,dx = \frac{4}{3} - \frac{5}{3}\,i.$$

24. On C, $x = \sin t$, $y = \cos t$, $0 \le t \le \pi/2$ or $z = ie^{-it}$, $dz = e^{-it}\,dt$,

$$\begin{aligned}\int_C (z^2 - z + 2)\,dz &= \int_0^{\pi/2} (-e^{-2it} - ie^{-it} + 2)e^{-it}\,dt = \int_0^{\pi/2} (-e^{-3it} - ie^{-2it} + 2e^{-it})\,dt \\ &= -\frac{1}{3}\,ie^{-3\pi i/2} + \frac{1}{2}e^{-\pi i} + 2ie^{-\pi i/2} + \frac{1}{3}\,i - \frac{1}{2} - 2i = \frac{4}{3} - \frac{5}{3}\,i.\end{aligned}$$

25. On C, $\left|\dfrac{e^z}{z^2+1}\right| \le \dfrac{|e^z|}{|z|^2 - 1} = \dfrac{e^5}{24}$. Thus $\left|\oint_C \dfrac{e^z}{z^2+1}\,dz\right| \le \dfrac{e^5}{24} \cdot 10\pi = \dfrac{5\pi}{12}\,e^5$.

26. On C, $\left|\dfrac{1}{z^2 - 2i}\right| \le \dfrac{1}{|z|^2 - |2i|} = \dfrac{1}{34}$. Thus $\left|\int_C \dfrac{1}{z^2 - 2i}\,dz\right| \le \dfrac{1}{34} \cdot \dfrac{1}{2}(12\pi) = \dfrac{3\pi}{17}$.

27. The length of the line segment from $z = 0$ to $z = 1 + i$ is $\sqrt{2}$. In addition, on this line segment

$$|z^2 + 4| \le |z|^2 + 4 \le |1 + i|^2 + 4 = 6.$$

Thus $\left|\int_C (z^2 + 4)\,dz\right| \le 6\sqrt{2}$.

28. On C, $\left|\dfrac{1}{z^3}\right| = \dfrac{1}{|z|^3} = \dfrac{1}{64}$. Thus $\left|\int_C \dfrac{1}{z^3}\,dz\right| \le \dfrac{1}{64} \cdot \dfrac{1}{4}(8\pi) = \dfrac{\pi}{32}$.

29. (a) $$\begin{aligned}\int_C dz &= \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta z_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} (z_k - z_{k-1}) \\ &= \lim_{\|P\|\to 0} [(z_1 - z_0) + (z_2 - z_1) + (z_3 - z_2) + \cdots + (z_{n-1} - z_{n-2}) + (z_n - z_{n-1})] \\ &= \lim_{\|P\|\to 0} (z_n - z_0) = z_n - z_0\end{aligned}$$

(b) With $z_n = -2i$ and $z_0 = 2i$, $\int_C dz = -2i - (2i) = -4i$.

30. With $z_k^* = z_k$,

$$\begin{aligned}\int_C z\,dz &= \lim_{\|P\|\to 0} \sum_{k=1}^{n} z_k(z_k - z_{k-1}) \\ &= \lim_{\|P\|\to 0} [(z_1^2 - z_1 z_0) + (z_2^2 - z_2 z_1) + \cdots + (z_n^2 - z_n z_{n-1})]. \qquad (1)\end{aligned}$$

With $z_k^* = z_{k-1}$,

$$\begin{aligned}\int_C z\,dz &= \lim_{\|P\|\to 0} \sum_{k=1}^{n} z_{k-1}(z_k - z_{k-1}) \\ &= \lim_{\|P\|\to 0} [(z_0 z_1 - z_0^2) + (z_1 z_2 - z_1^2) + \cdots + (z_{n-1} z_n - z_{n-1}^2)]. \qquad (2)\end{aligned}$$

Adding (1) and (2) gives

$$2\int_C z\,dz = \lim_{\|P\|\to 0} (z_n^2 - z_0^2) \quad \text{or} \quad \int_C z\,dz = \frac{1}{2}(z_n^2 - z_0^2).$$

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31. **(a)** $\displaystyle\int_C (6z+4)\,dz = 6\int_C z\,dz + 4\int_C dz = \frac{6}{2}[(2+3i)^2-(1+i)^2] + 4[(2+3i)-(1+i)] = -11+38i$

(b) Since the contour is closed, $z_0 = z_n$ and so

$$6\int_C z\,dz + 4\int_C dz = 6[z_0^2 - z_0^2] + 4[z_0 - z_0] = 0.$$

32. For $f(z) = 1/z$, $\overline{f(z)} = 1/\bar{z}$, so on $z = 2e^{it}$, $\bar{z} = 2e^{-it}$, $dz = 2ie^{it}\,dt$, and

$$\oint_C \overline{f(z)}\,dz = \int_0^{2\pi} \frac{1}{2e^{-it}} \cdot 2ie^{it}\,dt = \frac{1}{2}e^{2it}\Big|_0^{2\pi} = \frac{1}{2}[e^{4\pi i} - 1] = 0.$$

Thus circulation $= \text{Re}\left(\oint_C \overline{f(z)}\,dz\right) = 0$, net flux $= \text{Im}\left(\oint_C \overline{f(z)}\,dz\right) = 0$.

33. For $f(z) = 2z$, $\overline{f(z)} = 2\bar{z}$, so on $z = e^{it}$, $\bar{z} = e^{-it}$, $dz = ie^{it}\,dt$, and

$$\oint_C \overline{f(z)}\,dz = \int_0^{2\pi} (e^{-it})(ie^{it}\,dt) = 2i\int_0^{2\pi} dt = 4\pi i.$$

Thus circulation $= \text{Re}\left(\oint_C \overline{f(z)}\,dz\right) = 0$, net flux $= \text{Im}\left(\oint_C \overline{f(z)}\,dz\right) = 4\pi$.

34. For $f(z) = 1/(\overline{z-1})$, $\overline{f(z)} = 1/(z-1)$, so on $z - 1 = 2e^{it}$, $dz = 2ie^{it}dt$, and

$$\oint_C \overline{f(z)}\,dz = \int_0^{2\pi} \frac{1}{2e^{it}} \cdot 2ie^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

Thus circulation $= \text{Re}\left(\oint_C \overline{f(z)}\,dz\right) = 0$, net flux $= \text{Im}\left(\oint_C \overline{f(z)}\,dz\right) = 2\pi$.

35. For $f(z) = \bar{z}$, $\overline{f(z)} = z$ so on the square we have

$$\oint_C \overline{f(z)}\,dz = \int_{C_1} z\,dz + \int_{C_2} z\,dz + \int_{C_3} z\,dz + \int_{C_4} z\,dz$$

wheRe C_1 is $y = 0$, $0 \le x \le 1$, C_2 is $x = 1$, $0 \le y \le 1$, C_3 is $y = 1$, $0 \le x \le 1$, and C_4 is $x = 0$, $0 \le y \le 1$. Thus

$$\int_{C_1} z\,dz = \int_0^1 x\,dx = \frac{1}{2}$$

$$\int_{C_2} z\,dz = i\int_0^1 (1+iy)\,dy = -\frac{1}{2} + i$$

$$\int_{C_3} z\,dz = \int_1^0 (x+i)\,dx = -\frac{1}{2} - i$$

$$\int_{C_4} z\,dz = -\int_1^0 y\,dy = \frac{1}{2}$$

and so

$$\oint_C \overline{f(z)}\,dz = \frac{1}{2} + \left(-\frac{1}{2}+i\right) + \left(-\frac{1}{2}-i\right) + \frac{1}{2} = 0$$

$$\text{circulation} = \text{Re}\left(\oint_C \overline{f(z)}\,dz\right) = \text{Re}(0) = 0$$

$$\text{net flux} = \text{Im}\left(\oint_C \overline{f(z)}\,dz\right) = \text{Im}(0) = 0.$$

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Exercises 18.2

1. $f(z) = z^3 - 1 + 3i$ is a polynomial and so is an entire function.

2. z^2 is entire and $\dfrac{1}{z-4}$ is analytic within and on the circle $|z| = 1$.

3. $f(z) = \dfrac{z}{2z+3}$ is discontinuous at $z = -3/2$ but is analytic within and on the circle $|z| = 1$.

4. $f(z) = \dfrac{z-3}{z^2+2z+2}$ is discontinuous at $z = -1+i$ and at $z = -1-i$ but is analytic within and on the circle $|z| = 1$.

5. $f(z) = \dfrac{\sin z}{(z^2-25)(z^2+9)}$ is discontinuous at $z = \pm 5$ and at $z = \pm 3i$ but is analytic within and on the circle $|z| = 1$.

6. $f(z) = \dfrac{e^z}{2z^2+11z+15}$ is discontinuous at $z = -5/2$ and at $z = -3$ but is analytic within and on the circle $|z| = 1$.

7. $f(z) = \tan z$ is discontinuous at $z = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \ldots$ but is analytic within and on the circle $|z| = 1$.

8. $f(z) = \dfrac{z^2-9}{\cosh z}$ is discontinuous at $\dfrac{\pi}{2}i, \pm\dfrac{3\pi}{2}i, \ldots$ but is analytic within and on the circle $|z| = 1$.

9. By the principle of deformation of contours we can choose the more convenient circular contour C_1 defined by $|z| = 1$. Thus

$$\oint_C \frac{1}{z}\,dz = \oint_{C_1} \frac{1}{z}\,dz = 2\pi i$$

by (4) of Section 18.2.

10. By the principle of deformation of contours we can choose the more convenient circular contour C_1 defined by $|z-(-1-i)| = \dfrac{1}{16}$. Thus

$$\oint_C \frac{5}{z+1+i}\,dz = 5\oint_{C_1} \frac{1}{z-(-1-i)}\,dz = 5(2\pi i) = 10\pi i$$

by (4) of Section 18.2.

11. By Theorem 18.4 and (4) of Section 18.2,

$$\oint_C \left(z + \frac{1}{z}\right) dz = \oint_C z\,dz + \oint_C \frac{1}{z}\,dz = 0 + 2\pi i = 2\pi i.$$

12. By Theorem 18.4 and (4) of Section 18.2,

$$\oint_C \left(z + \frac{1}{z^2}\right) dz = \oint_C \frac{1}{z}\,dz + \oint_C \frac{1}{z^2}\,dz = 0 + 0 = 0.$$

13. Since $f(z) = \dfrac{z}{z^2-\pi^2}$ is analytic within and on C it follows from Theorem 18.4 that $\displaystyle\oint_C \frac{z}{z^2-\pi^2}\,dz = 0$.

14. By (4) of Section 18.2, $\displaystyle\oint_C \frac{10}{(z+i)^4}\,dz = 0$.

15. By partial fractions, $\displaystyle\oint_C \frac{2z+1}{z(z+1)}\,dz = \oint_C \frac{1}{z}\,dz + \oint_C \frac{1}{z+1}\,dz$.

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(a) By Theorem 18.4 and (4) of Section 18.2,

$$\oint_C \frac{1}{z}\,dz + \oint_C \frac{1}{z+1}\,dz = 2\pi i + 0 = 2\pi i.$$

(b) By writing $\displaystyle\oint_C = \oint_{C_1} + \oint_{C_2}$ where C_1 and C_2 are the circles $|z| = 1/2$ and $|z+1| = 1/2$, respectively, we have by Theorem 18.4 and (4) of Section 18.2,

$$\begin{aligned}\oint_C \frac{1}{z}\,dz + \oint_C \frac{1}{z+1}\,dz &= \oint_{C_1} \frac{1}{z}\,dz + \oint_{C_1} \frac{1}{z+1}\,dz + \oint_{C_2} \frac{1}{z}\,dz + \oint_{C_2} \frac{1}{z+1}\,dz \\ &= 2\pi i + 0 + 0 + 2\pi i = 4\pi i.\end{aligned}$$

(c) Since $f(z) = \dfrac{2z+1}{z(z+1)}$ is analytic within and on C it follows from Theorem 18.4 that

$$\oint_C \frac{2z+1}{z^2+z}\,dz = 0.$$

16. By partial fractions, $\displaystyle\oint_C \frac{2z}{z^2+3}\,dz = \oint_C \frac{1}{z+\sqrt{3}\,i}\,dz + \oint_C \frac{1}{z-\sqrt{3}\,i}\,dz.$

(a) By Theorem 18.4,

$$\oint_C \frac{1}{z+\sqrt{3}\,i}\,dz + \oint_C \frac{1}{z-\sqrt{3}\,i}\,dz = 0 + 0 = 0.$$

(b) By Theorem 18.4 and (4) of Section 18.2,

$$\oint_C \frac{1}{z+\sqrt{3}\,i}\,dz + \oint_C \frac{1}{z-\sqrt{3}\,i}\,dz = 0 + 2\pi i = 2\pi i.$$

(c) By writing $\displaystyle\oint_C = \oint_{C_1} + \oint_{C_2}$ where C_1 and C_2 are the circles $|z+\sqrt{3}\,i| = 1/2$ and $|z-\sqrt{3}\,i| = 1/2$, respectively, we have by Theorem 18.4 and (4) of Section 18.2,

$$\begin{aligned}\oint_C \frac{1}{z+\sqrt{3}\,i}\,dz + \oint_C \frac{1}{z-\sqrt{3}\,i}\,dz &= \oint_{C_1} \frac{1}{z+\sqrt{3}\,i}\,dz + \oint_{C_1} \frac{1}{z-\sqrt{3}\,i}\,dz + \oint_{C_2} \frac{1}{z+\sqrt{3}\,i}\,dz + \oint_{C_2} \frac{1}{z-\sqrt{3}\,i}\,dz \\ &= 2\pi i + 0 + 0 + 2\pi i = 4\pi i.\end{aligned}$$

17. By partial fractions, $\displaystyle\oint_C \frac{-3z+2}{z^2-8z+12}\,dz = \oint_C \frac{1}{z-2}\,dz - 4\oint_C \frac{1}{z-6}\,dz.$

(a) By Theorem 18.4 and (4) of Section 18.2,

$$\oint_C \frac{1}{z-2}\,dz - 4\oint_C \frac{1}{z-6}\,dz = 0 - 4(2\pi i) = -8\pi i.$$

(b) By writing $\displaystyle\oint_C = \oint_{C_1} + \oint_{C_2}$ where C_1 and C_2 are the circles $|z-2| = 1$ and $|z-6| = 1$, respectively, we have by Theorem 18.4 and (4) of Section 18.2,

$$\begin{aligned}\oint_C \frac{1}{z-2}\,dz - 4\oint_C \frac{1}{z-6}\,dz &= \oint_{C_1} \frac{1}{z-2}\,dz - 4\oint_{C_1} \frac{1}{z-6}\,dz + \oint_{C_2} \frac{1}{z-2}\,dz - 4\oint_{C_2} \frac{1}{z-6}\,dz \\ &= 2\pi i - 4(0) + 0 - 4(2\pi i) = -6\pi i.\end{aligned}$$

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18. **(a)** By writing $\oint_C = \oint_{C_1} + \oint_{C_2}$ where C_1 and C_2 are the circles $|z+2|=1$ and $|z-2i|=1$, respectively, we have by Theorem 18.4 and (4) of Section 18.2,

$$\oint_C \left(\frac{3}{z+2} - \frac{1}{z-2i}\right) dz = \oint_{C_1} \frac{3}{z+2}\, dz - \oint_{C_1} \frac{1}{z-2i}\, dz + \oint_{C_2} \frac{3}{z+2}\, dz - \oint_{C_2} \frac{1}{z-2i}\, dz$$
$$= 3(2\pi i) - 0 + 0 - 2\pi i = 4\pi i.$$

19. By partial fractions,

$$\oint_C \frac{z-1}{z(z-i)(z-3i)}\, dz = \frac{1}{3}\oint_C \frac{1}{z}\, dz + \left(-\frac{1}{2} + \frac{1}{2}\, i\right)\oint_C \frac{1}{z-i}\, dz + \left(\frac{1}{6} - \frac{1}{2}\, i\right)\oint_C \frac{1}{z-3i}\, dz.$$

By Theorem 18.4 and (4) of Section 18.2,

$$\oint_C \frac{z-1}{z(z-i)(z-3i)}\, dz = 0 + \left(-\frac{1}{2} + \frac{1}{2}\, i\right) 2\pi i + 0 = \pi(-1-i).$$

20. By partial fractions,

$$\oint_C \frac{1}{z^3+2iz^2}\, dz = \frac{1}{4}\oint_C \frac{1}{z}\, dz - \frac{1}{2}\, i\oint_C \frac{1}{z^2}\, dz - \frac{1}{4}\oint_C \frac{1}{z+2i}\, dz.$$

By Theorem 18.4 and (4) of Section 18.2,

$$\oint_C \frac{1}{z^3+2iz^2}\, dz = \frac{1}{4}\, 2\pi i - \frac{1}{2}\, i(0) - \frac{1}{4}\,(0) = \frac{\pi}{2}\, i.$$

21. We have $$\oint_C \frac{8z-3}{z^2-z}\, dz = \oint_{C_1} \frac{8z-3}{z^2-z}\, dz - \oint_{C_2} \frac{8z-3}{z^2-z}\, dz$$

where C_1 and C_2 are the closed portions of the curve C enclosing $z=0$ and $z=1$, respectively. By partial fractions, Theorem 18.4, and (4) of Section 18.2,

$$\oint_{C_1} \frac{8z-3}{z^2-z}\, dz = 5\oint_{C_1} \frac{1}{z-1}\, dz + 3\oint_{C_1} \frac{1}{z}\, dz = 5(0) + 3(2\pi i) = 6\pi i$$
$$\oint_{C_1} \frac{8z-3}{z^2-z}\, dz = 5\oint_{C_2} \frac{1}{z-1}\, dz + 3\oint_{C_2} \frac{1}{z}\, dz = 5(2\pi i) + 3(0) = 10\pi i.$$

Thus $$\oint_C \frac{8z-3}{z^2-z}\, dz = 6\pi i - 10\pi i = -4\pi i.$$

22. By choosing the more convenient contour C_1 defined by $|z-z_0| = r$ where r is small enough so that the circle C_1 lies entirely within C we can write

$$\oint_C \frac{1}{(z-z_0)^n}\, dz = \oint_{C_1} \frac{1}{(z-z_0)^n}\, dz.$$

Let $z - z_0 = re^{it}$, $0 \le t \le 2\pi$ and $dz = ire^{it}\, dt$. Then for $n = 1$:

$$\oint_{C_1} \frac{1}{z-z_0}\, dz = \int_0^{2\pi} \frac{1}{re^{it}}\, ire^{it}\, dt = i\int_0^{2\pi} dt = 2\pi i.$$

For $n \ne 1$:

$$\oint_{C_1} \frac{1}{(z-z_0)^n}\, dz = \frac{i}{r^{n-1}}\int_0^{2\pi} e^{(1-n)it}\, dt = \frac{i}{r^{n-1}}\, \frac{e^{(1-n)it}}{i(1-n)}\,\Bigg|_0^{2\pi} = \frac{1}{r^{n-1}(1-n)}[e^{2\pi(1-n)i} - 1] = 0$$

since $e^{2\pi(1-n)i} = 1$.

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23. Write $$\oint_C \left(\frac{e^z}{z+3} - 3\bar{z}\right) dz = \oint_C \frac{e^z}{z+3}\, dz - 3\oint_C \bar{z}\, dz.$$

By Theorem 18.4, $\displaystyle\oint_C \frac{e^z}{z+3}\, dz = 0$. However, since $\bar{z}$ is not analytic,

$$\oint_C \bar{z}\, dz = \int_0^{2\pi} e^{-it}(ie^{it}\, dt) = 2\pi i.$$

Thus $$\oint_C \left(\frac{e^z}{z+3} - 3\bar{z}\right) dz = 0 - 3(2\pi i) = -6\pi i.$$

24. Write $$\oint_C (z^2 + z + \mathrm{Re}(z))\, dz = \oint_C (z^2 + z)\, dz + \oint_C \mathrm{Re}(z)\, dz.$$

By Theorem 18.4, $\displaystyle\oint_C (z^2 + z)\, dz = 0$. However, since $\mathrm{Re}(z) = x$ is not analytic,

$$\oint_C x\, dz = \oint_{C_1} x\, dz + \oint_{C_2} x\, dz + \oint_{C_3} x\, dz$$

where C_1 is $y = 0$, $0 \le x \le 1$, C_2 is $x = 1$, $0 \le y \le 2$, and C_3 is $y = 2x$, $0 \le x \le 1$. Thus,

$$\oint_C x\, dz = \int_0^1 x\, dx + i\int_0^2 dy + (1+2i)\int_1^0 x\, dx = \frac{1}{2} + 2i - \frac{1}{2}(1+2i) = i.$$

Exercises 18.3

1. **(a)** Choosing $x = 0$, $-1 \le y \le 1$ we have $z = iy$, $dz = i\, dy$. Thus

$$\int_C (4z - 1)\, dz = i\int_{-1}^1 (4iy - 1)\, dy = -2i.$$

(b) $\displaystyle\int_C (4z-1)\, dz = \int_{-i}^{i} (4z-1)\, dz = 2z^2 - z\,\Big|_{-i}^{i} = -2i$

2. **(a)** Choosing the line $y = \frac{1}{3}x$, $0 \le x \le 3$ we have $z = x + \frac{1}{3}xi$, $dz = (1 + \frac{1}{3}i)\, dx$. Thus

$$\int_C e^z\, dz = \int_0^3 e^{(1+\frac{1}{3}i)x}\left(1 + \frac{1}{3}\, i\right) dx = e^{(1+\frac{1}{3}i)x}\,\Big|_0^3 = e^{3+i} - e^0 = (e^3\cos 1 - 1) + ie^3 \sin 1.$$

(b) $\displaystyle\int_C e^z\, dz = \int_0^{3+i} e^z\, dz = e^z\,\Big|_0^{3+i} = e^{3+i} - e^0 = (e^3\cos 1 - 1) + ie^3\sin 1$

3. The given integral is independent of the path. Thus

$$\int_C 2z\, dz = \int_{-2+7i}^{2-i} 2z\, dz = z^2\,\Big|_{-2+7i}^{2-i} = 48 + 24i.$$

4. The given integral is independent of the path. Thus

$$\int_C 6z^2\, dz = \int_2^{2-i} 6z^2\, dz = z^3\,\Big|_2^{2-i} = -15 - 24i.$$

5. $\displaystyle\int_0^{3+i} z^2\, dz = \frac{1}{3}z^3\,\Big|_0^{3+i} = 6 + \frac{26}{3}\, i$

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6. $\displaystyle\int_{-2i}^{1}(3z^2-4z+5i)\,dz = z^3-2z^2+5iz\,\Big|_{-2i}^{1} = -19-3i$

7. $\displaystyle\int_{1-i}^{1+i} z^3\,dz = \frac{1}{4}z^4\,\Big|_{1-i}^{1+i} = 0$

8. $\displaystyle\int_{-3i}^{2i}(z^3-z)\,dz = \frac{1}{4}z^4-\frac{1}{2}z^2\,\Big|_{-3i}^{2i} = \frac{123}{4}$

9. $\displaystyle\int_{-i/2}^{1-i}(2z+1)^2\,dz = \frac{1}{6}(2z+1)^3\,\Big|_{-i/2}^{1-i} = -\frac{7}{6}-\frac{22}{3}\,i$

10. $\displaystyle\int_{1}^{i}(iz+1)^3\,dz = \frac{1}{4i}(iz+1)^4\,\Big|_{1}^{i} = -i$

11. $\displaystyle\int_{i/2}^{i} e^{\pi z}\,dz = \frac{1}{\pi}\,e^{\pi z}\,\Big|_{i/2}^{i} = -\frac{1}{\pi}-\frac{1}{\pi}\,i$

12. $\displaystyle\int_{1-i}^{1+2i} ze^{z^2}\,dz = \frac{1}{2}e^{z^2}\,\Big|_{1-i}^{1+2i} = \frac{1}{2}[e^{-3+4i}-e^{-2i}] = \frac{1}{2}(e^{-3}\cos 4-\cos 2)+\frac{1}{2}(e^{-3}\sin 4+\sin 2)i = 0.1918+0.4358i$

13. $\displaystyle\int_{\pi}^{\pi+2i}\sin\frac{z}{2}\,dz = -2\cos\frac{z}{2}\,\Big|_{\pi}^{\pi+2i} = -2\left[\cos\left(\frac{\pi}{2}+i\right)-\cos\frac{\pi}{2}\right] = 2i\sin\frac{\pi}{2}\sinh 1 = 2.3504i$

14. $\displaystyle\int_{1-2i}^{\pi i}\cos z\,dz = \sin z\,\Big|_{1-2i}^{\pi i} = \sin\pi i-\sin(1-2i) = i\sinh\pi-[\sinh 1\cosh 2-i\cos 1\sinh 2]$

$$= -\sin 1\cosh 2+i(\sinh\pi+\cos 1\sinh 2) = -3.1658+13.5083i$$

15. $\displaystyle\int_{\pi i}^{2\pi i}\cosh z\,dz = \sinh z\,\Big|_{\pi i}^{2\pi i} = \sinh 2\pi i-\sinh\pi i = i\sin 2\pi-i\sin\pi = 0$

16. $\displaystyle\int_{i}^{1+\frac{\pi}{2}i}\sinh 3z\,dz = \frac{1}{3}\cosh 3z\,\Big|_{i}^{1+\frac{\pi}{2}i} = \frac{1}{3}\left[\cosh\left(3+\frac{3\pi}{2}\,i\right)-\cosh 3i\right]$

$$= \frac{1}{3}\left[\cosh 3\cos\frac{3\pi}{2}+i\sinh 3\sin\frac{3\pi}{2}-\cos 3\right] = -\frac{1}{3}\cos 3-\frac{1}{3}\,i\sinh 3 = 0.3300-3.3393i$$

17. $\displaystyle\int_{-4i}^{4i}\frac{1}{z}\,dz = \operatorname{Ln} z\,\Big|_{-4i}^{4i} = \operatorname{Ln}4i-\operatorname{Ln}(-4i) = \log_e 4+\frac{\pi}{2}\,i-\left(\log_e 4-\frac{\pi}{2}\,i\right) = \pi i$

18. $\displaystyle\int_{1+i}^{4+4i}\frac{1}{z}\,dz = \operatorname{Ln} z\,\Big|_{1+i}^{4+4i} = \operatorname{Ln}(4+4i)-\operatorname{Ln}(1+i) = \log_e 4\sqrt{2}+\frac{\pi}{4}\,i-\left(\log_e\sqrt{2}+\frac{\pi}{4}\,i\right) = \log_e 4 = 1.3863$

19. $\displaystyle\int_{-4i}^{4i}\frac{1}{z^2}\,dz = -\frac{1}{z}\,\Big|_{-4i}^{4i} = -\left[\frac{1}{4i}-\left(\frac{1}{-4i}\right)\right] = \frac{1}{2}\,i$

20. $\displaystyle\int_{1-i}^{1+\sqrt{3}\,i}\left(\frac{1}{z}+\frac{1}{z^2}\right)dz = \operatorname{Ln} z-\frac{1}{z}\,\Big|_{1-i}^{1+\sqrt{3}\,i} = \log_e 2+\frac{\pi}{3}\,i-\frac{1}{1+\sqrt{3}\,i}-\left(\log_e\sqrt{2}-\frac{\pi}{4}\,i-\frac{1}{1-i}\right)$

$$= \log_e\sqrt{2}+\frac{1}{4}+i\left(\frac{7\pi}{12}+\frac{\sqrt{3}}{4}+\frac{1}{2}\right) = 0.5966+2.7656i$$

21. Integration by parts gives

$$\int e^z\cos z\,dz = \frac{1}{2}\,e^z(\cos z+\sin z)+C$$

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and so

$$\int_{\pi}^{i} e^z \cos z\, dz = \frac{1}{2}\, e^z(\cos z + \sin z)\,\Big|_{\pi}^{i} = \frac{1}{2}[e^i(\cos i + \sin i) - e^{\pi}(\cos \pi + \sin \pi)]$$
$$= \frac{1}{2}[(\cos 1 \cosh 1 - \sin 1 \sinh 1 + e^{\pi}) + i(\cos 1 \sinh 1 + \sin 1 \cosh 1) = 11.4928 + 0.9667i.$$

22. Integration by parts gives

$$\int z \sin z\, dz = -z \cos z + \sin z + C$$

and so

$$\int_0^i z \sin z\, dz = -z \cos z + \sin z\,\Big|_0^i = -i \cos i + \sin i = -i \cosh 1 + i \sinh 1 = -0.3679i.$$

23. Integration by parts gives

$$\int z e^z\, dz = z e^z - e^z + C$$

and so

$$\int_i^{1+i} z e^z\, dz = e^z(z-1)\,\Big|_i^{1+i} = ie^{1+i} + e^i(1-i) = (\cos 1 + \sin 1 - e \sin 1) + i(\sin 1 - \cos 1 + e \cos 1) = -0.9056 + 1.7699i.$$

24. Integration by parts gives

$$\int z^2 e^z\, dz = z^2 e^z - 2z e^z + 2e^z + C$$

and so

$$\int_0^{\pi i} z^2 e^z\, dz = e^z(z^2 - 2z + 2)\,\Big|_0^{\pi i} = e^{\pi i}(-\pi^2 - 2\pi i + 2) - 2 = \pi^2 - 4 + 2\pi i.$$

Exercises 18.4

1. By Theorem 18.9, with $f(z) = 4$,

$$\oint_C \frac{4}{z - 3i}\, dz = 2\pi i \cdot 4 = 8\pi i.$$

2. By Theorem 18.10 with $f(z) = z^2$ and $f'(z) = 2z$,

$$\oint_C \frac{z^2}{(z - 3i)^2}\, dz = \frac{2\pi i}{1!}\, 2(3i) = -12\pi.$$

3. By Theorem 18.9 with $f(z) = e^z$,

$$\oint_C \frac{e^z}{z - \pi i}\, dz = 2\pi i e^{\pi i} = -2\pi i.$$

4. By Theorem 18.9 with $f(z) = 1 + 2e^z$,

$$\oint_C \frac{1 + 2e^z}{z}\, dz = 2\pi i(1 + 2e^0) = 6\pi i.$$

5. By Theorem 18.9 with $f(z) = z^2 - 3z + 4i$,

$$\oint_C \frac{z^2 - 3z + 4i}{z - (-2i)}\, dz = 2\pi i(-4 + 6i + 4i) = -\pi(20 + 8i).$$

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6. By Theorem 18.9 with $f(z) = \dfrac{1}{3}\cos z$,

$$\oint_C \frac{\dfrac{1}{3}\cos z}{z - \dfrac{\pi}{3}}\,dz = 2\pi i\left(\frac{1}{3}\cos\frac{\pi}{3}\right) = \frac{\pi}{3}\,i.$$

7. **(a)** By Theorem 18.9 with $f(z) = \dfrac{z^2}{z+2i}$,

$$\oint_C \frac{\dfrac{z^2}{z+2i}}{z-2i}\,dz = 2\pi i\left(-\frac{4}{4i}\right) = -2\pi.$$

(b) By Theorem 18.9 with $f(z) = \dfrac{z^2}{z-2i}$,

$$\oint_C \frac{\dfrac{z^2}{z-2i}}{z-(-2i)}\,dz = 2\pi i\left(\frac{-4}{-4i}\right) = 2\pi.$$

8. **(a)** By Theorem 18.9 with $f(z) = \dfrac{z^2+3z+2i}{z+4}$,

$$\oint_C \frac{\dfrac{z^2+3z+2i}{z+4}}{z-1}\,dz = 2\pi i\left(\frac{4+2i}{5}\right) = \pi\left(-\frac{4}{5} + \frac{8}{5}\,i\right).$$

(b) By Theorem 18.9 with $f(z) = \dfrac{z^2+3z+2i}{z-1}$,

$$\oint_C \frac{\dfrac{z^2+3z+2i}{z-1}}{z-(-4)}\,dz = 2\pi i\left(\frac{4+2i}{-5}\right) = \pi\left(\frac{4}{5} - \frac{8}{5}\,i\right).$$

9. By Theorem 18.9 with $f(z) = \dfrac{z^2+4}{z-i}$,

$$\oint_C \frac{\dfrac{z^2+4}{z-i}}{z-4i}\,dz = 2\pi i\left(-\frac{12}{3i}\right) = -8\pi.$$

10. By Theorem 18.9 with $f(z) = \dfrac{\sin z}{z+\pi i}$,

$$\oint_C \frac{\dfrac{\sin z}{z+\pi i}}{z-\pi i}\,dz = 2\pi i\left(\frac{\sin \pi i}{2\pi i}\right) = i\sinh\pi.$$

11. By Theorem 18.10 with $f(z) = e^{z^2}$, $f'(z) = 2ze^{z^2}$, and $f''(z) = 4z^2e^{z^2} + 2e^{z^2}$,

$$\oint_C \frac{e^{z^2}}{(z-i)^3}\,dz = \frac{2\pi i}{2!}[-4e^{-1} + 2e^{-1}] = -2\pi e^{-1}i.$$

12. By Theorem 18.10 with $f(z) = z$, $f'(z) = 1$, $f''(z) = 0$, and $f'''(z) = 0$,

$$\oint_C \frac{z}{(z-(-i))^4}\,dz = \frac{2\pi i}{3!}(0) = 0.$$

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13. By Theorem 18.10 with $f(z) = \cos 2z$, $f'(z) = -2\sin 2z$, $f''(z) = -4\cos 2z$, $f'''(z) = 8\sin 2z$, $f^{(4)}(z) = 16\cos 2z$,

$$\oint_C \frac{\cos 2z}{z^5}\,dz = \frac{2\pi i}{4!}(16\cos 0) = \frac{4\pi}{3}\,i.$$

14. By Theorem 18.10 with $f(z) = e^{-z}\sin z$, $f'(z) = e^{-z}\cos z - e^{-z}\sin z$, and $f''(z) = -2e^{-z}\cos z$,

$$\oint_C \frac{e^{-z}\sin z}{z^3}\,dz = \frac{2\pi i}{2!}(-2e^0\cos 0) = -2\pi i.$$

15. **(a)** By Theorem 18.9 with $f(z) = \dfrac{2z+5}{z-2}$,

$$\oint_C \frac{\dfrac{2z+5}{z-2}}{z}\,dz = 2\pi i\left(-\frac{5}{2}\right) = -5\pi i.$$

(b) Since the circle $|z-(-1)| = 2$ encloses only $z = 0$, the value of the integral is the same as in part **(a)**.

(c) From Theorem 18.9 with $f(z) = \dfrac{2z+5}{z}$,

$$\oint_C \frac{\dfrac{2z+5}{z}}{z-2}\,dz = 2\pi i\left(\frac{9}{2}\right) = 9\pi i.$$

(d) Since the circle $|z-(-2i)| = 1$ encloses neither $z = 0$ nor $z = 2$ it follows from the Cauchy-Goursat Theorem, Theorem 18.4, that

$$\oint_C \frac{2z+5}{z(z-2)}\,dz = 0.$$

16. By partial fractions,

$$\oint_C \frac{z}{(z-1)(z-2)}\,dz = 2\oint_C \frac{dz}{z-2} - \oint_C \frac{dz}{z-1}.$$

(a) By the Cauchy-Goursat Theorem, Theorem 18.4,

$$\oint_C \frac{z}{(z-1)(z-2)}\,dz = 0.$$

(b) As in part **(a)**, the integral is 0.

(c) By Theorem 18.4, $\displaystyle\oint_C \frac{dz}{z-2} = 0$ whereas by Theorem 18.9, $\displaystyle\oint_C \frac{dz}{z-1} = 2\pi i$. Thus

$$\oint_C \frac{z}{(z-1)(z-2)}\,dz = -2\pi i.$$

(d) By Theorem 18.9, $\displaystyle\oint_C \frac{dz}{z-1} = 2\pi i$ and $\displaystyle\oint_C \frac{dz}{z-2} = 2\pi i$. Thus

$$\oint_C \frac{z}{(z-1)(z-2)}\,dz = 2(2\pi i) - 2\pi i = 3\pi i.$$

17. **(a)** By Theorem 18.10 with $f(z) = \dfrac{z+2}{z-1-i}$ and $f'(z) = \dfrac{-3-i}{(z-1-i)^2}$,

$$\oint_C \frac{\dfrac{z+2}{z-1-i}}{z^2}\,dz = \frac{2\pi i}{1!}\left(\frac{-3-i}{(-1-i)^2}\right) = -\pi(3+i).$$

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(b) By Theorem 18.9 with $f(z) = \dfrac{z+2}{z^2}$,

$$\oint_C \frac{\dfrac{z+2}{z^2}}{z-(1+i)}\,dz = 2\pi i\left(\frac{3+i}{(1+i)^2}\right) = \pi(3+i).$$

18. **(a)** By Theorem 18.10 with $f(z) = \dfrac{1}{z-4}$, $f'(z) = -\dfrac{1}{(z-4)^2}$, and $f''(z) = \dfrac{2}{(z-4)^3}$,

$$\oint_C \frac{\dfrac{1}{z-4}}{z^3}\,dz = \frac{2\pi i}{2!}\left(\frac{2}{-64}\right) = -\frac{\pi}{32}\,i.$$

(b) By the Cauchy-Goursat Theorem, Theorem 18.4,

$$\oint_C \frac{1}{z^3(z-4)}\,dz = 0.$$

19. By writing
$$\oint_C \left(\frac{e^{2iz}}{z^4} - \frac{z^4}{(z-i)^3}\right)dz = \oint_C \frac{e^{2iz}}{z^4}\,dz - \oint_C \frac{z^4}{(z-i)^3}\,dz$$
we can apply Theorem 18.10 to each integral:

$$\oint_C \frac{e^{2iz}}{z^4}\,dz = \frac{2\pi i}{3!}(-8i) = \frac{8\pi}{3}, \qquad \oint_C \frac{z^4}{(z-i)^3}\,dz = \frac{2\pi i}{2!}(-12) = -12\pi i.$$

Thus
$$\oint_C \left(\frac{e^{2iz}}{z^4} - \frac{z^4}{(z-i)^3}\right)dz = \pi\left(\frac{8}{3} + 12i\right).$$

20. By writing
$$\oint_C \left(\frac{\cosh z}{(z-\pi)^3} - \frac{\sin^2 z}{(2z-\pi)^3}\right)dz = \oint_C \frac{\cosh z}{(z-\pi)^3}\,dz - \oint_C \frac{\frac{1}{8}\sin^2 z}{(z-\frac{\pi}{2})^3}\,dz$$
we apply Theorem 18.4 to the first integral and Theorem 18.10 to the second:

$$\oint_C \frac{\cosh z}{(z-\pi)^3}\,dz = 0, \qquad \oint_C \frac{\frac{1}{8}\sin^2 z}{(z-\frac{\pi}{2})^3}\,dz = \frac{2\pi i}{2!}\left(-\frac{1}{4}\sin^2\frac{\pi}{2}\right) = -\frac{\pi}{4}\,i.$$

Thus
$$\oint_C \left(\frac{\cosh z}{(z-\pi)^3} - \frac{\sin^2 z}{(2z-\pi)^3}\right)dz = \frac{\pi}{4}, i.$$

21. We have
$$\oint_C \frac{1}{z^3(z-1)^2}\,dz = \oint_{C_1} \frac{\dfrac{1}{(z-1)^2}}{z^3}\,dz + \oint_{C_2} \frac{\dfrac{1}{z^3}}{(z-1)^2}\,dz$$
where C_1 and C_2 are the circles $|z| = 1/3$ and $|z-1| = 1/3$, respectively. By Theorem 18.10,

$$\oint_{C_1} \frac{\dfrac{1}{(z-1)^2}}{z^3}\,dz = \frac{2\pi i}{2!}(6) = 6\pi i, \qquad \oint_{C_2} \frac{\dfrac{1}{z^3}}{(z-1)^2}\,dz = \frac{2\pi i}{1!}(-3) = -6\pi i.$$

Thus
$$\oint_C \frac{1}{z^3(z-1)^2}\,dz = 6\pi i - 6\pi i = 0.$$

22. We have
$$\oint_C \frac{1}{z^2(z^2+1)}\,dz = \oint_{C_1} \frac{\dfrac{1}{z^2(z+i)}}{z-i}\,dz + \oint_{C_2} \frac{\dfrac{1}{z^2+1}}{z^2}\,dz$$

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where C_1 and C_2 are the circles $|z-i| = 1/3$ and $|z| = 1/8$, respectively. By Theorems 18.9 and 18.10,

$$\oint_{C_1} \frac{\dfrac{1}{z^2(z+i)}}{z-i}\,dz = 2\pi i\left(\frac{1}{-2i}\right) = -\pi, \qquad \oint_{C_2} \frac{\dfrac{1}{z^2+1}}{z^2}\,dz = \frac{2\pi i}{1!}(0) = 0.$$

Thus $$\oint_C \frac{1}{z^2(z^2+1)}\,dz = -\pi.$$

23. We have $$\oint_C \frac{3z+1}{z(z-2)^2}\,dz = \oint_{C_1} \frac{\dfrac{3z+1}{z}}{(z-2)^2}\,dz - \oint_{C_2} \frac{\dfrac{3z+1}{(z-2)^2}}{z}\,dz$$

where C_1 and C_2 are the closed portions of the curve C enclosing $z = 2$ and $z = 0$, respectively. By Theorems 18.10 and 18.9,

$$\oint_{C_1} \frac{\dfrac{3z+1}{z}}{(z-2)^2}\,dz = \frac{2\pi i}{1!}\left(-\frac{1}{4}\right) = -\frac{\pi}{2}\,i, \qquad \oint_{C_2} \frac{\dfrac{3z+1}{(z-2)^2}}{z}\,dz = 2\pi i\left(\frac{1}{4}\right) = \frac{\pi}{2}\,i.$$

Thus $$\oint_C \frac{3z+1}{z(z-2)^2}\,dz = -\frac{\pi}{2}\,i - \frac{\pi}{2}\,i = -\pi i.$$

24. We have $$\oint_C \frac{e^{iz}}{(z^2+1)^2}\,dz = \oint_{C_1} \frac{\dfrac{e^{iz}}{(z+i)^2}}{(z-i)^2}\,dz - \oint_{C_2} \frac{\dfrac{e^{iz}}{(z-i)^2}}{(z-(-i))^2}\,dz$$

where C_1 and C_2 are the closed portions of the curve C enclosing $z = i$ and $z = -i$, respectively. By Theorem 18.10,

$$\oint_{C_1} \frac{\dfrac{e^{iz}}{(z+i)^2}}{(z-i)^2}\,dz = \frac{2\pi i}{1!}\left(\frac{-4e^{-1}}{-8i}\right) = \pi e^{-1}, \qquad \oint_{C_2} \frac{\dfrac{e^{iz}}{(z-i)^2}}{(z-(-i))^2}\,dz = \frac{2\pi i}{1!}\left(\frac{0}{8i}\right) = 0.$$

Thus $$\oint_C \frac{e^{iz}}{(z^2+1)^2}\,dz = \pi e^{-1}.$$

Chapter 18 Review Exercises

1. True **2.** False **3.** True **4.** True

5. 0 **6.** $\pi(-16+8i)$ **7.** $\pi(6\pi - i)$ **8.** a constant function

9. True (Use partial fractions and write the given integral as two integrals.)

10. True

11. integer not equal to -1; -1

12. 12π

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13. Since $f(z) = z$ is entire, $\displaystyle\int_C (x+iy)\,dz$ is independent of the path C. Thus

$$\oint_C (x+iy)\,dz = \int_{-4}^{3} z\,dz = \frac{z^2}{2}\,\Big|_{-4}^{3} = -\frac{7}{2}\,.$$

14. We have $$\int_C (x-iy)\,dz = \int_{C_1} (x-iy)\,dz + \int_{C_2} (x-iy)\,dz + \int_{C_3} (x-iy)\,dz$$

On C_1, $x = 4$, $0 \le y \le 2$, $z = 4+iy$, $dz = i\,dy$,

$$\int_{C_1} (4-iy)i\,dy = i\int_0^2 (4-iy)\,dy = i\left(4y - \frac{i}{2}y^2\right)\,\Big|_0^2 = 2+8i.$$

On C_2, $y = 2$, $-4 \le x \le 3$, $z = x+2i$, $dz = dx$,

$$\int_{C_2} (x-2i)\,dx = \int_{-4}^{3} (x-2i)\,dx = \frac{1}{2}x^2 - 2ix\,\Big|_{-4}^{3} = -\frac{7}{2} - 14i.$$

On C_3, $x = 3$, $0 \le y \le 2$, $z = 3+iy$, $dz = i\,dy$,

$$\int_{C_3} (3-iy)i\,dy = i\int_2^0 (3-iy)\,dy = i\left(3y - \frac{i}{2}y^2\right)\,\Big|_2^0 = -2-6i.$$

Thus $$\int_C (x-iy)\,dz = 2+8i-\frac{7}{2}-14i-2-6i = -\frac{7}{2}-12i.$$

15. $\displaystyle\int_C |z^2|\,dz = \int_0^2 (t^4+t^2)\,dt + 2i\int_0^2 (t^5+t^3)\,dt = \frac{136}{15} + \frac{88}{3}\,i$

16. $\displaystyle\int_C e^{\pi z}\,dz = \frac{1}{\pi}\int_i^{1+i} e^{\pi z}(\pi\,dz) = \frac{1}{\pi}e^{\pi z}\,\Big|_i^{1+i} = \frac{1}{\pi}(1-e^{\pi})$

17. By the Cauchy-Goursat Theorem, Theorem 18.4, $\displaystyle\oint_C e^{\pi z}\,dz = 0$.

18. $\displaystyle\int_{3i}^{1-i} (4z-6)\,dz = 2z^2 - 6z\,\Big|_{3i}^{1-i} = 12+20i$

19. $\displaystyle\int_C \sin z\,dz = \int_1^{1+4i} \sin z\,dz = -\cos z\,\Big|_1^{1+4i} = \cos 1 - \cos(1+4i) = -14.2144 + 22.9637i$

20. $\displaystyle\int_C (4z^3+3z^2+2z+1)\,dz = \int_0^{2i} (4z^3+3z^2+2z+1)\,dz = z^4+z^3+z^2+z\,\Big|_0^{2i} = 12-6i$

21. On $|z| = 1$, let $z = e^{it}$, $dz = ie^{it}\,dt$, so that

$$\oint_C (z^{-2}+z^{-1}+z+z^2)\,dz = i\int_0^{2\pi} (e^{-2it}+e^{-it}+e^{it}+e^{2it})e^{it}\,dt = -e^{-it} + it + \frac{1}{2}e^{2it} + \frac{1}{3}e^{3it}\,\Big|_0^{2\pi} = 2\pi i.$$

22. By partial-fractions and Theorem 18.9,

$$\oint_C \frac{3z+4}{z^2-1}\,dz = \frac{7}{2}\oint_C \frac{1}{z-1}\,dz - \frac{1}{2}\oint_C \frac{1}{z-(-1)}\,dz = \frac{7}{2}(2\pi i) - \frac{1}{2}(2\pi i) = 6\pi i.$$

23. By Theorem 18.10 with $f(z) = e^{-2z}$, $f'(z) = -2e^{-2z}$, $f''(z) = 4e^{-2z}$, and $f'''(z) = -8e^{-2z}$,

$$\oint_C \frac{e^{-2z}}{z^4}\,dz = \frac{2\pi i}{3!}(-8) = -\frac{8\pi}{3}\,i.$$

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24. By Theorem 18.10 with $f(z) = \dfrac{\cos z}{z-1}$ and $f'(z) = \dfrac{\sin z - \cos z - z\sin z}{(z-1)^2}$,

$$\oint_C \frac{\dfrac{\cos z}{z-1}}{z^2}\,dz = \frac{2\pi i}{1!}\left(\frac{-1}{1}\right) = -2\pi i.$$

25. By Theorem 18.9 with $f(z) = \dfrac{1}{2(z+3)}$,

$$\oint_C \frac{\dfrac{1}{2(z+3)}}{(z-(-1/2))}\,dz = 2\pi i\left(\frac{1}{5}\right) = \frac{2\pi}{5}\,i.$$

26. Since the function $f(z) = z/\sin z$ is analytic within and on the given simple closed contour C, it follows from the Cauchy-Goursat Theorem, Theorem 18.4, that

$$\oint_C z\csc z\,dz = 0.$$

27. Using the principle of deformation of contours we choose C to be the more convenient circular contour $|z+i| = \frac{1}{4}$. On this circle $z = -i + \frac{1}{4}e^{it}$ and $dz = \frac{1}{4}ie^{it}\,dt$. Thus

$$\oint_C \frac{z}{z+i}\,dz = i\int_0^{2\pi}\left(\frac{1}{4}e^{it} - i\right)dt = 2\pi.$$

28. **(a)** By Theorem 18.9 with $f(z) = \dfrac{e^{i\pi z}}{2(z-2)}$,

$$\oint_C \frac{\dfrac{e^{i\pi z}}{2(z-2)}}{z-1/2}\,dz = 2\pi i\left(\frac{e^{i\pi/2}}{-3}\right) = \frac{2\pi}{3}.$$

(b) By Theorem 18.9 with $f(z) = \dfrac{e^{i\pi z}}{2z-1}$,

$$\oint_C \frac{\dfrac{e^{i\pi z}}{2z-1}}{z-2}\,dz = 2\pi i\left(\frac{e^{2\pi i}}{3}\right) = \frac{2\pi}{3}\,i.$$

(c) By the Cauchy-Goursat Theorem, Theorem 18.4,

$$\oint_C \frac{e^{i\pi z}}{2z^2 - 5z + 2}\,dz = 0.$$

29. For $f(z) = z^n g(z)$ we have $f'(z) = z^n g'(z) + nz^{n-1}g(z)$ and so

$$\frac{f'(z)}{f(z)} = \frac{z^n g'(z) + nz^{n-1}g(z)}{z^n g(z)} = \frac{g'(z)}{g(z)} + \frac{n}{z}.$$

Thus by Theorem 18.4 and (4) of Section 18.2,

$$\oint_C \frac{f'(z)}{f(z)}\,dz = \oint_C \frac{g'(z)}{g(z)}\,dz + n\oint_C \frac{1}{z}\,dz = 0 + n(2\pi i) = 2n\pi i.$$

30. We have

$$\left|\int_C \operatorname{Ln}(z+1)\,dz\right| \le |\max \text{ of } \operatorname{Ln}(z+1) \text{ on } C| \cdot 2,$$

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where 2 is the length of the line segment. Now

$$|\text{Ln}(z+1)| \le |\log_e(z+1)| + |\text{Arg}(z+1)|.$$

But $\max \text{Arg}(z+1) = \pi/4$ when $z = i$ and $\max|z+1| = \sqrt{10}$ when $z = 2+i$. Thus,

$$\left|\int_C \text{Ln}(z+1)\,dz\right| \le \left(\frac{1}{2}\log_e 10 + \frac{\pi}{4}\right)2 = \log_e 10 + \frac{\pi}{2}.$$

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19 Series and Residues

Exercises 19.1

1. $5i$, -5, $-5i$, 5, $5i$

2. $2-i$, 1, $2+i$, 3, $2-i$

3. 0, 2, 0, 2, 0

4. $1+i$, $2i$, $-2+2i$, -4, $-4-4i$

5. Converges. To see this write the general term as $\dfrac{3i+2/n}{1+i}$.

6. Converges. To see this write the general term as $\left(\dfrac{2}{5}\right)^n \dfrac{1+n2^{-n}i}{1+3n5^{-n}i}$.

7. Converges. To see this write the general term as $\dfrac{(i+2/n)^2}{i}$.

8. Diverges. To see this consider the term $\dfrac{n}{n+1}\,i^n$ and take n to be an odd positive integer.

9. Diverges. To see this write the general term as $\sqrt{n}\left(1+\dfrac{1}{\sqrt{n}}\,i^n\right)$.

10. Converges. The real part of the general term converges to 0 and the imaginary part of the general term converges to π.

11. $\mathrm{Re}(z_n) = \dfrac{8n^2+n}{4n^2+1} \to 2$ as $n\to\infty$, and $\mathrm{Im}(z_n) = \dfrac{6n^2-4n}{4n^2+1} \to \dfrac{3}{2}$ as $n\to\infty$.

12. Write $z_n = \left(\dfrac{1}{4}+\dfrac{1}{4}\,i\right)^n$ in polar form as $z_n = \left(\dfrac{\sqrt{2}}{4}\right)^n \cos n\theta + i\left(\dfrac{\sqrt{2}}{4}\right)^n \sin n\theta$. Now

$$\mathrm{Re}(z_n) = \left(\frac{\sqrt{2}}{4}\right)^n \cos n\theta \to 0 \text{ as } n\to\infty \quad \text{and} \quad \mathrm{Im}(z_n) = \left(\frac{\sqrt{2}}{4}\right)^n \sin n\theta \to 0 \text{ as } n\to\infty$$

since $\sqrt{2}/4 < 1$.

13. $S_n = \dfrac{1}{1+2i} - \dfrac{1}{2+2i} + \dfrac{1}{2+2i} - \dfrac{1}{3+2i} + \dfrac{1}{3+2i} - \dfrac{1}{4+2i} + \cdots + \dfrac{1}{n+2i} - \dfrac{1}{n+1+2i} = \dfrac{1}{1+2i} - \dfrac{1}{n+1+2i}$

Thus, $\lim\limits_{n\to\infty} S_n = \dfrac{1}{1+2i} = \dfrac{1}{5} - \dfrac{2}{5}\,i$.

14. By partial fractions, $\dfrac{i}{k(k+1)} = \dfrac{i}{k} - \dfrac{i}{k+1}$ and so

$$S_n = i - \frac{i}{2} + \frac{i}{2} - \frac{i}{3} + \frac{i}{3} - \frac{i}{4} + \cdots + \frac{i}{n} - \frac{i}{n+1} = i - \frac{i}{n+1}\,.$$

Thus $\lim\limits_{n\to\infty} S_n = i$.

15. We identify $a=1$ and $z=1-i$. Since $|z| = \sqrt{2} > 1$ the series is divergent.

16. We identify $a = 4i$ and $z = 1/3$. Since $|z| = 1/3 < 1$ the series converges to

$$\frac{4i}{1-1/3} = 6i.$$

17. We identify $a = i/2$ and $z = i/2$. Since $|z| = 1/2 < 1$ the series converges to

$$\frac{i/2}{1-i/2} = -\frac{1}{5} + \frac{2}{5}\,i.$$

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18. We identify $a = 1/2$ and $z = i$. Since $|z| - 1$ the series is divergent.

19. We identify $a = 3$ and $z = 2/(1+2i)$. Since $|z| = 2/\sqrt{5} < 1$ the series converges to

$$\frac{3}{1 - \dfrac{2}{1+2i}} = \frac{9}{5} - \frac{12}{5}\,i.$$

20. We identify $a = -1/(1+i)$ and $z = i/(1+i)$. Since $|z| = 1/\sqrt{2} < 1$ the series converges to

$$\frac{-\dfrac{1}{1+i}}{1 - \dfrac{i}{1+i}} = -1.$$

21. From
$$\lim_{n\to\infty} \left| \frac{\dfrac{1}{(1-2i)^{n+2}}}{\dfrac{1}{(1-2i)^{n+1}}} \right| = \frac{1}{|1-2i|} = \frac{1}{\sqrt{5}}$$

we see that the radius of convergence is $R = \sqrt{5}$. The circle of convergence is $|z - 2i| = \sqrt{5}$.

22. From
$$\lim_{n\to\infty} \left| \frac{\dfrac{1}{n+1}\left(\dfrac{i}{1+i}\right)^{n+1}}{\dfrac{1}{n}\left(\dfrac{i}{1+i}\right)^{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} \left| \frac{i}{1+i} \right| = \frac{1}{\sqrt{2}}$$

we see that the radius of convergence is $R = \sqrt{2}$. The circle of convergence is $|z| = \sqrt{2}$.

23. From
$$\lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}}{(n+1)2^{n+1}}}{\dfrac{(-1)^n}{n2^n}} \right| = \lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{1}{2}$$

we see that the radius of convergence is $R = 2$. The circle of convergence is $|z - 1 - i| = 2$.

24. From
$$\lim_{n\to\infty} \left| \frac{\dfrac{1}{(n+1)^2(3+4i)^{n+1}}}{\dfrac{1}{n^2(3+4i)^n}} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 \frac{1}{|3+4i|} = \frac{1}{5}$$

we see that the radius of convergence is $R = 5$. The circle of convergence is $|z + 3i| = 5$.

25. From
$$\lim_{n\to\infty} \sqrt[n]{|1+3i|^n} = |1+3i| = \sqrt{10}$$

we see that the radius of convergence is $R = 1/\sqrt{10}$. The circle of convergence is $|z - i| = 1/\sqrt{10}$.

26. From
$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^n}\right|} = \lim_{n\to\infty} \frac{1}{n} = 0$$

we see that the radius of convergence is ∞. The power series with center 0 converges absolutely for all z.

27. From
$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{5^{2n}}\right|} = \lim_{n\to\infty} \frac{1}{25} = \frac{1}{25}$$

we see that the radius of convergence is $R = 25$. The circle of convergence is $|z - 4 - 3i| = 25$.

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28. From $$\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\left(\frac{1+2i}{2}\right)^n\right|}=\lim_{n\to\infty}\left|\frac{1+2i}{2}\right|=\frac{\sqrt{5}}{2}$$

we see that the radius of convergence is $R=2/\sqrt{5}$. The circle of convergence is $|z+2i|=2/\sqrt{5}$.

29. The circle of convergence is $|z-i|=2$. Since the series of absolute values

$$\sum_{k=1}^{\infty}\left|\frac{(z-1)^k}{k2^k}\right|=\sum_{k=1}^{\infty}\frac{|z-1|^k}{k2^k}=\sum_{k=1}^{\infty}\frac{2^k}{k2^k}=\sum_{k=1}^{\infty}\frac{1}{k}$$

is the divergent harmonic series. But $z=-2+i$ is on the circle of convergence and $(z-i)^k=(-2)^k$. The series

$$\sum_{k=1}^{\infty}\frac{(-2)^k}{k2^k}=\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$$

is convergent.

30. **(a)** The circle of convergence is $|z|=1$. Since the series of absolute values

$$\sum_{k=1}^{\infty}\left|\frac{z^k}{k^2}\right|=\sum_{k=1}^{\infty}\frac{|z|^k}{k^2}=\sum_{k=1}^{\infty}\frac{1}{k^2}$$

converges, the given series is absolutely convergent for every z on $|z|=1$. Since absolute convergence implies convergence, the given series converges for all z on $|z|=1$.

(b) The circle of convergence is $|z|=1$. On the circle, $n|z|^n\to\infty$ as $n\to\infty$. This implies $nz^n\not\to 0$ as $n\to\infty$. Thus by Theorem 19.3 the series is divergent for every z on the circle $|z|=1$.

Exercises 19.2

1. $\dfrac{z}{1+z}=z[1-z+z^2-z^3+\cdots]=z-z^2+z^3-z^4+\cdots=\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}z^k;\quad R=1$

2. $\dfrac{1}{4-2z}=\dfrac{1}{4}\cdot\left[1+\dfrac{z}{2}+\dfrac{z^2}{2^2}+\dfrac{z^3}{2^3}+\cdots\right]=\dfrac{1}{4}\displaystyle\sum_{k=0}^{\infty}\frac{z^k}{2^k};\quad R=2$

3. Differentiating $\dfrac{1}{1+2z}=1-2z+2^2z^2-2^3z^3+\cdots$ gives $\dfrac{-2}{(1+2z)^2}=-2+2\cdot 2^2z-3\cdot 2^3z^2+\cdots$. Thus

$$\frac{1}{(1+2z)}=1-2\cdot(2z)+3\cdot(2z)^2-\cdots=\sum_{k=1}^{\infty}(-1)^{k-1}k(2z)^{k-1}.\ R=1/2$$

4. Using the binomial series gives

$$\frac{z}{(1-z)}=z\left[1+3z+\frac{3\cdot 4}{2!}z+\frac{3\cdot 4\cdot 5}{3!}z^3+\cdots\right]=z+3z^2+\frac{3\cdot 4}{2!}z^3+\frac{3\cdot 4\cdot 5}{3!}z^4+\cdots.\ R=1$$

5. Replacing z in $e^z=\displaystyle\sum_{k=0}^{\infty}\frac{z^k}{k!}$ by $-2z$ gives $e^{-2z}=\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(2z)^k.\quad R=\infty$

6. Replacing z in $e^z=\displaystyle\sum_{k=0}^{\infty}\frac{z^k}{k!}$ by $-z^2$ and multiplying the result by z gives $ze^{-z^2}=\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}z^{2k+1}.\quad R=\infty$

7. Subtracting the series for e^z and e^{-z} gives $\sinh z=\dfrac{1}{2}(e^z-e^{-z})=\displaystyle\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!}.\quad R=\infty$

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8. Adding the series for e^z and e^{-z} gives $\cosh z = \dfrac{1}{2}(e^z + e^{-z}) = \displaystyle\sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}$. $R = \infty$

9. Replacing z in $\cos z = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$ by $z/2$ gives $\cos \dfrac{z}{2} = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{z}{2}\right)^{2k}$. $R = \infty$

10. Replacing z in $\sin z = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$ by $3z$ gives $\sin 3z = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{(3z)^{2k+1}}{(2k+1)!}$. $R = \infty$

11. Replacing z in $\sin z = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$ by z^2 gives $\sin z^2 = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{z^{4k+2}}{(2k+1)!}$. $R = \infty$

12. Using the identity $\cos z = \dfrac{1}{2}(1 + \cos 2z)$ and the series $\cos z = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$ gives

$$\cos^2 z = \frac{1}{2} + \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \frac{(2z)^{2k}}{(2k)!} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k-1}}{(2k)!} z^{2k}. \; R = \infty$$

13. Using (6) of Section 19.1,

$$\frac{1}{z} = \frac{1}{1 + (z-1)} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots = \sum_{k=0}^{\infty} (-1)^k (z-1)^k. \; R = 1$$

14. Using (6) of Section 19.1,

$$\frac{1}{z} = \frac{1}{1 + i + (z-1-i)} = \frac{1}{1+i} \cdot \frac{1}{1 + \dfrac{z-1-i}{1+i}} = \frac{1}{1+i} \left[1 - \frac{(z-1-i)}{1+i} + \frac{(z-1-i)^2}{(1+i)^2} - \frac{(z-1-i)^3}{(1+i)^3} + \cdots\right]$$

$$= \frac{1}{1+i} - \frac{(z-1-i)}{(1+i)^2} + \frac{(z-1-i)^2}{(1+i)^3} - \frac{(z-1-i)^3}{(1+i)^4} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{(z-1-i)^k}{(1+i)^{k+1}}. \; R = \sqrt{2}$$

15. Using (5) of Section 19.1,

$$\frac{1}{3-z} = \frac{1}{3 - 2i - (z - 2i)} = \frac{1}{3-2i} \cdot \frac{1}{1 - \dfrac{z-2i}{3-2i}}$$

$$= \frac{1}{3-2i} \left[1 + \frac{z-2i}{3-2i} + \frac{(z-2i)^2}{(3-2i)^2} + \frac{(z-2i)^3}{(3-2i)^3} + \cdots\right]$$

$$= \frac{1}{3-2i} + \frac{z-2i}{(3-2i)^2} + \frac{(z-2i)^2}{(3-2i)^3} + \frac{(z-2i)^3}{(3-2i)^4} + \cdots = \sum_{k=0}^{\infty} \frac{(z-2i)^k}{(3-2i)^{k+1}}. \; R = \sqrt{13}$$

16. Using (6) of Section 19.1,

$$\frac{1}{1+z} = \frac{1}{1 - i + z + i} = \frac{1}{1-i} \cdot \frac{1}{1 + \dfrac{z+i}{1-i}} = \frac{1}{1-i} \left[1 - \frac{z+i}{1-i} + \frac{(z+i)^2}{(1-i)^2} - \frac{(z+i)^3}{(1-i)^3} + \cdots\right]$$

$$= \frac{1}{1-i} - \frac{z+i}{(1-i)^2} + \frac{(z+i)^2}{(1-i)^3} - \frac{(z+i)^3}{(1-i)^4} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{(z+i)^k}{(1-i)^{k+1}}. \; R = \sqrt{2}$$

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17. Using (5) of Section 19.1,

$$\frac{z-1}{3-z} = (z-1)\cdot\frac{1}{2-(z-1)} = \frac{(z-1)}{2}\cdot\frac{1}{1-\dfrac{z-1}{2}} = \frac{z-1}{2}\left[1+\frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}+\cdots\right]$$

$$= \frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}+\frac{(z-1)^4}{2^4}+\cdots = \sum_{k=1}^{\infty}\frac{(z-1)^k}{2^k}. \quad R=2$$

18. Using (5) of Section 19.1,

$$\frac{1+z}{1-z} = -1+\frac{2}{1-z} = -1+\frac{2}{1-i-(z-i)} = -1+\frac{2}{1-i}\cdot\frac{1}{1-\dfrac{z-i}{1-i}}$$

$$= -1+\frac{2}{1-i}\left[1+\frac{z-i}{1-i}+\frac{(z-i)^2}{(1-i)^2}+\frac{(z-i)^3}{(1-i)^3}+\cdots\right]$$

$$= -1+\frac{2}{1-i}+\frac{2(z-i)}{(1-i)^2}+\frac{2(z-i)^2}{(1-i)^3}+\frac{2(z-i)^3}{(1-i)^4}+\cdots = -1+\sum_{k=0}^{\infty}\frac{2(z-i)^k}{(1-i)^{k+1}}. \quad R=\sqrt{2}$$

19. Using (8) of Section 19.2,

$$\cos z = \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2\cdot 1!}\left(z-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{2\cdot 2!}\left(z-\frac{\pi}{4}\right)^2+\frac{\sqrt{2}}{2\cdot 3!}\left(z-\frac{\pi}{4}\right)^3+\cdots. \quad R=\infty$$

20. Using the identity $\sin z = \cos(z-\pi/2)$ and (14) of Section 19.2, $\sin z = \displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{(z-\frac{\pi}{2})^{2k}}{(2k)!}$. $R=\infty$

21. Using $e^z = e^{3i}\cdot e^{z-3i}$ and (12) of Section 19.2, $e^z = e^{3i}\displaystyle\sum_{k=0}^{\infty}\frac{(z-3i)^k}{k!}$. $R=\infty$

22. Using $(z-1)e^{-2z} = e^2(z-1)e^{-2(z-1)}$ and (12) of Section 19.2,

$$(z-1)e^{-2z} = e^2\sum_{k=0}^{\infty}\frac{(-1)^k 2^k}{k!}(z-1)^{k+1}. \quad R=\infty$$

23. Using (8) of Section 19.2, $\tan z = z+\dfrac{1}{3}z^3+\dfrac{2}{15}z^5+\cdots$.

24. Using (8) of Section 19.2, $e^{1/(1+z)} = e-ez+\dfrac{3e}{2}z^2-\cdots$.

25. Using (5) of Section 19.1,

$$f(z) = \frac{1}{z-2i}-\frac{1}{z-i} = -\frac{1}{2i}\cdot\frac{1}{1-z/2i}+\frac{1}{i}\,\frac{1}{1-z/i}$$

$$= -\frac{1}{2i}\left(1+\frac{z}{2i}+\frac{z}{(2i)^2}+\frac{z^3}{(2i)^3}+\cdots\right)+\frac{1}{i}\left(1+\frac{z}{i}+\frac{z^2}{i^2}+\frac{z^3}{i^3}+\cdots\right) = -\frac{i}{2}-\frac{3}{4}z+\frac{7i}{8}z^2+\frac{15}{16}z^3-\cdots.$$

The radius of convergence is $R=1$.

26. Using (5) and (6) of Section 19.1,

$$f(z) = \frac{2}{z+1}-\frac{1}{z-3} = 2\cdot\frac{1}{1+z}+\frac{1}{3}\cdot\frac{1}{1-z/3} = 2(1-z+z^2-z^3+\cdots)+\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots\right)$$

$$= \frac{7}{3}-\frac{17}{9}z+\frac{55}{27}z^2-\frac{161}{81}z^3+\cdots.$$

27. The distance from $2+5i$ to i is $|2+5i-i| = |2+4i| = 2\sqrt{5}$.

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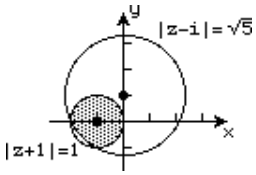
28. The distance from πi to 0 is $|\pi i| = \pi$.

29. The Taylor series are

$$f(z) = \sum_{k=0}^{\infty} (-1)^k (z+1)^k, \; R = 1; \quad \text{and} \quad f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{(z-i)^k}{(2+i)^{k+1}}, \quad R = \sqrt{5}.$$

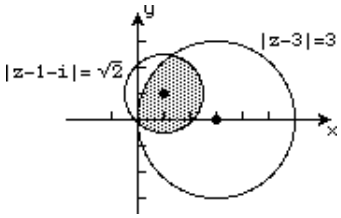


30. The series are

$$f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{(z-3)^k}{3^{k+1}}, \quad R = 3$$

and

$$f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{(z-1-i)^k}{(1+i)^{k+1}}, \quad R = \sqrt{2}.$$



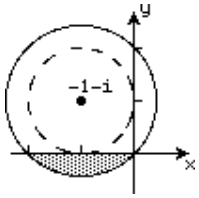
31. **(a)** The distance from z_0 to the branch cut is one unit.

(b) The first term of the series defined by (8) of this section is

$$f(-1+i) = \text{Ln}(-1+i) = \log_e \sqrt{2} + \frac{3\pi}{4}\, i = \frac{1}{2} \log_e 2 + \frac{3\pi}{4}\, i.$$

The subsequent terms of the series come from $f'(z) = \dfrac{1}{z}$, $f''(z) = -\dfrac{1}{z^2}$, and so on, evaluated at $-1+i$.

(c) The series converges within the circle $|z+1-i| = \sqrt{2}$. Although the series converges in the shaded region, it does not converge to (or represent) $\text{Ln}\, z$ in this region.



32. **(a)** $R = 1$, which is the distance from the origin to $z = -1$.

(b) Using (8) of this section (or integrating the series for $1/(1+z)$ we obtain for $R = 1$,

$$\text{Ln}(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, z^k.$$

(c) By replacing z in part (b) by $-z$ we obtain for $R = 1$,

$$\text{Ln}(1-z) = -\sum_{k=0}^{\infty} \frac{z^k}{k}.$$

(d) One way of obtaining the Maclaurin series for $\text{Ln}\left(\dfrac{1+z}{1-z}\right)$ is to use (8) of this section. Alternatively, let us write

$$\text{Ln}\left(\frac{1+z}{1-z}\right) = \text{Ln}(1+z) - L(1-z)$$

and subtract the series in parts **(b)** and **(c)**. This gives for the common circle of convergence $|z| = 1$,

$$\text{Ln}\left(\frac{1+z}{1-z}\right) = 2z + \frac{2}{3}z^3 + \frac{2}{5}z^5 + \frac{2}{7}z^7 + \cdots = 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)}\, z^{2k+1}.$$

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But recall that in general $\text{Ln}(z_1/z_2) \neq \text{Ln}z_1 - \text{Ln}z_2$ since $\text{Ln}z_1$ and $\text{Ln}z_2$ could differ by a constant multiple of i. That is $\text{Ln}z_1 - \text{Ln}z_2 = Ci$, for some C. So

$$\text{Ln}\left(\frac{1+z}{1-z}\right) = \text{Ln}(1+z) - \text{Ln}(1-z) - Ci.$$

When $z = 0$ we obtain $\text{Ln}\,1 = \text{Ln}\,1 - \text{Ln}\,1 - Ci$. Since $\text{Ln}\,1 = 0$ we get $C = 0$.

33. From $e^z \approx 1 + z + \dfrac{z^2}{2}$ we obtain

$$e^{(1+i)/10} \approx 1 + \frac{1+i}{10} + \frac{(1+i)^2}{100} = 1.1 + 0.12i.$$

34. From $\sin z \approx z - \dfrac{z^3}{6}$ we obtain

$$\sin\left(\frac{1+i}{10}\right) \approx \frac{1+i}{10} - \frac{1}{6}\left(\frac{1+i}{10}\right)^3 = \frac{1}{10} + \frac{1}{10}\,i - \frac{1}{6}\left(\frac{-2+2i}{1000}\right) = \frac{301}{3000} + \frac{299}{3000}\,i.$$

35. Using the series $e^z = \displaystyle\sum_{k=0}^{\infty} \frac{z^k}{k!}$ we obtain $e^{-t^2} = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{k!}$. Thus

$$\frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^z t^{2k}\,dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)}\, z^{2k+1}.$$

36. $e^{iz} = \displaystyle\sum_{k=0}^{\infty} \frac{(iz)^k}{k!} = 1 + i\,\frac{z}{1!} - \frac{z^2}{2!} - i\,\frac{z^3}{3!} + \frac{z^4}{4!} + i\,\frac{z^5}{5!} - \frac{z^6}{6!} - i\,\frac{z^7}{7!} + \cdots$

$$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) + i\left(\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) = \cos z + i \sin z$$

Exercises 19.3

1. $f(z) = \dfrac{1}{z}\left(1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \cdots\right) = \dfrac{1}{z} - \dfrac{z}{2!} + \dfrac{z^3}{4!} - \dfrac{z^5}{6!} + \cdots$

2. $f(z) = \dfrac{1}{z^5}\left[z - \left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \cdots\right)\right] = \dfrac{1}{3!z^2} - \dfrac{1}{5!} + \dfrac{z^2}{7!} - \dfrac{z^4}{9!} + \cdots$

3. $f(z) = 1 - \dfrac{1}{1!z^2} + \dfrac{1}{2!z^4} - \dfrac{1}{3!z^6} + \cdots$

4. $f(z) = \dfrac{1}{z^2}\left[1 - \left(1 + \dfrac{z}{1!} + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots\right)\right] = -\dfrac{1}{1!z} - \dfrac{1}{2!} - \dfrac{z}{3!} - \dfrac{z^2}{4!} - \cdots$

5. $f(z) = \dfrac{e \cdot e^{z-1}}{z-1} = \dfrac{e}{z-1}\left(1 + \dfrac{(z-1)}{1!} + \dfrac{(z-1)^2}{2!} + \dfrac{(z-1)^3}{3!} + \cdots\right) = \dfrac{e}{z-1} + \dfrac{e}{1!} + \dfrac{e(z-1)}{2!} + \dfrac{e(z-1)^2}{3!} + \cdots$

6. $f(z) = z\left(1 - \dfrac{1}{2!z^2} + \dfrac{1}{4!z^4} - \dfrac{1}{6!z^6} + \cdots\right) = z - \dfrac{1}{2!z} + \dfrac{1}{4!z^3} - \dfrac{1}{6!z^5} + \cdots$

7. $f(z) = -\dfrac{1}{3z} \cdot \dfrac{1}{1 - \dfrac{z}{3}} = -\dfrac{1}{3z}\left[1 + \dfrac{z}{3} + \dfrac{z^2}{3^2} + \dfrac{z^3}{3^3} + \cdots\right] = -\dfrac{1}{3z} - \dfrac{1}{3^2} - \dfrac{z}{3^3} - \dfrac{z^2}{3^4} - \cdots$

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8. $f(z) = \dfrac{1}{z^2} \cdot \dfrac{1}{1 - \dfrac{3}{z}} = \dfrac{1}{z^2}\left[1 + \dfrac{3}{z} + \dfrac{3^2}{z^2} + \dfrac{3^3}{z^3} + \cdots\right] = \dfrac{1}{z^2} + \dfrac{3}{z^3} + \dfrac{3^2}{z^4} + \dfrac{3^3}{z^5} + \cdots$

9. $f(z) = \dfrac{1}{z-3} \cdot \dfrac{1}{3+z-3} = \dfrac{1}{3(z-3)} \cdot \dfrac{1}{1 + \dfrac{z-3}{3}} = \dfrac{1}{3(z-3)}\left[1 - \dfrac{z-3}{3} + \dfrac{(z-3)^2}{3^2} - \dfrac{(z-3)^3}{3^3} + \cdots\right]$

$= \dfrac{1}{3(z-3)} - \dfrac{1}{3^2} + \dfrac{z-3}{3^3} - \dfrac{(z-3)^2}{3^4} + \cdots$

10. $f(z) = \dfrac{1}{z-3} \cdot \dfrac{1}{z-3+3} = \dfrac{1}{(z-3)^2} \cdot \dfrac{1}{1 + \dfrac{3}{z-3}} = \dfrac{1}{(z-3)^2}\left[1 - \dfrac{3}{z-3} + \dfrac{3^2}{(z-3)^2} - \dfrac{3^3}{(z-3)^3} + \cdots\right]$

$= \dfrac{1}{(z-3)^2} - \dfrac{3}{(z-3)^3} + \dfrac{3^2}{(z-3)^4} - \dfrac{3^3}{(z-3)^5} + \cdots$

11. $f(z) = \dfrac{1}{3}\left[\dfrac{1}{z-3} - \dfrac{1}{z}\right] = \dfrac{1}{3}\left[\dfrac{1}{z-4+1} - \dfrac{1}{4+z-4}\right] = \dfrac{1}{3}\left[\dfrac{1}{z-4} \cdot \dfrac{1}{1 + \dfrac{1}{z-4}} - \dfrac{1}{4} \cdot \dfrac{1}{1 + \dfrac{z-4}{4}}\right]$

$= \dfrac{1}{3}\left[\dfrac{1}{z-4}\left(1 - \dfrac{1}{z-4} + \dfrac{1}{(z-4)^2} - \dfrac{1}{(z-4)^3} + \cdots\right) - \dfrac{1}{4}\left(1 - \dfrac{z-4}{4} + \dfrac{(z-4)^2}{4^2} - \dfrac{(z-4)^3}{4^3} + \cdots\right)\right]$

$= \cdots - \dfrac{1}{3(z-4)^2} + \dfrac{1}{3(z-1)} - \dfrac{1}{12} + \dfrac{z-4}{3 \cdot 4^2} - \dfrac{(z-4)^2}{3 \cdot 4^3} + \cdots$

12. $f(z) = \dfrac{1}{3}\left[\dfrac{1}{z-3} - \dfrac{1}{z}\right] = \dfrac{1}{3}\left[\dfrac{1}{-4+z+1} - \dfrac{1}{z+1-1}\right] = \dfrac{1}{3}\left[-\dfrac{1}{4} \cdot \dfrac{1}{1 - \dfrac{z+1}{4}} - \dfrac{1}{z+1} \cdot \dfrac{1}{1 - \dfrac{1}{z+1}}\right]$

$= \dfrac{1}{3}\left[-\dfrac{1}{4}\left(1 + \dfrac{z+1}{4} + \dfrac{(z+1)^2}{4^2} + \dfrac{(z+1)^3}{4^3} + \cdots\right) - \dfrac{1}{z+1}\left(1 + \dfrac{1}{z+1} + \dfrac{1}{(z+1)^2} + \dfrac{1}{(z+1)^3} + \cdots\right)\right]$

$= \cdots - \dfrac{1}{(z+1)^2} - \dfrac{1}{z+1} - \dfrac{1}{12} - \dfrac{z+1}{3 \cdot 4^2} - \dfrac{(z+1)^2}{3 \cdot 4^3} - \cdots$

13. $f(z) = \dfrac{1}{z-2} - \dfrac{1}{z-1} = -\dfrac{1}{2} \cdot \dfrac{1}{1 - \dfrac{z}{2}} - \dfrac{1}{z} \cdot \dfrac{1}{1 - \dfrac{1}{z}}$

$= -\dfrac{1}{2}\left(1 + \dfrac{z}{2} + \dfrac{z^2}{2^2} + \dfrac{z^3}{2^3} + \cdots\right) - \dfrac{1}{z}\left(1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \dfrac{1}{z^3} + \cdots\right) = \cdots - \dfrac{1}{z^2} - \dfrac{1}{z} - \dfrac{1}{2} - \dfrac{z}{2^2} - \dfrac{z^2}{2^3} - \cdots$

14. $f(z) = \dfrac{1}{z-2} - \dfrac{1}{z-1} = \dfrac{1}{z} \cdot \dfrac{1}{1 - \dfrac{2}{z}} - \dfrac{1}{z} \cdot \dfrac{1}{1 - \dfrac{1}{z}} = \dfrac{1}{z}\left(1 + \dfrac{2}{z} + \dfrac{2^2}{z^2} + \dfrac{2^3}{z^3} + \cdots\right) - \dfrac{1}{z}\left(1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \dfrac{1}{z^3} + \cdots\right)$

$= \dfrac{1}{z^2} + \dfrac{2^2 - 1}{z^3} + \dfrac{2^3 - 1}{z^4} + \dfrac{2^4 - 1}{z^5} + \cdots$

15. $f(z) = \dfrac{1}{z-1} \cdot \dfrac{-1}{1-(z-1)} = \dfrac{-1}{z-1}[1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots] = -\dfrac{1}{z-1} - 1 - (z-1) - (z-1)^2 - \cdots$

16. $f(z) = \dfrac{1}{z-2} \cdot \dfrac{1}{1+(z-2)} = \dfrac{1}{z-2}[1 - (z-2) + (z-2)^2 - (z-2)^3 + \cdots] = \dfrac{1}{z-2} - 1 + (z-2) - (z-2)^2 + \cdots$

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17. $f(z) = \dfrac{1/3}{z+1} + \dfrac{2/3}{z-2} = \dfrac{1}{3(z+1)} + \dfrac{2}{3}\cdot\dfrac{1}{-3+(z+1)} = \dfrac{1}{3(z+1)} - \dfrac{2}{9}\cdot\dfrac{1}{1-\dfrac{z+1}{3}}$

$= \dfrac{1}{3(z+1)} - \dfrac{2}{9}\left[1 + \dfrac{z+1}{3} + \dfrac{(z+1)^2}{3^2} + \dfrac{(z+1)^3}{3^3} + \cdots\right]$

$= \dfrac{1}{3(z+1)} - \dfrac{2}{9} - \dfrac{2(z+1)}{3^3} - \dfrac{2(z+1)^2}{3^4} - \cdots$

18. $f(z) = \dfrac{1}{3(z+1)} + \dfrac{2}{3}\cdot\dfrac{1}{(z+1)-3} = \dfrac{1}{3(z+1)} + \dfrac{2}{3(z+1)}\cdot\dfrac{1}{1-\dfrac{3}{z+1}}$

$= \dfrac{1}{3(z+1)} + \dfrac{2}{3(z+1)}\left(1 + \dfrac{3}{z+1} + \dfrac{3^2}{(z+1)^2} + \dfrac{3^3}{(z+1)^3} + \cdots\right)$

$= \dfrac{1}{z+1} + \dfrac{2}{(z+1)^2} + \dfrac{2\cdot 3}{(z+1)^3} + \dfrac{2\cdot 3^2}{(z+1)^4} + \cdots$

19. $f(z) = \dfrac{1/3}{z+1} + \dfrac{2/3}{z-2} = \dfrac{1}{3z}\cdot\dfrac{1}{1+\dfrac{1}{z}} - \dfrac{1}{3}\cdot\dfrac{1}{1-\dfrac{z}{2}} = \dfrac{1}{3z}\left(1 - \dfrac{1}{z} + \dfrac{1}{z^2} - \dfrac{1}{z^3} + \cdots\right) - \dfrac{1}{3}\left(1 + \dfrac{z}{2} + \dfrac{z^2}{2^2} + \dfrac{z^3}{2^3} + \cdots\right)$

$= \cdots - \dfrac{1}{3z^2} + \dfrac{1}{3z} - \dfrac{1}{3} - \dfrac{z}{3\cdot 2} - \dfrac{z^2}{3\cdot 2^2} - \cdots$

20. $f(z) = \dfrac{2/3}{z-2} + \dfrac{1}{3}\dfrac{1}{3+(z-2)} = \dfrac{2/3}{z-2} + \dfrac{1}{9}\cdot\dfrac{1}{1+\dfrac{z-2}{3}} = \dfrac{2/3}{z-2} + \dfrac{1}{9}\left(1 + \dfrac{z-2}{3} + \dfrac{(z-2)^2}{3^2} + \dfrac{(z-2)^3}{3^3} + \cdots\right)$

$= \dfrac{2}{3(z-2)} + \dfrac{1}{9} + \dfrac{z-2}{3^3} + \dfrac{(z-2)^2}{3^4} + \cdots$

21. $f(z) = \dfrac{1}{z}(1-z)^{-2} = \dfrac{1}{z}\left(1 + (-2)(-z) + \dfrac{(-2)(-3)}{z!}(-z)^2 + \dfrac{(-2)(-3)(-4)}{3!}(-z)^3 + \cdots\right) = \dfrac{1}{z} + 2 + 3z + 4z^2 + \cdots$

22. $f(z) = \dfrac{1}{z^3(1-\dfrac{1}{z})^2} = \dfrac{1}{z^3}\left(1 - \dfrac{1}{z}\right)^{-2}$

$= \dfrac{1}{z^3}\left(1 + (-2)\left(-\dfrac{1}{z}\right) + \dfrac{(-2)(-3)}{2!}\left(-\dfrac{1}{z}\right)^2 + \dfrac{(-2)(-3)(-4)}{3!}\left(-\dfrac{1}{z}\right)^3 + \cdots\right)$

$= \dfrac{1}{z^3} + \dfrac{2}{z^4} + \dfrac{3}{z^5} + \dfrac{4}{z^6} + \cdots$

23. $f(z) = \dfrac{1}{(z-2)[1+(z-2)]^3} = \dfrac{1}{z-2}[1+(z-2)]^{-3}$

$= \dfrac{1}{z-2}\left(1 + (-3)(z-2) + \dfrac{(-3)(-4)}{2!}(z-2)^2 + \dfrac{(-3)(-4)(-5)}{3!}(z-2)^3 + \cdots\right)$

$= \dfrac{1}{z-2} - 3 + 6(z-2) - 10(z-2)^2 + \cdots$

24. $f(z) = \dfrac{1}{(z-3)^3}\cdot\dfrac{-1}{1-(z-1)} = \dfrac{-1}{(z-1)^3}[1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots]$

$= -\dfrac{1}{(z-1)^3} - \dfrac{1}{(z-1)^2} - \dfrac{1}{z-1} - 1 - (z-1) - \cdots$

25. $f(z) = \dfrac{3}{z} + \dfrac{4}{z-1} = \dfrac{3}{z} - 4\cdot\dfrac{1}{1-z} = \dfrac{3}{z} - 4(1 + z + z^2 + z^3 + \cdots) = \dfrac{3}{z} - 4 - 4z - 4z^2 - \cdots$

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26. $f(z) = \dfrac{4}{z-1} + 3 \cdot \dfrac{1}{1+(z-1)} = \dfrac{4}{z-1} + 3(1-(z-1)+(z-1)^2-(z-1)^3+\cdots) = \dfrac{4}{z-1} + 3 - 3(z-1) + 3(z-1)^2 - \cdots$

27. $f(z) = z + \dfrac{2}{z-2} = 1 + (z-1) + \dfrac{2}{-1+z-1} = 1 + (z-1) + \dfrac{2}{z-1} \cdot \dfrac{1}{1 - \dfrac{1}{z-1}}$

$$= 1 + (z-1) + \frac{2}{z-1}\left(1 + \frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \cdots\right) = \cdots + \frac{2}{(z-1)^2} + \frac{2}{z-1} + 1 + (z-1)$$

28. $f(z) = z + \dfrac{2}{z-2} = \dfrac{2}{z-2} + 2 + (z-2)$

Exercises 19.4

1. Using $e^{2z} = \displaystyle\sum_{k=0}^{\infty} \frac{2^k z^k}{k!}$ we obtain

$$\frac{e^{2z}-1}{z} = \frac{\left(1 + \dfrac{2}{1!}z + \dfrac{2^2}{2!}z^2 + \dfrac{2^3}{3!}z^3 + \cdots\right) - 1}{z} = \frac{1}{z}\left(\frac{2}{1!}z + \frac{2^2}{2!}z^2 + \frac{2^3}{3!}z^3 + \cdots\right) = \frac{2}{1!} + \frac{2^2}{2!}z + \frac{2^3}{3!}z^2 + \cdots.$$

From the form of the last series we see that $z = 0$ is a removable singularity. Define $f(0) = 2$.

2. Using $\sin 4z \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{(4z)^{2k+1}}{(2k+1)!}$ we obtain

$$\frac{\sin 4z - 4z}{z^2} = \frac{\left(\dfrac{4}{1!}z - \dfrac{4^3}{3!}z^3 + \dfrac{4^5}{5!}z^5 - \dfrac{4^7}{7!}z^7 + \cdots\right) - 4z}{z^2} = \frac{1}{z^2}\left(-\frac{4^3}{3!}z^3 + \frac{4^5}{5!}z^5 - \frac{4^7}{7!}z^7 + \cdots\right)$$

$$= -\frac{4^3}{3!}z + \frac{4^5}{5!}z^3 - \frac{4^7}{7!}z^5 + \cdots.$$

From the form of the last series we see that $z = 0$ is a removable singularity. Define $f(0) = 0$.

3. Since $f(-2+i) = f'(-2+i) = 0$ and $f''(z) = 2$ for all z, $z = -2+i$ is a zero of order two.

4. Write $f(z) = z^4 - 16 = (z^2-4)(z^2+4) = (z-2)(z+2)(z-2i)(z+2i)$ to see that 2, -2, $2i$, and $-2i$ are zeros of f. Now $f'(z) = 4z^3$ and $f'(2) \neq 0$, $f'(-2) \neq 0$, $f'(2i) \neq 0$, and $f'(-2i) \neq 0$. This indicates that each zero is of order one.

5. Write $f(z) = z^2(z^2+1) = z^2(z-i)(z+i)$ to see that 0, i, and $-i$ are zeros of f. Now $f'(z) = 4z^3 + 2z$ and $f'(i) \neq 0$ and $f'(-i) \neq 0$. This indicates that $z = i$ and $z = -i$ are zeros of order one. However $f'(0) = 0$, but $f''(0) = 2 \neq 0$. Hence $z = 0$ is a zero of order two.

6. Write $f(z) = (z^2+9)/z = (z-3i)(z+3i)/z$ to see that $3i$ and $-3i$ are zeros of f. Now $f'(z) = 1 - 9/z^2$ and $f'(3i) = f'(-3i) = 2 \neq 0$. This indicates that each zero is of order one.

7. Write $f(z) = e^z(e^z - 1)$ to see that $2n\pi i$, $n = 0, \pm 1, \pm 2, \ldots$ are zeros of f. Now $f'(z) = 2e^{2z} - e^z$ and $f'(2n\pi i) = 2e^{4n\pi i} - e^{2n\pi i} = 1 \neq 0$. This indicates that each zero is of order one.

8. The zeros of f are the zeros of $\sin z$, that is, $n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. From $f'(z) = 2\sin z \cos z$ we see $f'(n\pi) = 0$. From $f''(z) = 2(-\sin^2 z + \cos^2 z)$ we see $f''(n\pi) \neq 0$. This indicates that each zero is of order two.

9. From $$f(z) = z(1 - \cos z^2) = z\left(-\frac{z^4}{2!} + \frac{z^8}{4!} - \cdots\right) = z^5\left(-\frac{1}{2!} + \frac{z^4}{4!} - \cdots\right)$$

we see that $z = 0$ is a zero of order five.

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10. From $$f(z) = z - \sin z = \frac{z^3}{3!} - \frac{z^5}{5!} + \cdots = z^3\left(\frac{1}{3!} - \frac{z^2}{5!} + \cdots\right)$$ we see that $z = 0$ is a zero of order three.

11. From $$f(z) = 1 - e^{z-1} = -\frac{z-1}{1!} - \frac{(z-1)^2}{2!} - \cdots = (z-1)\left(1 - \frac{z-1}{2!} - \cdots\right)$$ we see that $z = 1$ is a zero of order one.

12. From the series $e^z = -\sum_{k=0}^{\infty} \frac{(z-\pi i)^k}{k!}$ centered at πi and

$$f(z) = 1 - \pi i + z + e^z = 1 - \pi i + z + \left(-1 - \frac{z-\pi i}{1!} - \frac{(z-\pi i)^2}{2!} - \frac{(z-\pi i)^3}{3!} - \cdots\right)$$
$$= -\frac{(z-\pi i)^2}{2!} - \frac{(z-\pi i)^3}{3!} - \cdots = (z-\pi i)^2\left(-\frac{1}{2!} - \frac{z-\pi i}{3!} - \cdots\right)$$

we see that $z = \pi i$ is a zero of order two.

13. From $$f(z) = \frac{3z-1}{[(z-(-1+2i)][z-(-1-2i)]}$$ and Theorem 19.11 we see that $-1+2i$ and $-1-2i$ are simple poles.

14. From $f(z) = \dfrac{5z^2-6}{z^2}$ and Theorem 19.11 we see that 0 is a pole of order two.

15. From $f(z) = \dfrac{1+4i}{(z+2)(z+i)^4}$ and Theorem 19.11 we see that -2 is a simple pole and $-i$ is a pole of order four.

16. From $$f(z) = \frac{z-1}{(z+1)^2\left[z-(\frac{1}{2}+\frac{\sqrt{3}}{2}i)\right]\left[z-(\frac{1}{2}-\frac{\sqrt{3}}{2}i)\right]}$$ and Theorem 19.11 we see that -1 is a pole of order two and $\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$ and $\frac{1}{2} - \frac{\sqrt{3}}{2}\,i$ are simple poles.

17. Since $\sin z$ and $\cos z$ are analytic at $n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, $\sin z$ has zeros of order one at $n\pi$, and $\cos n\pi \neq 0$, it follows from Theorem 19.11 that the numbers $n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ are simple poles of $f(z) = \tan z$.

18. From $z^2 \sin \pi z = z^3\left(\pi - \dfrac{\pi^3 z^2}{3!} + \cdots\right)$ we see $z = 0$ is a zero of order three. From $f(z) = \dfrac{\cos \pi z}{z^2 \sin \pi z}$ and Theorem 19.11 we see 0 is a pole of order three. The numbers n, $n \pm 1$, ± 2, $\ldots$ are simple poles.

19. From the Laurent series

$$f(z) = \frac{1-\cosh z}{z^4} = \frac{1 - \left(1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \dfrac{z^6}{6!} + \cdots\right)}{z^4} = -\frac{1}{2!z^2} - \frac{1}{4!} - \frac{z^2}{6!} - \cdots$$

we see that 0 is a pole of order two.

20. From the Laurent series

$$f(z) = \frac{e^z}{z^2} = \frac{\left(1 + \dfrac{z}{1!} + \dfrac{z^2}{2!} + \cdots\right)}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \cdots$$

we see that 0 is a pole of order two.

21. From $1 - e^z = 1 - \left(1 + \dfrac{z}{1!} + \dfrac{z^2}{2!} + \cdots\right) = z\left(-1 - \dfrac{z}{2!} - \cdots\right)$ we see that $z = 0$ is a zero of order one. By periodicity of e^z it follows that $z = 2n\pi i$, $n = 0, \pm 1, \pm 2, \ldots$ are zeros of order one. From $f(z) = \dfrac{1}{1-e^z}$ and Theorem 19.11 we see that the numbers $2n\pi i$, $n = 0, \pm 1, \pm 2, \ldots$ are simple poles.

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22. $z = 0$ is a removable singularity of the function $(\sin z)/z$. From $f(z) = \dfrac{\sin z}{z(z-1)}$ we see that only 1 is a (simple) pole.

23. The function $f(z) = \dfrac{\sin(1/z)}{\cos(1/z)}$ fails to be defined at $z = 0$ and at the solutions of $\cos\dfrac{1}{z} = 0$, that is, at $\dfrac{1}{z} = (2n+1)\dfrac{\pi}{2}$, $n = 0, \pm1, \pm2, \ldots$. Since $z = \dfrac{2}{(2n+1)\pi}$, $n = 0, \pm1, \pm2, \ldots$ we see that in any neighborhood of $z = 0$ there are points at which f is not defined and thus not analytic. Hence $z = 0$ is a non-isolated singularity.

24. From the Laurent series

$$f(z) = z^3\left[\frac{1}{z} - \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \frac{1}{5!}\left(\frac{1}{z}\right)^5 - \frac{1}{7!}\left(\frac{1}{z}^7\right) + \cdots\right] = z^2 - \frac{1}{3!} + \frac{1}{5!z^2} - \frac{1}{7!z^4} + \cdots, \quad 0 < |z|,$$

we see that the principal part contains an infinite number of nonzero terms. Hence $z = 0$ is an essential singularity.

Exercises 19.5

1. $f(z) = \dfrac{2}{5(z-1)} \cdot \dfrac{1}{1 + \dfrac{z-1}{5}} = \dfrac{2}{5(z-1)}\left(1 - \dfrac{z-1}{5} + \dfrac{(z-1)^2}{5^2} - \dfrac{(z-1)^3}{5^3} + \cdots\right)$

$= \dfrac{2/5}{z-1} - \dfrac{2}{25} + \dfrac{2(z-1)}{5^3} - \dfrac{2(z-1)^2}{5^4} + \cdots$

$\text{Res}(f(z), 1) = 2/5$

2. $f(z) = \dfrac{1}{z^3}(1-z)^{-3} = \dfrac{1}{z^3}\left(1 + (-3)(-z) + \dfrac{(-3)(-4)}{2!}(-z)^2 + \dfrac{(-3)(-4)(-5)}{3!}(-z)^3 + \cdots\right)$

$= \dfrac{1}{z^3} + \dfrac{3}{z^2} + \dfrac{6}{z} + 10 + \cdots$

$\text{Res}(f(z), 0) = 6$

3. $f(z) = -\dfrac{3}{z} - \dfrac{1}{z-2} = -\dfrac{3}{z} + \dfrac{1}{2} \cdot \dfrac{1}{1 - \dfrac{z}{2}} = -\dfrac{3}{z} + \dfrac{1}{2}\left(1 + \dfrac{z}{2} + \dfrac{z^2}{2^2} + \dfrac{z^3}{2^3} + \cdots\right) = -\dfrac{3}{z} + \dfrac{1}{2} + \dfrac{z}{2^2} + \dfrac{z^2}{2^3} + \cdots$

$\text{Res}(f(z), 0) = -3$

4. $f(z) = (z+3)^2\left(\dfrac{2}{z+3} - \dfrac{2^3}{3!(z+3)^3} + \dfrac{2^5}{5!(z+3)^5} + \cdots\right) = \cdots + \dfrac{2^5}{5!(z+3)^3} - \dfrac{2^3}{3!(z+3)} + 2(z+3)$

$\text{Res}(f(z), -3) = -\dfrac{4}{3}$

5. $f(z) = e^{-2/z^2} = \displaystyle\sum_{k=0}^{\infty} \frac{(-2/z^2)^k}{k!} = \cdots - \frac{2^3}{3!z^6} + \frac{2^2}{2!z^4} - \frac{2}{1!z^2} + 1; \quad \text{Res}(f(z), 0) = 0$

6. $f(z) = \dfrac{e^{-2}}{(z-2)^2}\, e^{-(z-2)} = \dfrac{e^{-2}}{(z-2)^2}\left(1 - \dfrac{z-2}{1!} + \dfrac{(z-2)^2}{2!} - \dfrac{(z-2)^3}{3!} + \cdots\right)$

$= \dfrac{e^{-2}}{(z-2)^2} - \dfrac{e^{-2}}{z-2} + \dfrac{e^{-2}}{2} - \dfrac{e^{-2}(z-2)}{3!} + \cdots$

$\text{Res}(f(z), 2) = -e^{-2}$

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7. $\text{Res}(f(z), 4i) = \lim\limits_{z\to 4i}(z-4i)\cdot\dfrac{z}{(z-4i)(z+4i)} = \lim\limits_{z\to 4i}\dfrac{z}{z+4i} = \dfrac{1}{2}$

$\text{Res}(f(z), -4i) = \lim\limits_{z\to -4i}(z+4i)\cdot\dfrac{z}{(z-4i)(z+4i)} = \lim\limits_{z\to -4i}\dfrac{z}{z-4i} = \dfrac{1}{2}$

8. $\text{Res}(f(z), 1/2) = \lim\limits_{z\to 1/2}(z-1/2)\dfrac{4z+8}{2(z-1/2)} = \lim\limits_{z\to 1/2}(2z+4) = 5$

9. $\text{Res}(f(z), 1) = \lim\limits_{z\to 1}(z-1)\dfrac{1}{z^2(z+2)(z-1)} = \lim\limits_{z\to 1}\dfrac{1}{z^2(z+2)} = \dfrac{1}{3}$

$\text{Res}(f(z), -2) = \lim\limits_{z\to -2}(z+2)\dfrac{1}{z^2(z+2)(z-1)} = \lim\limits_{z\to -2}\dfrac{1}{z^2(z-1)} = -\dfrac{1}{12}$

$\text{Res}(F(z), 0) = \dfrac{1}{1!}\lim\limits_{z\to 0}\dfrac{d}{dz}\left[z^2\cdot\dfrac{1}{z^2(z+2)(z-1)}\right] = \lim\limits_{z\to 0}\dfrac{-2z-1}{(z+2)^2(z-1)^2} = -\dfrac{1}{4}$

10. $\text{Res}(f(z), 1+i) = \dfrac{1}{1!}\lim\limits_{z\to 1+i}\dfrac{d}{dz}\left[(z-1-i)^2\cdot\dfrac{1}{(z-1-i)^2(z-1+1)^2}\right] = \lim\limits_{z\to 1+i}\dfrac{-2}{(z-1+i)^3} = -\dfrac{1}{4}\,i$

$\text{Res}(f(z), 1-i) = \dfrac{1}{1!}\lim\limits_{z\to 1-i}\dfrac{d}{dz}\left[(z-1+i)^2\cdot\dfrac{1}{(z-1-i)^2(z-1+i)^2}\right] = \lim\limits_{z\to 1-i}\dfrac{-2}{(z-1-i)^3} = \dfrac{1}{4}\,i$

11. $\text{Res}(f(z), -1) = \lim\limits_{z\to -1}(z+1)\cdot\dfrac{5z^2-4z+3}{(z+1)(z+2)(z+3)} = \lim\limits_{z\to -1}\dfrac{5z^2-4z+3}{(z+2)(z+3)} = 6$

$\text{Res}(f(z), -2) = \lim\limits_{z\to -2}(z+2)\cdot\dfrac{5z^2-4z+3}{(z+1)(z+2)(z+3)} = \lim\limits_{z\to -2}\dfrac{5z^2-4z+3}{(z+1)(z+3)} = -31$

$\text{Res}(f(z), -3) = \lim\limits_{z\to -3}(z+3)\cdot\dfrac{5z^2-4z+3}{(z+1)(z+2)(z+3)} = \lim\limits_{z\to -3}\dfrac{5z^2-4z+3}{(z+1)(z+2)} = 30$

12. $\text{Res}(f(z), -3) = \lim\limits_{z\to -3}(z+3)\cdot\dfrac{2z-1}{(z-1)^4(z+3)} = \lim\limits_{z\to -3}\dfrac{2z-1}{(z-1)^4} = -\dfrac{7}{256}$

$\text{Res}(f(z), 1) = \dfrac{1}{3!}\lim\limits_{z\to 1}\dfrac{d^3}{dz^3}\left[(z-1)^4\cdot\dfrac{2z-1}{(z-1)^4(z+3)}\right] = \dfrac{1}{6}\lim\limits_{z\to 1}\dfrac{-42}{(z+3)^4} = -\dfrac{7}{256}$

13. $\text{Res}(f(z), 0) = \dfrac{1}{1!}\lim\limits_{z\to 0}\dfrac{d}{dz}\left[z^2\cdot\dfrac{\cos z}{z^2(z-\pi)^3}\right] = \lim\limits_{z\to 0}\dfrac{-(z-\pi)\sin z - 3\cos z}{(z-\pi)^4} = -\dfrac{3}{\pi^4}$

$\text{Res}(f(z), \pi) = \dfrac{1}{2!}\lim\limits_{z\to \pi}\dfrac{d^2}{dz^2}\left[(z-\pi)^3\cdot\dfrac{\cos z}{z^2(z-\pi)^3}\right] = \dfrac{1}{2}\lim\limits_{z\to \pi}\dfrac{-z^2\cos z + 4z\sin z + 6\cos z}{z^4} = \dfrac{\pi^2-6}{2\pi^4}$

14. Using $\dfrac{d}{dz}(e^z - 1) = e^z$ and the result in (4),

$$\text{Res}(f(z), 2n\pi i) = \left.\frac{e^z}{e^z}\right|_{z=2n\pi i} = 1.$$

15. Using $\dfrac{d}{dz}\cos z = -\sin z$ and the result in (4),

$$\text{Res}\left(f(z), (2n+1)\frac{\pi}{2}\right) = \left.\frac{1}{-\sin z}\right|_{z=(2n+1)\frac{\pi}{2}} = \frac{1}{-\sin(2n+1)\frac{\pi}{2}} = (-1)^{n+1}.$$

16. $z = 0$ is a pole of order two. Thus by (2) and L'Hôpital's rule,

$$\text{Res}(f(z), 0) = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left[z^2\cdot\frac{1}{z\sin z}\right] = \lim_{z\to 0}\frac{\sin z - z\cos z}{\sin^2 z} = \lim_{z\to 0}\frac{\cos z + z\sin z - \cos z}{2\sin z\cos z} = \lim_{z\to 0}\frac{z}{2\cos z} = 0.$$

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For the simple poles at $z = n\pi$, $n = \pm 1, \pm 2, \ldots$ we have from (4),

$$\text{Res}(f(z), n\pi) = \frac{1}{z\cos z + \sin z}\bigg|_{z=n\pi} = \frac{(-1)^n}{n\pi}.$$

17. **(a)** $\displaystyle\oint_C \frac{1}{(z-1)(z+2)^2}\,dz = 0$ by Theorem 18.4.

(b) $\displaystyle\oint_C \frac{1}{(z-1)(z+2)^2}\,dz = 2\pi i\,\text{Res}(f(z), 1) = \frac{2\pi}{9}\,i$

(c) $\displaystyle\oint_C \frac{1}{(z-1)(z+2)^2}\,dz = 2\pi i\,[\text{Res}(f(z), 1) + \text{Res}(f(z), -2)] = 2\pi i\left[\frac{1}{9} + \left(-\frac{1}{9}\right)\right] = 0$

18. **(a)** $\displaystyle\oint_C \frac{z+1}{z^2(z-2i)}\,dz = 2\pi i\,\text{Res}(f(z), 0) = \pi\left(-1 + \frac{1}{2}\,i\right)$

(b) $\displaystyle\oint_C \frac{z+1}{z^2(z-2i)}\,dz = 2\pi i\,\text{Res}(f(z), 2i) = \pi\left(1 - \frac{1}{2}\,i\right)$

(c) $\displaystyle\oint_C \frac{z+1}{z^2(z-2i)}\,dz = 2\pi i[\text{Res}(f(z), 0) + \text{Res}(f(z), 2i)] = 2\pi i\left[\frac{1}{4} + \frac{1}{2}\,i + \left(-\frac{1}{4} - \frac{1}{2}\,i\right)\right] = 0$

19. **(a)** From the Laurent series $z^3 e^{-1/z^2} = \cdots - \dfrac{1}{3!z^3} + \dfrac{1}{2!z} - z + z^3$ we see $\text{Res}(f(z), 0) = 1/2$. Hence

$$\oint_C z^3 e^{-1/z^2}\,dz = 2\pi i\,\text{Res}(f(z), 0) = \pi i.$$

(b) $\displaystyle\oint_C z^3 e^{-1/z^2}\,dz = 2\pi i\,\text{Res}(f(z), 0) = \pi i$

(c) $\displaystyle\oint_C z^3 e^{1/z^2}\,dz = 0$ by Theorem 18.4.

20. **(a)** $\displaystyle\oint_C \frac{1}{z\sin z}\,dz = 0$ by Theorem 18.4.

(b) $z = 0$ is a pole of order two (see Problem 16). Thus

$$\oint_C \frac{1}{z\sin z}\,dz = 2\pi i\,\text{Res}(f(z), 0) = 2\pi i(0) = 0.$$

(c) $\displaystyle\oint_C \frac{1}{z\sin z}\,dz = 2\pi i[\text{Res}(f(z), -\pi) + \text{Res}(f(z), 0) + \text{Res}(f(z), \pi)] = 2\pi i\left[\frac{1}{\pi} + 0 + \left(-\frac{1}{\pi}\right)\right] = 0$

21. $\displaystyle\oint_C \frac{1}{z^2 + 4z + 13}\,dz = 2\pi i\,\text{Res}(f(z), -2 + 3i) = \frac{\pi}{3}$

22. $\displaystyle\oint_C 1z^3(z-1)^4\,dz = 2\pi i\,\text{Res}(f(z), 1) = -20\pi i$

23. $\displaystyle\oint_C \frac{z}{z^4 - 1}\,dz = 2\pi i[\text{Res}(f(z), -1) + \text{Res}(f(z), 1) + \text{Res}(f(z), -i) + \text{Res}(f(z), i)] = 2\pi i\left[\frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4}\right] = 0$

24. $\displaystyle\oint_C \frac{z}{(z+1)(z^2+1)}\,dz = 2\pi i[\text{Res}(f(z), i) + \text{Res}(f(z), -i)] = 2\pi i\left[\frac{1}{4} - \frac{1}{4}\,i + \frac{1}{4} + \frac{1}{4}\,i\right] = \pi i$

25. $\displaystyle\oint_C \frac{ze^z}{z^2 - 1}\,dz = 2\pi i[\text{Res}(f(z), 1) + \text{Res}(f(z), -1)] = 2\pi i\left[\frac{e}{2} + \frac{e^{-1}}{2}\right] = 2\pi i\cosh 1$

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26. $\displaystyle\oint_C \frac{e^z}{z^3+2z^2}\,dz = 2\pi i[\text{Res}(f(z),0)+\text{Res}(f(z),-2)] = 2\pi i\left[\frac{1}{2}+\frac{e^{-2}}{4}\right] = \pi i\left(1+\frac{1}{2}\,e^{-2}\right)$

27. $\displaystyle\oint_C \frac{\tan z}{z}\,dz = 2\pi i\,\text{Res}\left(f(z),\frac{\pi}{2}\right)) = -4i$. Note: $z=0$ is not a pole. See Example 1, Section 19.4.

28. $\displaystyle\oint_C \frac{\cot \pi z}{z^2}\,dz = 2\pi i\,\text{Res}(f(z),0) = 2\pi i\left(-\frac{\pi}{3}\right) = -\frac{2\pi^2}{3}\,i$

Note: $z=0$ is a pole of order three. Use L'Hôpital's rule (or *Mathematica*) to show that

$$\text{Res}(f(z),0) = \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}z\cot \pi z = \frac{1}{2}\lim_{z\to 0}[-2\pi\csc^2\pi z + 2\pi^2 z\cot \pi z\csc^2 \pi z] = \frac{1}{2}\left(-\frac{2\pi}{3}\right) = -\frac{\pi}{3}\,.$$

29. $\displaystyle\oint_C \cot \pi z\,dz = 2\pi i[\text{Res}(f(z),1)+\text{Res}(f(z),2)+\text{Res}(f(z),3)] = 2\pi i\left[\frac{1}{\pi}+\frac{1}{\pi}+\frac{1}{\pi}\right] = 6i$

30. $\displaystyle\oint_C \frac{2z-1}{z^2(z^3+1)}\,dz = 2\pi i\left[\text{Res}(f(z),0)+\text{Res}(f(z),-1)+\text{Res}\left(f(z),\frac{1}{2}+\frac{\sqrt{3}}{2}\,i\right)\right]$

$$= 2\pi i\left[2+(-1)+\left(-\frac{1}{2}-\frac{1}{6}\sqrt{3}\,i\right)\right] = \pi\left(\frac{\sqrt{3}}{3}+i\right)$$

31. $\displaystyle\oint_C \frac{e^{iz}+\sin z}{(z-\pi)^4}\,dz = 2\pi i\,\text{Res}(f(z),\pi) = \pi\left(-\frac{1}{3}+\frac{1}{3}\,i\right)$

32. $\displaystyle\oint_C \frac{\cos z}{(z-1)^2(z^2+9)}\,dz = 2\pi i\,\text{Res}(f(z),1) = 2\pi i(-0.02\cos 1 - 0.1\sin 1) = -0.5966i$

Exercises 19.6

1. $\displaystyle\int_0^{2\pi}\frac{d\theta}{1+\frac{1}{2}\sin\theta} = \oint_C \frac{4}{z^2+4iz-1}\,dz = (4)2\pi i\,\text{Res}(f(z),(\sqrt{3}-2)i) = \frac{4\pi}{\sqrt{3}}$

2. $\displaystyle\int_0^{2\pi}\frac{d\theta}{10-6\cos\theta} = \frac{1}{2}\cdot\left(\frac{-2}{i}\right)\oint_C \frac{dz}{(3z-1)(z-3)} = (i)2\pi i\,\text{Res}\left(f(z),\frac{1}{3}\right) = \frac{\pi}{4}$

3. $\displaystyle\int_0^{2\pi}\frac{\cos\theta}{3+\sin\theta}\,d\theta = \oint_C \frac{z^2+1}{z(z^2+6iz-1)}\,dz = 2\pi i[\text{Res}(f(z),0)+\text{Res}(f(z),-3+2\sqrt{2}\,i)] = 0$

4. $\displaystyle\int_0^{2\pi}\frac{1}{1+3\cos^2\theta}\,d\theta = \frac{4}{i}\oint_C \frac{z}{3z^4+10z^2+3}\,dz = \left(\frac{4}{i}\right)2\pi i\left[\text{Res}\left(f(z),(\frac{\sqrt{3}}{3}\,i\right)+\text{Res}\left(f(z),-\frac{\sqrt{3}}{3}\,i\right)\right] = \pi$

5. $\displaystyle\int_0^{\pi}\frac{d\theta}{2-\cos\theta} = \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{2-\cos\theta} = -\frac{1}{i}\oint_C \frac{dz}{z^2-4z+1} = \left(-\frac{1}{i}\right)2\pi i\,\text{Res}(f(z),2-\sqrt{3}\,) = \frac{\pi}{\sqrt{3}}$

6. $\displaystyle\int_0^{\pi}\frac{d\theta}{1+\sin^2\theta} = \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{1+\sin^2\theta} = -\frac{2}{i}\oint_C \frac{z}{z^4-6z^2+1}\,dz$

$$= \left(-\frac{2}{i}\right)2\pi i[\text{Res}(f(z),\sqrt{3-2\sqrt{2}}\,)+\text{Res}(f(z),-\sqrt{3-2\sqrt{2}}\,)] = \frac{\pi}{\sqrt{2}}$$

7. $\displaystyle\int_0^{2\pi}\frac{\sin^2\theta}{5+4\cos\theta}\,d\theta = -\frac{1}{4i}\oint_C \frac{(z^2-1)^2}{z^2(2z^2+5z+2)}\,dz = \left(-\frac{1}{4i}\right)2\pi i\left[\text{Res}(f(z),0)+\text{Res}\left(f(z),-\frac{1}{2}\right)\right] = \frac{\pi}{4}$

8. $\displaystyle\int_0^{2\pi}\frac{\cos^2\theta}{3-\sin\theta}\,d\theta = \frac{1}{2i}\oint_C \frac{z^4+2z^2+1}{z^2(iz^2+6z-i)}\,dz = \left(\frac{1}{2i}\right)2\pi i[\text{Res}(f(z),0)+\text{Res}(f(z),3-2\sqrt{2}\,)i)] = \pi[6-4\sqrt{2}\,]$

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9. We use $\cos 2\theta = (z^2 + z^{-2})/2$.

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta}\, d\theta = \frac{i}{2}\oint_C \frac{z^4+1}{z^2(2z^2-5z+2)}\, dz = \left(\frac{i}{2}\right) 2\pi i \left[\operatorname{Res}(f(z), 0) + \operatorname{Res}\left(f(z), \frac{1}{2}\right)\right] = \frac{\pi}{6}$$

10. $\displaystyle\int_0^{2\pi} \frac{1}{\cos\theta + 2\sin\theta + 3}\, d\theta = \frac{2}{i}\oint_C \frac{1}{(1-2i)z^2 + 6z + 1 + 2i} = \left(\frac{2}{i}\right) 2\pi i \operatorname{Res}\left(f(z), -\frac{1}{5} - \frac{2}{5}\, i\right) = \pi$

11. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2 - 2x + 2}\, dx = 2\pi i \operatorname{Res}(f(z), 1+i) = \pi$

12. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2 - 2x + 25}\, dx = 2\pi i \operatorname{Res}(f(z), 1 + 2\sqrt{6}\, i) = \frac{\pi}{2\sqrt{6}}$

13. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\, dx = 2\pi i \operatorname{Res}(f(z), 2i) = \frac{\pi}{16}$

14. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\, dx = 2\pi i \operatorname{Res}(f(z), i) = \frac{\pi}{2}$

15. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3}\, dx = 2\pi i \operatorname{Res}(f(z), i) = \frac{3\pi}{8}$

16. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+4)^3}\, dx = 0$ (The integrand is an odd function)

17. $\displaystyle\int_{-\infty}^{\infty} \frac{2x^2 - 1}{x^4 + 5x^2 + 4}\, dx = 2\pi i[\operatorname{Res}(f(z), i) + \operatorname{Res}(f(z), 2i)] = \frac{\pi}{2}$

18. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2(x^2+9)} = 2\pi i[\operatorname{Res}(f(z), i) + \operatorname{Res}(f(z), 3i)] = \frac{5\pi}{96}$

19. $\displaystyle\int_0^{\infty} \frac{x^2+1}{x^4+1}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\, dx = \pi i \operatorname{Res}(f(z), i) = \frac{\pi}{\sqrt{2}}$

20. $\displaystyle\int_0^{\infty} \frac{1}{x^6+1}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^6+1}\, dx = \pi i \left[\operatorname{Res}\left(f(z), \frac{\sqrt{3}}{2} + \frac{1}{2}\, i\right) + \operatorname{Res}(f(z), i) + \operatorname{Res}\left(f(z), \frac{-\sqrt{3}}{2} + \frac{1}{2}\, i\right)\right] = \frac{\pi}{3}$

21. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\, dx = 2\pi i \operatorname{Res}(f(z), i) = \pi e^{-1}$. Therefore, $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx = \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\, dx\right) = \pi e^{-1}$.

22. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+1}\, dx = 2\pi i \operatorname{Res}(f(z), i) = \pi e^{-2}$. Therefore, $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+1}\, dx = \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+1}\, dx\right) = \pi e^{-2}$.

23. $\displaystyle\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+1}\, dx = 2\pi i \operatorname{Res}(f(z), i) = \pi e^{-1} i$. Therefore, $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}\, dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+1}\, dx\right) = \pi e^{-1}$.

24. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)^2}\, dx = 2\pi i \operatorname{Res}(f(z), 2i) = \frac{3e^{-2}}{16}\,\pi$; $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+4)^2}\, dx = \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)^2}\, dx\right) = \frac{3e^{-2}}{16}\,\pi$.

Therefore

$$\int_0^{\infty} \frac{\cos x}{(x^2+4)^2}\, dx = \frac{1}{2}\left(\frac{3e^{-2}}{16}\,\pi\right) = \frac{3e^{-2}}{32}\pi.$$

25. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{3ix}}{(x^2+1)^2}\, dx = 2\pi i \operatorname{Res}(f(z), i) = 2\pi e^{-3}$; $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2}\, dx = \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{3ix}}{(x^2+1)^2}\, dx\right) = 2\pi e^{-3}$.

Therefore

$$\int_0^{\infty} \frac{\cos 3x}{(x^2+1)^2}\, dx = \frac{1}{2}(2\pi e^{-3}) = \pi e^{-3}.$$

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26. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4x+5}\,dx = 2\pi i \operatorname{Res}(f(z), -2+i) = \pi e^{-1-2i}$. Therefore

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4x+5}\,dx\right) = -\pi e^{-1}\sin 2$$

27. $$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^4+1}\,dx &= 2\pi i\left[\operatorname{Res}\left(f(z), \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,i\right) + \operatorname{Res}\left(f(z), -\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,i\right)\right] \\ &= 2\pi i\left[\left(-\frac{\sqrt{2}}{8}-\frac{\sqrt{2}}{8}\,i\right)e^{(-\sqrt{2}+\sqrt{2}\,i)} + \left(\frac{\sqrt{2}}{8}-\frac{\sqrt{2}}{8}\,i\right)e^{(-\sqrt{2}-\sqrt{2}\,i)}\right] \\ &= \pi e^{-\sqrt{2}}\left[\frac{\sqrt{2}}{2}\cos\sqrt{2}+\frac{\sqrt{2}}{2}\sin\sqrt{2}\right]\end{aligned}$$

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^4+1}\,dx = \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^4+1}\,dx\right) = \pi e^{-\sqrt{2}}\left[\frac{\sqrt{2}}{2}\cos\sqrt{2}+\frac{\sqrt{2}}{2}\sin\sqrt{2}\right]$$

Therefore

$$\int_0^{\infty} \frac{\cos 2x}{x^4+1}\,dx = \pi e^{-\sqrt{2}}\frac{\sqrt{2}}{4}(\cos\sqrt{2}+\sin\sqrt{2}\,).$$

28. $$\begin{aligned}\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^4+1}\,dx &= 2\pi i\left[\operatorname{Res}\left(f(z), \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,i\right) + \operatorname{Res}\left(f(z), -\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,i\right)\right] \\ &= 2\pi i\left[-\frac{i}{4}e^{(-1/\sqrt{2}+i/\sqrt{2})} + \frac{i}{4}e^{(-1/\sqrt{2}-i/\sqrt{2})}\right] = \left(\pi e^{-1/\sqrt{2}}\sin\frac{1}{\sqrt{2}}\right)i\end{aligned}$$

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^4+1}\,dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^4+1}\,dx\right) = \pi e^{-1/\sqrt{2}}\sin\frac{1}{\sqrt{2}}$$

Therefore

$$\int_0^{\infty} \frac{x\sin x}{x^4+1}\,dx = \frac{\pi}{2}\,e^{-1/\sqrt{2}}\sin\frac{1}{\sqrt{2}}\,.$$

29. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)(x^2+9)}\,dx = 2\pi i[\operatorname{Res}(f(z), i) + \operatorname{Res}(f(z), 3i)] = 2\pi i\left[-\frac{i}{16}\,e^{-1} + \frac{i}{48}\,e^{-3}\right] = \frac{1}{8}\,e^{-1} - \frac{1}{24}\,e^{-3}$

30. $\displaystyle\int_{-\infty}^{\infty} \frac{xe^{ix}}{(x^2+1)(x^2+4)}\,dx = 2\pi i[\operatorname{Res}(f(z), i) + \operatorname{Res}(f(z), 2i)] = 2\pi i\left[\frac{1}{6}\,e^{-1} - \frac{1}{6}\,e^{-2}\right] = \frac{\pi}{3}(e^{-1}-e^{-2})i;$

$$\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)(x^2+4)}\,dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{(x^2+1)(x^2+4)}\,dx\right) = \frac{\pi}{3}(e^{-1}-e^{-2}).$$

Therefore

$$\int_0^{\infty} \frac{x\sin x}{(x^2+1)(x^2+4)}\,dx = \frac{1}{2}\left[\frac{\pi}{3}(e^{-1}-e^{-2})\right] = \frac{\pi}{6}(e^{-1}-e^{-2}).$$

31. Consider the contour integral $\displaystyle\oint_C \frac{e^{iz}}{z}\,dz$. The function $f(z) = \dfrac{1}{z}$ has a simple pole at $z = 0$. If we use the contour C shown in Figure 19.14, it follows from the Cauchy-Goursat Theorem that

$$\oint_C = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_r^R = 0.$$

Taking limits as $R \to \infty$ and as $r \to 0$ and using Theorem 19.17 we then find

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx - \pi i\operatorname{Res}(f(z)e^{iz}, 0) = 0 \quad \text{or} \quad \text{P.V.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \pi i.$$

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Equating the imaginary parts of $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x}\, dx = 0 + \pi i$ gives

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

32. Consider the contour integral $\displaystyle\oint_C \frac{e^{iz}}{z(z^2+1)}\, dz$. The function $f(z) = \dfrac{1}{z(z^2+1)}$ has simple poles at $z = 0$ and at $z = i$. If we use the contour C shown in Figure 19.14, it follows from Theorem 19.14 that

$$\oint_C = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_r^R = 2\pi i \operatorname{Res}(f(z), i).$$

Taking limits as $R \to \infty$ and as $r \to 0$ and using Theorem 19.17 we then find

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}\, dx - \pi i \operatorname{Res}(f(z)e^{iz}, 0) = 2\pi i \operatorname{Res}(f(z)e^{iz}, i)$$

or

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}\, dx = \pi i + 2\pi i \left(-\frac{e^{-1}}{2}\right).$$

Equating the imaginary parts of $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x(x^2+1)}\, dx = 0 + \pi(1 - e^{-1})i$ gives

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\, dx = \pi(1 - e^{-1}).$$

33. $\displaystyle\int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2}{i}\oint_C \frac{z}{(z^2 + 2az + 1)^2}\, dz \quad (C \text{ is } |z| = 1) \quad = \dfrac{2}{i}\displaystyle\oint_C \frac{z}{(z - r_1)^2(z - r_2)^2}\, dz$

where $r_1 = -a + \sqrt{a^2 - 1}$, $r_2 = -a - \sqrt{a^2 - 1}$. Now

$$\oint_C \frac{z}{(z - r_1)^2(z - r_2)^2}\, dz = 2\pi i \operatorname{Res}(f(z), r_1) = 2\pi i \frac{a}{4(\sqrt{a^2 - 1}\,)^3} = \frac{a\pi}{2(\sqrt{a^2 - 1}\,)^3}\, i.$$

Thus

$$\int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{2}{i} \cdot \frac{a\pi}{2(\sqrt{a^2 - 1}\,)^3}\, i = \frac{a\pi}{(\sqrt{a^2 - 1}\,)^3}.$$

When $a = 2$ we obtain

$$\int_0^{\pi} \frac{d\theta}{(2 + \cos\theta)^2} = \frac{2\pi}{(\sqrt{3}\,)^3} \qquad \text{and so} \qquad \int_0^{2\pi} \frac{d\theta}{(2 + \cos\theta)^2} = \frac{4\pi}{3\sqrt{3}}.$$

34. $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta = \frac{i}{2b}\oint_C \frac{z^2 - 1}{z^2(z - r_1)(z - r_2)}\, dz \quad (C \text{ is } |z| = 1)$ where $r_1 = (-a + \sqrt{a^2 - b^2}\,)/b$, $r_2 = (-a - \sqrt{a^2 - b^2}\,)/b$. Now

$$\oint_C \frac{z^2 - 1}{z^2(z - r_1)(z - r_2)}\, dz = 2\pi i[\operatorname{Res}(f(z), 0) + \operatorname{Res}(f(z), r_1)] = 2\pi i \left[-\frac{2a}{b} + \frac{2\sqrt{a^2 - b^2}}{b}\right].$$

Thus

$$\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta = \frac{2\pi}{b^2}(a - \sqrt{a^2 - b^2}\,), \quad a > b > 0.$$

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When $a = 5$, $b = 4$ we obtain

$$\int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta}\, d\theta = \frac{2\pi}{16}(5 - \sqrt{9}) = \frac{\pi}{4}.$$

35. Consider the contour integral $\displaystyle\oint_C \frac{e^{az}}{1+e^z}\, dz$. The function $f(z) = \dfrac{e^{az}}{1+e^z}$ has simple poles at $z = \pi i$, $3\pi i$, $5\pi i$, ... in the upper plane. Using the contour in Figure 19.15 we have from Theorem 19.14

$$\oint_C = \int_{-r} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 2\pi i \operatorname{Res}(f(z), \pi i) = -2\pi i e^{a\pi i}.$$

On C_2, $z = r + iy$, $0 \le y \le \pi$, $dz = i\, dy$,

$$\left| \int_{C_2} \frac{e^{az}}{1+e^z}\, dz \right| \le \frac{e^{ar}}{e^r - 1}(2\pi).$$

Because $0 < a < 1$, this last expression goes to 0 as $r \to \infty$. On C_3, $z = x + 2\pi i$, $-r \le x \le r$, $dz = dx$,

$$\int_{C_3} \frac{e^{az}}{1+e^z}\, dz = \int_r^{-r} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\, dx = -e^{2a\pi i} \int_{-r}^{r} \frac{e^{ax}}{1+e^x}\, dx.$$

On C_4, $z = -r + iy$, $0 \le y \le 2\pi$, $dz = i\, dy$,

$$\left| \int_{C_4} \frac{e^{az}}{1+e^z}\, dz \right| \le \frac{e^{-ar}}{1 - e^{-r}}(2\pi).$$

Because $0 < a$, this last expression goes to 0 as $r \to \infty$. Hence

$$\int_{-r}^{r} \frac{e^{ax}}{1+e^x}\, dx - e^{2a\pi i} \int_{-r}^{r} \frac{e^{ax}}{1+e^x}\, dx = -2\pi i e^{a\pi i}$$

gives, as $r \to \infty$,

$$(1 - e^{2a\pi i}) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx = -2\pi i e^{a\pi i}.$$

That is

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx = -\frac{2\pi i e^{a\pi i}}{1 - e^{2a\pi i}} = \frac{\pi}{\dfrac{e^{a\pi i} - e^{-a\pi i}}{2i}} = \frac{\pi}{\sin ax}.$$

36. Using the Fourier sine transform with respect to y the partial differential equation becomes $\dfrac{d^2U}{dx^2} - \alpha^2 U = 0$ and so

$$U(x, \alpha) = c_1 \cosh \alpha x + c_2 \sinh \alpha x.$$

The boundary condition $u(0, y)$ becomes $U(0, \alpha) = 0$ and so $c_1 = 0$. Thus $U(x, \alpha) = c_2 \sinh \alpha x$. Now to evaluate

$$U(\pi, \alpha) = \int_0^{\infty} \frac{2y}{y^4 + 4} \sin \alpha y\, dy = \int_{-\infty}^{\infty} \frac{y}{y^4 + 4} \sin \alpha y\, dy$$

we use the contour integral $\displaystyle\int_C \frac{z e^{i\alpha z}}{z^4 + 4}\, dz$ and

$$\int_{-\infty}^{\infty} \frac{x e^{i\alpha x}}{x^4 + 4}\, dx = 2\pi i[\operatorname{Res}(f(z), 1+i) + \operatorname{Res}(f(z), -1+i)] = 2\pi i \left[-\frac{1}{8} i e^{(-1+i)\alpha} + \frac{1}{8} i e^{(-1-i)\alpha} \right] = \frac{\pi}{2}(e^{-\alpha} \sin \alpha) i$$

$$\int_{-\infty}^{\infty} \frac{x \sin \alpha x}{x^4 + 4}\, dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{x e^{i\alpha x}}{x^4 + 4}\, dx \right) = \frac{\pi}{2} e^{-\alpha} \sin \alpha.$$

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Finally, $U(\pi,\alpha) = \dfrac{\pi}{2}e^{-\alpha}\sin\alpha = c_2\sinh\alpha\pi$ gives $c_2 = \dfrac{\pi}{2}\,\dfrac{e^{-\alpha}\sin\alpha}{\sinh\alpha\pi}$. Hence $U(x,\alpha) = \dfrac{\pi}{2}\,\dfrac{e^{-\alpha}\sin\alpha}{\sinh\alpha\pi}\sinh\alpha x$ and

$$u(x,y) = \int_0^\infty \frac{e^{-\alpha}\sin\alpha}{\sinh\alpha\pi}\sinh\alpha x\sin\alpha y\,d\alpha.$$

Chapter 19 Review Exercises

1. True **2.** False **3.** False **4.** True

5. True **6.** True **7.** True **8.** five

9. $1/\pi$ **10.** three; $-1/6$ **11.** $|z-i| = \sqrt{5}$ **12.** False

13. $\dfrac{e^{z(1+i)}+e^{z(1-i)}}{2} = \dfrac{1}{2}\left(1+z(1+i)+\dfrac{z^2}{2!}(1+i)^2+\cdots\right)+\dfrac{1}{2}\left(1+z(1-i)+\dfrac{z^2}{2!}(1-i)^2+\cdots\right)$

$$= 1+z\left[\frac{(1+i)+(1-i)}{2}\right]+\frac{z^2}{2!}\left[\frac{(1+i)^2+(1-i)^2}{2}\right]+\cdots = 1+\sum_{k=1}^{\infty}\frac{(\sqrt{2}\,)^k\cos\frac{k\pi}{4}}{k!}\,z^k$$

Here we have used $(1+i)^n = (\sqrt{2}\,)^n e^{n\pi i/4}$ and $(1-i)^n = (\sqrt{2}\,)^n e^{-n\pi i/4}$ so that

$$\frac{(1+i)^n+(1-i)^n}{2} = (\sqrt{2}\,)^n\left[\frac{e^{n\pi i/4}+e^{-n\pi i/4}}{2}\right] = (\sqrt{2}\,)^n\cos\frac{n\pi}{4}.$$

14. $\sin\dfrac{\pi}{z} = 0$ implies $z = \dfrac{1}{n}$, $n = \pm1, \pm2, \ldots$. All singularities are isolated except the singularity $z = 0$.

15. $f(z) = \dfrac{1}{z^4}\left[1-\left(1+\dfrac{iz}{1!}+\dfrac{i^2z^2}{2!}+\dfrac{i^3z^3}{3!}+\dfrac{i^4z^4}{4!}+\cdots\right)\right] = -\dfrac{i}{z^3}+\dfrac{1}{2!z^2}+\dfrac{i}{3!z}-\dfrac{1}{4!}-\dfrac{iz}{5!}+\cdots$

16. $e^{z/(z-2)} = e\cdot e^{2/(z-2)} = e\left(1+\dfrac{2}{z-2}+\dfrac{2^2}{2!(z-2)^2}+\dfrac{2^3}{3!(z-2)^3}+\cdots\right) = e\displaystyle\sum_{k=0}^{\infty}\frac{2^k}{k!}(z-2)^{-k}$

17. $(z-i)^2\sin\dfrac{1}{z-i} = (z-i)^2\left[\dfrac{1}{z-i}-\dfrac{1}{3!(z-i)^3}+\dfrac{1}{5!(z-i)^5}-\cdots\right] = \cdots+\dfrac{1}{5!(z-i)^3}-\dfrac{1}{3!(z-i)}+(z-i)$

18. $\dfrac{1-\cos z^2}{z^5} = \dfrac{1}{z^5}\left[1-\left(1-\dfrac{z^4}{2!}+\dfrac{z^8}{4!}-\dfrac{z^{12}}{6!}+\dfrac{z^{16}}{8!}-\cdots\right)\right] = \dfrac{1}{2!z}-\dfrac{z^3}{4!}+\dfrac{z^7}{6!}-\dfrac{z^{11}}{8!}+\cdots$

19. **(a)** $f(z) = \dfrac{1}{z-3}-\dfrac{1}{z-1} = \dfrac{1}{1-z}-\dfrac{1}{3}\cdot\dfrac{1}{1-\frac{z}{3}} = (1+z+z^2+z^4+\cdots)-\dfrac{1}{3}\left(1+\dfrac{z}{3}+\dfrac{z^2}{3^2}+\dfrac{z^3}{3^3}+\cdots\right)$

$$= \frac{2}{3}+\frac{8}{9}z+\frac{26}{27}z^2$$

(b) $f(z) = -\dfrac{1}{z}\cdot\dfrac{1}{1-\frac{1}{z}}-\dfrac{1}{3}\cdot\dfrac{1}{1-\frac{z}{3}} = -\dfrac{1}{z}\left(1+\dfrac{1}{z}+\dfrac{1}{z^2}+\dfrac{1}{z^3}+\cdots\right)-\dfrac{1}{3}\left(1+\dfrac{z}{3}+\dfrac{z^2}{3^2}+\dfrac{z^3}{3^3}+\cdots\right)$

$$= \cdots-\frac{1}{z^3}-\frac{1}{z^2}-\frac{1}{z}-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\cdots$$

(c) $f(z) = -\dfrac{1}{z}\cdot\dfrac{1}{1-\frac{1}{z}}+\dfrac{1}{z}\cdot\dfrac{1}{1-\frac{3}{z}} = -\dfrac{1}{z}\left(1+\dfrac{1}{z}+\dfrac{1}{z^2}+\dfrac{1}{z^3}+\cdots\right)+\dfrac{1}{z}\left(1+\dfrac{3}{z}+\dfrac{3^2}{z^2}+\dfrac{3^3}{z^3}+\cdots\right)$

$$= \frac{2}{z^2}+\frac{8}{z^3}+\frac{26}{z^4}+\cdots$$

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(d) $f(z) = -\dfrac{1}{z-1} - \dfrac{1}{2}\cdot\dfrac{1}{1-\dfrac{z-1}{2}} = -\dfrac{1}{z-1} - \dfrac{1}{2}\left(1 + \dfrac{z-1}{2} + \dfrac{(z-1)^2}{2^2} + \dfrac{(z-1)^3}{3!} + \cdots\right)$

$= -\dfrac{1}{z-1} - \dfrac{1}{2} - \dfrac{z-1}{2^2} - \dfrac{(z-1)^2}{2^3} - \cdots$

20. **(a)** $f(z) = \dfrac{1}{25}\left(1-\dfrac{z}{5}\right)^{-2} = \dfrac{1}{25}\left[1 + (-2)\left(-\dfrac{z}{5}\right) + \dfrac{(-2)(-3)}{2!}\left(-\dfrac{z}{5}\right)^2 + \dfrac{(-2)(-3)(-4)}{3!}\left(-\dfrac{z}{5}\right)^3 + \cdots\right]$

$= \dfrac{1}{25} + 2\,\dfrac{z}{5^3} + 3\,\dfrac{z^2}{5^4} + 4\,\dfrac{z^3}{5^5} + \cdots$

(b) $(z-5)^{-2} = \dfrac{1}{z^2}\left(1-\dfrac{5}{z}\right)^{-2} = \dfrac{1}{z^2}\left[1 + (-2)\left(-\dfrac{5}{z}\right) + \dfrac{(-2)(-3)}{2!}\left(-\dfrac{5}{z}\right)^2 + \dfrac{(-2)(-3)(-4)}{3!}\left(-\dfrac{5}{z}\right)^3 + \cdots\right]$

$= \dfrac{1}{z^2} + 2\,\dfrac{5}{z^3} + 3\,\dfrac{5^2}{z^4} + 4\,\dfrac{5^3}{z^5} + \cdots$

(c) $\dfrac{1}{(z-5)^2}$ is the Laurent series.

21. $\displaystyle\oint_C \frac{2z+5}{z(z+2)(z-1)^4}\,dz = 2\pi i[\operatorname{Res}(f(z),0) + \operatorname{Res}(f(z),-2)] = \frac{404}{81}\,\pi i$

22. $\displaystyle\oint_C \frac{z^2}{(z-1)^3(z^2+4)}\,dx = 2\pi i \operatorname{Res}(f(z),1) = \frac{8\pi}{125}\,i$

23. $\displaystyle\oint_C \frac{1}{2\sin z - 1}\,dz = 2\pi i \operatorname{Res}\left(f(z), \frac{\pi}{6}\right) = \frac{2\pi}{\sqrt{3}}\,i$

24. $\displaystyle\oint_C \frac{z+1}{\sinh z}\,dz = 2\pi i[\operatorname{Res}(f(z),0) + \operatorname{Res}(f(z),\pi i)] = 2\pi i[1 + (-\pi i - 1)] = 2\pi^2$

25. $\displaystyle\oint_C \frac{e^{2z}}{z^4+2z^3+2z^2}\,dz = 2\pi i[\operatorname{Res}(f(z),0) + \operatorname{Res}(f(z),-1+i) + \operatorname{Res}(f(z),-1-i)]$

$= 2\pi i\left[\dfrac{1}{2} + \dfrac{e^{-2}}{4}(\cos 2 + i\sin 2) + \dfrac{e^{-2}}{4}(\cos 2 - i\sin 2)\right] = \pi(1 + e^{-2}\cos 2)i$

26. $\displaystyle\oint_C \frac{1}{z^4-2z^2+4}\,dz = 2\pi i\left[\operatorname{Res}\left(f(z), \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}\,i\right) + \operatorname{Res}\left(f(z), -\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}\,i\right)\right] = \frac{\pi}{2\sqrt{2}}$

27. $\displaystyle\oint_C \frac{1}{z(e^z-1)}\,dz = 2\pi i \operatorname{Res}(f(z),0) = -\pi i$. Note: $z = 0$ is a pole of order two, and so

$$\operatorname{Res}(f(z),0) = \lim_{z\to 0}\frac{d}{dz}\,z^2\cdot\frac{1}{z^2\left(1 + \dfrac{z}{2!} + \dfrac{z^2}{3!} + \cdots\right)} = \lim_{z\to 0} -\frac{\left(\dfrac{1}{2!} + \dfrac{2z}{3!} + \cdots\right)}{\left(1 + \dfrac{z}{2!} + \dfrac{z^2}{3!} + \cdots\right)^2} = -\frac{1}{2}$$

28. $\displaystyle\oint_C \frac{z}{(z-1)(z+1)^{10}}\,dz = 2\pi i[\operatorname{Res}(f(z),1) + \operatorname{Res}(f(z),-1)] = 2\pi i\left[\frac{1}{2^{10}} + \left(-\frac{1}{2^{10}}\right)\right] = 0$

29. Using two integrals,

$$\oint_C ze^{3/z}\,dz + \oint_C \frac{\sin z}{z^2(z-\pi)^3}\,dz = 2\pi i \operatorname{Res}(f(z),0) + 2\pi i[\operatorname{Res}(f(z),0) + \operatorname{Res}(f(z),\pi)]$$

$$= 2\pi i\cdot\frac{9}{2} + 2\pi i\left[-\frac{1}{\pi^3} + \frac{2}{\pi^3}\right] = \left(9\pi + \frac{2}{\pi^2}\right)i.$$

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Note: In the first integral $z = 0$ is an essential singularity and the residue is obtained from the Laurent series

$$ze^{3/z} = \cdots + \frac{3^3}{3!z^2} + \frac{3^2}{2!z} + 3 + z.$$

30. $\displaystyle\oint_C \csc \pi z \, dz = 2\pi i[\text{Res}(f(z), 0) + \text{Res}(f(z), 1) + \text{Res}(f(z), 2)] = 2\pi i \left[\frac{1}{\pi} + \left(-\frac{1}{\pi}\right) + \frac{1}{\pi}\right] = 2i$

31. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 2x + 2)(x^2 + 1)^2} \, dx = 2\pi i[\text{Res}(f(z), -1 + i) + \text{Res}(f(z), i)] = 2\pi i \left[\frac{3}{25} - \frac{4}{25}\, i - \frac{3}{25} + \frac{9}{100}\, i\right] = \frac{7\pi}{50}$

32. $\displaystyle\int_{-\infty}^{\infty} \frac{x + ai}{x^2 + a^2} \, e^{ix} \, dx = \int_{-\infty}^{\infty} \frac{x \cos x - a \sin x}{x^2 + a^2} \, dx + i \int_{-\infty}^{\infty} \frac{a \cos x + x \sin x}{x^2 + a^2} \, dx = 0 + 2\pi i \, \text{Res}(f(z), ai) = 2\pi i e^{-a}$

Thus

$$\int_{-\infty}^{\infty} \frac{a \cos x + x \sin x}{x^2 + a^2} \, dx = 2\pi e^{-a}.$$

33. $\displaystyle\int_0^{2\pi} \frac{\cos^2 \theta}{2 + \sin \theta} \, d\theta = \frac{1}{2} \oint_C \frac{z^4 + 2z^2 + 1}{z^2(z^2 + 4iz - 1)} \, dz \ (C \text{ is } |z| = 1) \ = \pi i[\text{Res}(f(z), 0) + \text{Res}(f(z), (-2 + \sqrt{3}\,)i)]$

$$= \pi i[-4i + 2\sqrt{3}\, i] = (4 - 2\sqrt{3}\,)\pi$$

[Note: The answer in the text is correct but not simplified.]

34. $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos \theta} \, d\theta = -\frac{1}{2i} \oint_C \frac{z^6 + 1}{z^3(z - 2)(2z - 1)} \, dz \ (C \text{ is } |z| = 1) \ = -\pi \left[\text{Res}(f(z), 0) + \text{Res}\left(f(z), \frac{1}{2}\right)\right]$

$$= -\pi \left[\frac{21}{8} + \left(-\frac{65}{24}\right)\right] = \frac{\pi}{12}$$

35. The integrand of $\displaystyle\oint_C \frac{1 - e^{iz}}{z^2} \, dz$ has a simple pole at $z = 0$. Using a contour as in Figure 19.14 of Section 19.6 we have

$$\oint_C = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_R} + \int_r^R = 0.$$

By taking limits as $R \to 0$ and as $r \to 0$, and using Theorem 19.17 we find

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1 - e^{ix}}{x^2} \, dx - \pi i \, \text{Res}(f(z), 0) = 0.$$

Thus

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1 - \cos x - i \sin x}{x^2} \, dx = \pi.$$

Equating real parts gives

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} \, dx = \pi.$$

Finally

$$\int_0^{\infty} \frac{1 - \cos x}{x^2} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} \, dx = \frac{\pi}{2}.$$

36. We have

$$Ce^{-a^2 z^2} e^{ibz} \, dz = \int_{-r}^{r} + \int_{C_1} + \int_{C_2} + \int_{C_3} = 0$$

by the Cauchy-Goursat Theorem. Therefore

$$\int_{-r}^{r} = -\int_{C_1} - \int_{C_2} - \int_{C_3}.$$

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Let C_1 and C_3 denote the vertical sides of the rectangle. By the ML-inequality, $\displaystyle\int_{C_1} \to 0$ and $\displaystyle\int_{C_3} \to 0$ as $r \to \infty$. On C_2, $z = x + \dfrac{b}{2a^2}\,i$, $-r \le x \le r$, $dz = dx$,

$$\int_{-\infty}^{\infty} e^{-ax^2} e^{ibx}\,dx = -\int_{\infty}^{-\infty} e^{-a^2\left(x+\frac{b}{2a^2}i\right)^2} e^{ib\left(x+\frac{b}{2a^2}i\right)}\,dx = \int_{-\infty}^{\infty} e^{-a^2x^2} e^{-b^2/4a^2}\,dx$$

$$\int_{-\infty}^{\infty} e^{-ax^2}(\cos bx + i\sin bx)\,dx = e^{-b^2/4a^2}\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx.$$

Using the given value of $\displaystyle\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx$ and equating real and imaginary parts gives

$$\int_{-\infty}^{\infty} e^{-ax^2}\cos bx\,dx = \frac{\sqrt{\pi}}{a}\,e^{-b^2/4a^2} \qquad \text{and so} \qquad \int_{0}^{\infty} e^{-ax^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2a}\,e^{-b^2/4a^2}.$$

37. $a_k = \dfrac{1}{2\pi i}\displaystyle\oint_C \frac{e^{(u/2)(z-z^{-1})}}{z^{k+1}}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{(u/2)(e^{it}-e^{-it})}}{(e^{it})^{k+1}}\,ie^{it}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{(u/2)(2i\sin t)}e^{-kit}\,dt$

$= \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{-i(kt-u\sin t)}dt = \frac{1}{2\pi}\int_0^{2\pi}[\cos(kt-u\sin t) - i\sin(kt-u\sin t)]\,dt = \frac{1}{2\pi}\int_0^{2\pi}\cos(kt-u\sin t)\,dt$

since $\displaystyle\int_0^{2\pi}\sin(kt-u\sin t)\,dt = 0$. (To obtain this last result, expand the integrand and let $t = 2\pi - x$.)

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20 Conformal Mappings and Applications

Exercises 20.1

1. For $w = \dfrac{1}{z}$, $u = \dfrac{x}{x^2+y^2}$ and $v = \dfrac{-y}{x^2+y^2}$. If $y = x$, $u = \dfrac{1}{2}\dfrac{1}{x}$, $v = -\dfrac{1}{2}\dfrac{1}{x}$, and so $v = -u$. The image is the line $v = -u$ (with the origin $(0, 0)$ excluded.)

2. If $y = 1$, $u = \dfrac{x}{x^2+1}$ and $v = \dfrac{-1}{x^2+1}$. It follows that $u^2 + v^2 = \dfrac{1}{x^2+1} = -v$ and so $u^2 + (v + \frac{1}{2})^2 = (\frac{1}{2})^2$. This is a circle with radius $r = \frac{1}{2}$ and center at $(0, -\frac{1}{2}) = -\frac{1}{2}\,i$. The circle can also be described by $|w + \frac{1}{2}\,i| = \frac{1}{2}$.

3. For $w = z^2$, $u = x^2 - y^2$ and $v = 2xy$. If $xy = 1$, $v = 2$ and so the hyperbola $xy = 1$ is mapped onto the line $v = 2$.

4. If $x^2 - y^2 = 4$, $u = 4$ and so the hyperbola $x^2 - y^2 = 4$ is mapped onto the vertical line $u = 4$.

5. For $w = \operatorname{Ln} z$, $u = \log_e |z|$ and $v = \operatorname{Arg} z$. The semi-circle $|z| = 1$, $y > 0$ may also be described by $r = 1$, $0 < \theta < \pi$. Therefore $u = 0$ and $0 < v < \pi$. The image is therefore the open line segment from $z = 0$ to $z = \pi i$.

6. If $\theta = \pi/4$, then $v = \theta = \pi/4$. In addition $u = \log_e r$ will vary from $-\infty$ to ∞. The image is therefore the horizontal line $v = \pi/4$.

7. For $w = z^{1/2} = (re^{i\theta})^{1/2} = r^{1/2}e^{i\theta/2}$ and $\theta = \theta_0$, $w = \sqrt{r}\,e^{i\theta_0/2}$. Therefore $\operatorname{Arg} w = \theta_0/2$ and so the image is the ray $\theta = \theta_0/2$.

8. If $r = 2$ and $0 \le \theta \le \frac{\pi}{2}$, $w = \sqrt{2}\,e^{i\theta/2}$. Therefore $|w| = \sqrt{2}$ and $0 \le \operatorname{Arg} w \le \pi/4$. This image is a circular arc.

9. For $w = e^z$, $u = e^x \cos y$ and $v = e^x \sin y$. Therefore if $e^x \cos y = 1$, $u = 1$. The curve $e^x \cos y = 1$ is mapped into the line $u = 1$. Since $v = \dfrac{\sin y}{\cos y} = \tan y$, v varies from $-\infty$ to ∞ and the image is the line $u = 1$.

10. If $w = z + \dfrac{1}{z}$ and $z = e^{it}$, $w = e^{it} - e^{-it} = 2\cos t$. Therefore $u = 2\cos t$ and $v = 0$ and so the image is the closed interval $[-2, 2]$ on the u-axis.

11. The first quadrant may be described by $r > 0$, $0 < \theta < \pi/2$. If $w = 1/z$ and $z = re^{i\theta}$, $w = \frac{1}{r}e^{-i\theta}$. Therefore $\operatorname{Arg} w = -\theta$ and so $-\pi/2 < \operatorname{Arg} w < 0$. The image is therefore the fourth quadrant.

12. For $w = \dfrac{1}{z}$, $u = \dfrac{x}{x^2+y^2}$ and $v = \dfrac{-y}{x^2+y^2}$. The line $y = 0$ is mapped to the line $v = 0$, and, from Problem 2, the line $y = 1$ is mapped onto the circle $|w + \frac{1}{2}\,i| = \frac{1}{2}$. Since $f(\frac{1}{2}\,i) = -2i$, the region $0 \le y \le 1$ is mapped onto the points in the half-plane $v \le 0$ which are on or outside the circle $|w + \frac{1}{2}\,i| = \frac{1}{2}$. (The image does not include the point $w = 0$.)

13. Since $w = e^{x+iy} = e^x e^{iy}$ and $\pi/4 \le y \le \pi/2$, $\pi/4 \le \operatorname{Arg} w \le \pi/2$ and $|w| = e^x$. The image is therefore the angular wedge defined by $\pi/4 \le \operatorname{Arg} w \le \pi/2$.

14. Since $w = e^{x+iy} = e^x e^{iy}$ and $0 \le x \le 1$, $0 \le y \le \pi$, we have $|w| = e^x$ and $\operatorname{Arg} w = y$. Therefore $1 \le |w| \le e$ and $0 \le \operatorname{Arg} w \le \pi$. These inequalities define a semi-angular region in the w-plane.

15. The mapping $w = z + 4i$ is a translation which maps the circle $|z| = 1$ to a circle of radius $r = 1$ and with center $w = 4i$. This circle may be described by $|w - 4i| = 1$.

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16. If $w = 2z - 1$ and $|z| = 1$, then, since $z = \dfrac{w+1}{2}$, $\left|\dfrac{w+1}{2}\right| = 1$ or $|w+1| = 2$. The image is a circle with center at $w = -1$ and with radius $r = 2$.

17. The mapping $w = iz$ is a rotation through 90° since $i = e^{i\pi/2}$. Therefore the strip $0 \le y \le 1$ is rotated through 90° and so the strip $-1 \le u \le 0$ is the image in the w-plane.

18. Since $w = (1+i)z = \sqrt{2}\, e^{i\pi/4} z$, the mapping is the composite of a rotation through 45° and a magnification by $\alpha = \sqrt{2}$. The image of the first quadrant is therefore the angular wedge $\pi/4 \le \operatorname{Arg} w \le 3\pi/4$.

19. The power function $w = z^3$ changes the opening of the wedge $0 \le \operatorname{Arg} z \le \pi/4$ by a factor of 3. Therefore the image region is $0 \le \operatorname{Arg} w \le 3\pi/4$.

20. The power function $w = z^{1/2}$ changes the opening of the wedge $0 \le \operatorname{Arg} z \le \pi/4$ by a factor of 1/2. Therefore the image region is $0 \le \operatorname{Arg} w \le \pi/8$.

21. We first let $z_1 = z - i$ to map the region $1 \le y \le 4$ to the region $0 \le y_1 \le 3$. We then let $w = e^{-i\pi/2} z_1$ to rotate this strip through −90°. Therefore $w = -i(z-i) = -iz - 1$ maps $1 \le y \le 4$ to the strip $0 \le u \le 3$.

22. The mapping $w = z - i$ lowers the strip $1 \le y \le 4$ one unit so that the image is $0 \le v \le 3$ in the w-plane.

23. We first let $z_1 = z - 1$ to map the disk $|z-1| \le 1$ to the disk $|z_1| \le 1$. We then use the magnification $w = 2z_1$ to obtain $|w| \le 2$ as the image. The composite of these two mappings is $w = 2(z-1)$.

24. The mapping $w = iz$ will rotate the strip $-1 \le x \le 1$ through 90° so that the strip $-1 \le v \le 1$ results.

25. We first use $z_1 = e^{-i\pi/4} z$ to rotate the wedge $\pi/4 \le \operatorname{Arg} z \le \pi/2$ to the wedge $0 \le \operatorname{Arg} z_1 \le \pi/4$. The power function $w = z_1^4$ then changes the opening of this wedge by a factor of 4 so that the strip $0 \le \operatorname{Arg} w \le \pi$ results. The composite of these two mappings is $w = (e^{-\pi/4i} z)^4 = e^{-\pi i} z^4 = -z^4$.

26. The magnification $z_1 = \frac{\pi}{4} z$ maps the strip $0 \le y \le 4$ to the strip $0 \le y_1 \le \pi$. By Example 1, Section 20.1, $w = e^{z_1}$ maps this strip onto the upper half-plane $v \ge 0$. The composite of these two mappings is $w = e^{\frac{\pi}{4} z}$.

27. By Example 1, Section 20.1, $z_1 = e^z$ maps the strip $0 \le y \le \pi$ onto the upper half-plane $y_1 \ge 0$, or $0 \le \operatorname{Arg} z_1 \le \pi$. The power function $w = z_1^{3/2}$ changes the opening of this wedge by a factor of 3/2 so the wedge $0 \le \operatorname{Arg} w \le 3\pi/2$ results. The composite of these two mappings is $w = (e^z)^{3/2} = e^{3z/2}$.

28. The power function $z_1 = z^{2/3}$ maps the wedge $0 \le \operatorname{Arg} z \le 3\pi/2$ to the upper half-plane $0 \le \operatorname{Arg} z_1 \le \pi$. We then let $w = e^{-i\pi/2} z_1 + 2 = -iz_1 + 2$ to rotate the upper half-plane through −90° and then translate 2 units to the right. Therefore the composite function is $w = -iz^{2/3} + 2$ and the image region is the half-plane $u \ge 2$.

29. We may obtain the image region by (1) rotating R through 180° and (2) raising the resulting region one unit in the vertical direction. Therefore $w = e^{i\pi} z + i = -z + i$.

30. The mapping $z_1 = -(z - \pi i)$ lowers R by π units in the vertical direction and then rotates the resulting region through 180°. The image region R_1 is upper half-plane $y_1 \ge 0$. By Example 1, Section 20.1, $w = \operatorname{Ln} z_1$ maps R_1 onto the strip $0 \le v \le \pi$. The composite of these two mappings is $w = \operatorname{Ln}(\pi i - z)$.

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Exercises 20.2

1. Since $f'(z) = 3(z^2 - 1)$, f is conformal at all points except $z = \pm 1$.

2. Since $f'(z) = -\sin z$, f is conformal at all points except $z = \pm n\pi$.

3. $f'(z) = 1 + e^z$ and $1 + e^z = 0$ for $z = \pm i \pm 2n\pi i$. Therefore f is conformal except for $z = \pi i \pm 2n\pi i$.

4. $f(z) = z + \operatorname{Ln} z + 1$ is analytic for all z except $z = y \leq 0$. But $f'(z) = 1 + 1/z$ and $1 + 1/z = 0$ for $z = -1$. Therefore $f(z)$ is conformal at all points except those on the branch cut $y \leq 0$.

5. $f(z) = (z^2 - 1)^{1/2} = e^{\frac{1}{2}\operatorname{Ln}(z^2-1)}$ is analytic for all z outside the interval $[-1, 1]$ on the real axis. This follows from the fact that $z^2 - 1 = (x^2 - y^2 - 1) + 2xyi$ and so we must exclude values of z for which $v = 2xy = 0$ and $u = x^2 - y^2 - 1 \leq 0$. Therefore $y = 0$ and $x^2 \leq 1$. $f'(z) = z/(z^2 - 1)^{1/2}$ is non-zero outside this interval. Therefore f is conformal except for $z = x$, $-1 \leq x \leq 1$.

6. The function $f(z) = \pi i - \frac{1}{2}[\operatorname{Ln}(z + 1) + \operatorname{Ln}(z - 1)]$ is analytic except on the branch cut $x - 1 \leq 0$ or $x \leq 1$, and

$$f'(z) = -\frac{1}{2}\left(\frac{1}{z+1} + \frac{1}{z-1}\right) = -\frac{z}{z^2 - 1}$$

is non-zero for $z \neq 0, \pm 1$. Therefore f is conformal except for $z = x$, $x \leq 1$.

7. $f(z) = \cos z = \sin(\pi/2 - z)$ is the composite of $z_1 = \pi/2 - z$ and $w = \sin z$. The strip $0 \leq x \leq \pi$ is mapped onto the strip $-\pi/2 \leq x \leq \pi/2$ by $z_1 = \pi/2 - z$ and $w = \sin z$ maps this strip onto the region shown in Figure 20.11(b). The horizontal segment $z(t) = t + ib$, $0 < t < \pi$ is first mapped to the horizontal segment $z_i(t) = (\pi/2 - t) - ib$, $0 < t < \pi$. This latter segment is mapped onto the lower or upper portion of the ellipse

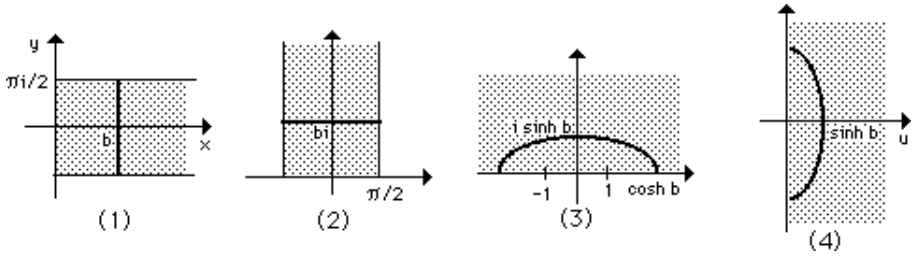
$$\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1$$

according to whether $b > 0$ or $b < 0$. See Figure 20.11.

8. $f(z) = \sinh z = -i \sin(iz)$ is the composite of $z_1 = iz$, $z_2 = \sin z_1$, and $w = -iz_2$. The strip $-\pi/2 \leq y \leq \pi/2$, $x \geq 0$ is rotated through 90° so that the image is the strip $-\pi/2 \leq x \leq \pi/2$, $y \geq 0$. This latter strip is mapped to the upper half-plane $y_2 \geq 0$ by $z_2 = \sin z_1$, and $w = -iz_2$ rotates this upper half-plane through −90°. The final image region is the half-plane $u \geq 0$. A vertical line segment in the original strip is mapped to the right hand side of the ellipse

$$\frac{u^2}{\sinh^2 b} + \frac{v^2}{\cosh^2 b} = 1.$$

See the figures below.



9. $f(z) = (\sin z)^{1/4}$ is the composite of $z_1 = \sin z$ and $w = z_1^{1/4}$. The region $-\pi/2 \leq x \leq \pi/2$, $y \geq 0$ is mapped to the upper half-plane $y_2 \geq 0$ by $z_1 = \sin z$ (See Example 2) and the power function $w = z_1^{1/4}$ maps this upper half-plane to the angular wedge $0 \leq \operatorname{Arg} w \leq \pi/4$. The real interval $[-\pi/2, \pi/2]$ is first mapped to $[-1, 1]$ and then to the union of the line segments from $e^{i\frac{\pi}{4}}$ to 0 and 0 to 1. See the figures on the following page.

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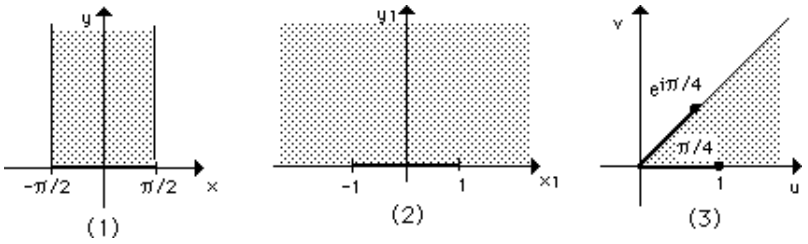
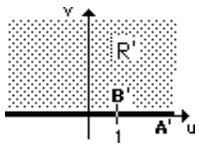
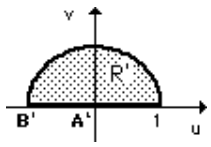


10. From Example 3, $u = (r + 1/r)\cos\theta$ and $v = (r - 1/r)\sin\theta$. If $r = 1$, $u = 2\cos\theta$ and $v = 0$. Therefore the image of the circle $|z| = 1$ is the real interval $[-2, 2]$. If $0 < r < 1$, $u^2/a^2 + v^2/b^2 = 1$ where $a = r + 1/r > 2$ and $b = 1/r - r$. The resulting ellipse together with $[-2, 2]$ fill up the entire w-plane. Therefore the image of the region $|z| \le 1$ is the entire w-plane. If $z = x$ and $-1 \le x \le 1$, $w = x + 1/x$. Therefore w is real and, from an analysis of the graph of $w = x + 1/x$ for $-1 \le x \le 1$, $|w| \ge 2$. Therefore the segment $[-1, 1]$ (with 0 excluded) is mapped on those points $w = u$ on the u-axis for which $|u| \ge 2$.

11. Using H-4 with $a = 2$, $w = \cos(\pi z/2)$ maps R onto the target region R'. The image of AB is shown in the figure.

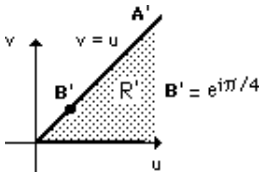


12. Using C-3 $w = e^z$ maps R onto the target region R'. The image of AB is shown in the figure.



13. Using H-5, $z_1 = \left(\dfrac{1+z}{1-z}\right)^2$ maps R onto the upper half-plane $y_1 \ge 0$, and $w = z_1^{1/4}$ maps this half-plane onto the target region R'. The image of AB is shown in the figure, and

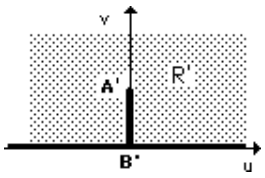
$$w = \left(\frac{1+z}{1-z}\right)^{1/2}.$$



14. Using H-1, $z_1 = i\,\dfrac{1-z}{1+z}$ maps R onto the upper half-plane $y_1 \ge 0$, and using M-4 with $a = 1$, $w = (z_1^2 - 1)^{1/2}$ maps this upper half-plane onto the target region R'. Therefore

$$w = \left[-\left(\frac{1-z}{1+z}\right)^2 - 1\right]^{1/2}$$

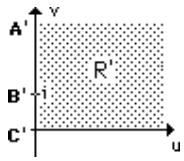
and the image of AB is shown in the figure.



15. Using H-6, $z_1 = \dfrac{e^{\pi/z} + e^{-\pi/z}}{e^{\pi/z} - e^{-\pi/z}}$ maps R onto the upper half-plane $y_1 \ge 0$, and $w = z_1^{1/2}$ maps this half-plane onto the target region R'. Therefore

$$w = \left(\frac{e^{\pi/z} + e^{-\pi/z}}{e^{\pi/z} - e^{-\pi/z}}\right)^{1/2}$$

and the image of AB is shown in the figure.



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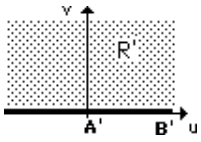
16. We can translate R to the origin, magnify by 2, and then use H-1 to reach the target region R'. Therefore $z_1 = 2(z - \frac{1}{2})$, $w = i\,\dfrac{1-z}{1+z}$ and so



$$w = i\,\frac{1-(2z-1)}{1+(2z-1)} = i\,\frac{1-z}{z}\,.$$

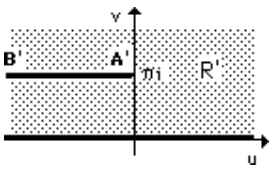
The image of AB is shown in the figure.

17. $z_1 = \text{Ln}z$ maps R onto the horizontal strip $0 \le y_1 \le \pi$, and, to prepare this strip for mapping by the sine function, we let $z_2 = -iz_1 - \pi/2$ to obtain the vertical strip $-\pi/2 \le x \le \pi/2$. Finally $w = \sin z_2$ maps this vertical strip onto the target region R'. Therefore

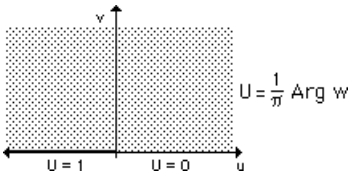
$$w = \sin\left(-i\text{Ln}\,z - \frac{\pi}{2}\right).$$

The image of AB is the real interval $(-\infty, -1]$.

18. Using E-9, $z_1 = \cosh z$ maps R onto the upper half-plane $y_1 \ge 0$. Using M-7, $w = z_1 + \text{Ln}\,z_1 + 1$ maps this half-plane onto the target region R'. Therefore $w = \cosh z + \text{Ln}(\cosh z) + 1$ and the image of AB is shown in the figure.



In Exercises 19-22, we find a conformal mapping $w = f(z)$ that maps the given region R onto the upper half-plane $v \ge 0$ and transfers the boundary conditions so that the resulting Dirichlet problem is as shown in the figure.



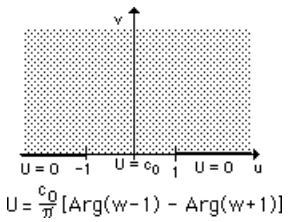
19. $f(z) = z^4$ and so $u = U(f(z)) = \dfrac{1}{\pi}\text{Arg}\,z^4$. The solution may also be written as $u(r,\theta) = 4\theta/\pi$.

20. $f(z) = \left(\dfrac{1+z}{1-z}\right)^2$, using H-5, and so $u = U(f(z)) = \dfrac{1}{\pi}\text{Arg}\left(\dfrac{1+z}{1-z}\right)^2$.

21. $f(z) = i\,\dfrac{1-z}{1+z}$, using H-1, and so $u = U(f(z)) = \dfrac{1}{\pi}\text{Arg}\left(\dfrac{1-z}{1+z}\right)$. The solution may also be written as $u(x,y) = \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{1-x^2-y^2}{2y}\right)$.

22. $f(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$, using H-3 with $a = 1$, and so $u = U(f(z)) = \dfrac{1}{\pi}\text{Arg}\,\dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$. The solution may also be written as $u(x,y) = \dfrac{1}{\pi}\tan^{-1}\left[\dfrac{y}{x}\,\dfrac{x^2+y^2-1}{x^2+y^2+1}\right]$.

In Exercises 23-26, we find a conformal mapping $w = f(z)$ that maps the given region R onto the upper half-plane $v \ge 0$ and transfers the boundary conditions so that the resulting Dirichlet problem is as shown in the figure.



23. $f(z) = z^2$ and $c_0 = 1$. Therefore $u = \dfrac{1}{\pi}[\text{Arg}(z^2-1) - \text{Arg}(z^2+1)]$.

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24. The mapping $z_1 = z^2$ maps R onto the region R_1 defined by $y_1 \geq 0$, $|z_1| \geq 1$ and shown in H-3, and $w = \dfrac{1}{2}\left(z_1 + \dfrac{1}{z_1}\right)$ maps R_1 onto the upper half-plane $v \geq 0$. Letting $c_0 = 5$,

$$u = \frac{5}{\pi}\left[\operatorname{Arg}\left(\frac{1}{2}\left[z^2 + \frac{1}{z^2}\right] - 1\right) - \operatorname{Arg}\left(\frac{1}{2}\left[z^2 + \frac{1}{z^2}\right] + 1\right)\right].$$

25. $f(z) = e^{\pi z}$, using H-2 with $a = 1$. Letting $c_0 = 10$,

$$u = \frac{10}{\pi}[\operatorname{Arg}(e^{\pi z} - 1) - \operatorname{Arg}(e^{\pi z} + 1)].$$

26. $f(z) = \cos(\pi z/2)$ using H-4 with $a = 2$. Letting $c_0 = 4$,

$$u = \frac{4}{\pi}[\operatorname{Arg}(\cos(\pi z/2) - 1) - \operatorname{Arg}(\cos(\pi z/2) + 1)].$$

27. **(a)** If $u = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^4 \phi}{\partial x^4} + 2\,\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

since ϕ is assumed to be biharmonic.

(b) If $g = u + iv$, then $\phi = \operatorname{Re}(\bar{z}g(z)) = xu + yv$.

$$\frac{\partial^2 \phi}{\partial x^2} = 3\,\frac{\partial u}{\partial x} + x\,\frac{\partial^2 u}{\partial x^2} + y\,\frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 \phi}{\partial y^2} = 2\,\frac{\partial v}{\partial y} + x\,\frac{\partial^2 u}{\partial y^2} + y\,\frac{\partial^2 v}{\partial y^2}.$$

Since u and v are harmonic and $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\,\frac{\partial u}{\partial x} + 2\,\frac{\partial v}{\partial y} = 4\,\frac{\partial u}{\partial x}.$$

Now $u_1 = \dfrac{\partial u}{\partial x}$ is also harmonic and so $\dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{\partial^2 u_1}{\partial y^2} = 0$. But

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = \frac{1}{4}\left[\frac{\partial^4 \phi}{\partial x^4} + 2\,\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4}\right]$$

and so ϕ is biharmonic.

Exercises 20.3

1. **(a)** For $T(z) = i/z$, $T(0) = \infty$, $T(1) = i$, and $T(\infty) = 0$.

(b) If $|z| = 1$, $|w| = |i/z| = 1/|z| = 1$. Therefore the image of the circle $|z| = 1$ is the circle $|w| = 1$ in the w-plane. The circle $|z - 1| = 1$ passes through the pole at $z = 0$ and so the image is a line. Since $T(2) = \frac{1}{2}i$ and $T(1 + i) = \frac{1}{2} + \frac{1}{2}i$, the image is the line $v = \frac{1}{2}$.

(c) The disk $|z| \leq 1$ is mapped onto the disk $|w| \geq 1$.

2. **(a)** For $T(z) = \dfrac{1}{z - 1}$, $T(0) = -1$, $T(1) = \infty$, and $T(\infty) = 0$.

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(b) The circle $|z| = 1$ passes through the pole at $z = 1$ and so the image is a line. Since $T(-1) = -\frac{1}{2}$ and $T(i) = -\frac{1}{2} - \frac{1}{2}\,i$, the image is the line $u = -\frac{1}{2}$. If $|z-1| = 1$, $|w| = 1/|z-1| = 1$ and the image is the circle $|w| = 1$ in the w-plane.

(c) Since $T(0) = -1$, the image of the disk $|z| \le 1$ is the half-plane $u = -\frac{1}{2}$.

3. (a) For $T(z) = \dfrac{z+1}{z-1}$, $T(0) = -1$, $T(1) = \infty$, and $T(\infty) = 1$.

(b) The circle $|z| = 1$ passes through the pole at $z = 1$ and so the image is a line. Since $T(-1) = 0$ and $T(i) = -i$, the image is the line $u = 0$. If $|z-1| = 1$,

$$|w-1| = \left|\frac{z+1}{z-1} - 1\right| = \frac{2}{|z-1|} = 2$$

and so the image is the circle $|w-1| = 2$ in the w-plane.

(c) Since $T(0) = -1$, the image of the disk $|z| \le 1$ is the half-plane $u \le 0$.

4. (a) For $T(z) = \dfrac{z-i}{z}$, $T(0) = \infty$, $T(1) = 1 - i$, and $T(\infty) = 1$.

(b) $T(i) = 0$, $T(1) = 1 - i$ and $T(-i) = 2$. The image of $|z| = 1$ is therefore a circle which passes through 0, $1 - i$, and 2. This is the circle $|w-1| = 1$. The circle $|z-1| = 1$ passes through the pole at $z = 0$ and so the image is a line. Since $T(2) = 1 - \frac{1}{2}\,i$ and $T(1+i) = \frac{1}{2} - \frac{1}{2}\,i$, the image is the line $v = -\frac{1}{2}$.

(c) $T(\frac{1}{2}) = 1 - 2i$ which is exterior to the circle $|w-1| = 1$. Therefore the image of $|z| \le 1$ is $|w-1| \ge 1$, the exterior of the circle $|w-1| = 1$ in the w-plane.

5. $S^{-1}(T(z)) = \dfrac{az+b}{cz+d}$ where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \operatorname{adj}\left(\begin{bmatrix} i & 1 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ i & -1 \end{bmatrix} = \begin{bmatrix} -1-i & 1 \\ -2 & -i \end{bmatrix}.$$

Therefore

$$S^{-1}(T(z)) = \frac{(-1-i)z+1}{-2z-i} = \frac{(1+i)z-1}{2z+i} \quad \text{and} \quad S^{-1}(w) = \frac{-w-1}{-w+i} = \frac{w+1}{w-i}.$$

6. $S^{-1}(T(z)) = \dfrac{az+b}{cz+d}$ where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \operatorname{adj}\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right)\begin{bmatrix} i & 0 \\ 1 & -2i \end{bmatrix} = \begin{bmatrix} -1+i & 2i \\ 2-i & -4i \end{bmatrix}.$$

Therefore

$$S^{-1}(T(z)) = \frac{(-1+i)z+2i}{(2-i)z-4i} \quad \text{and} \quad S^{-1}(w) = \frac{w-1}{-w+2}.$$

7. $S^{-1}(T(z)) = \dfrac{az+b}{cz+d}$ where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \operatorname{adj}\left(\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} 2 & -3 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -1 & 0 \end{bmatrix}.$$

Therefore

$$S^{-1}(T(z)) = \frac{-3}{-z} = \frac{3}{z} \quad \text{and} \quad S^{-1}(w) = \frac{-w+2}{-w+1} = \frac{w-2}{w-1}.$$

8. $S^{-1}(T(z)) = \dfrac{az+b}{cz+d}$ where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \operatorname{adj}\left(\begin{bmatrix} 2-i & 0 \\ 1 & -1-i \end{bmatrix}\right)\begin{bmatrix} 1 & -1+i \\ i & -2 \end{bmatrix} = \begin{bmatrix} -1-i & 2 \\ 2i & -3+i \end{bmatrix}.$$

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Therefore

$$S^{-1}(T(z)) = \frac{(-1-i)z+2}{2iz-3+i} \quad \text{and} \quad S^{-1}(w) = \frac{(-1-i)w}{-w+2-i} = \frac{(1+i)w}{w-2+1}\,.$$

9. $T(z) = \dfrac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$ maps z_1, z_2, z_3 to 0, 1, ∞. Therefore $T(z) = \dfrac{(z+1)(-2)}{(z-2)(1)} = -2\,\dfrac{z+1}{z-2}$ maps -1, 0, 2 to 0, 1, ∞.

10. $T(z) = \dfrac{(z-i)i}{(z+i)(-i)} = -\dfrac{z-i}{z+i}$ maps i, 0, $-i$ to 0, 1, ∞.

11. $S(w) = \dfrac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$ maps w_1, w_2, w_3 to 0, 1, ∞ and so S' maps 0, 1, ∞ to w_1, w_2, w_3. Therefore $z = \dfrac{(w-0)(i-2)}{(w-2)(i-0)}$ and so $w = \dfrac{2z}{z-1-2i}$ maps 0, 1, ∞ to 0, i, 2.

12. As in Exercise 11, $z = \dfrac{(w-1-i)(-1+i)}{(w-1+i)(-1-i)}$ and, solving for w, $w = \dfrac{2z-2}{(1+i)z-1+i}$ maps 0, 1, ∞ to $1+i$, 0, $1-i$.

13. Using the cross-ratio formula (7),

$$\frac{(w-i)(1)}{w(1)} = \frac{(z+1)(-1)}{(z-1)(1)}$$

and so $w = \dfrac{i}{2}\,\dfrac{z-1}{z}$ maps -1, 0, 1 to i, ∞, 0.

14. Using the cross-ratio formula (7),

$$\frac{(1)(-i-1)}{(w-1)(1)} = \frac{(z+1)(-1)}{(z-1)(1)}$$

and so $w = \dfrac{(2+i)z-i}{z+1}$ maps -1, 0, 1 to ∞, $-i$, 1.

15. Using the cross-ratio formula (7),

$$S(w) = \frac{(w+1)(-3)}{(w-3)(1)} = \frac{(z-1)(2i)}{(z+i)(i-1)} = T(z).$$

We can solve for w to obtain

$$w = 3\,\frac{(1+i)z+(1-i)}{(-3+5i)z-3-5i}\,.$$

Alternatively we can apply the matrix method to compute $w = S^{-1}(T(z))$.

16. Using the cross-ratio formula (7),

$$S(w) = \frac{(w-i)(-i-1)}{(w-1)(-2i)} = \frac{(z-1)(2i)}{(z+i)(i-1)} = T(z).$$

We can solve for w to obtain

$$w = \frac{(1+2i)z-i}{iz+1-2i}\,.$$

Alternatively we can apply the matrix method to compute $w = S^{-1}(T(z))$.

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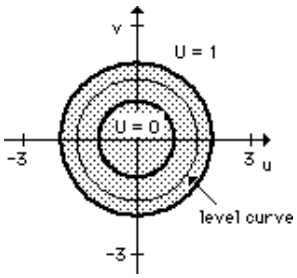
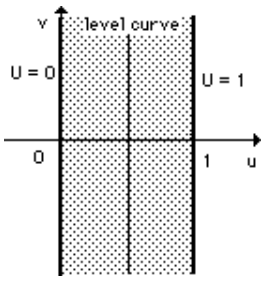
17. From Example 2, $z = \dfrac{w+2}{w-1}$ maps the annular region $1 < |w| < 2$ onto the region R and the circle $|w| = 1$ corresponds to the line $x = -1/2$. Solving for w, $w = \dfrac{z+2}{z-1}$ maps R onto the annulus and the transferred boundary conditions are shown in the figure. The solution to this new Dirichlet problem is $U = \log_e r / \log_e 2$ and so



$$u = U(f(z)) = \frac{1}{\log_e 2} \log_e \left| \frac{z+2}{z-1} \right|$$

is the solution to the Dirichlet problem in Figure 20.37. The level curves are the images of the level curves of U, $|w| = r$ for $1 < r < 2$ under the mapping $z = \dfrac{w+2}{w+1}$. Since these circles do not pass through the pole at $w = 1$, the images are circles.

18. The mapping $T(z) = \dfrac{1}{2}\dfrac{z+1}{z}$ maps -1, 1, 0 to 0, 1, ∞ and maps each of the two circles in R to lines since both circles pass through the pole at $z = 0$. Since $T(\frac{1}{2} + \frac{1}{2}\,i) = 1 - i$ and $T(1) = 1$, the circle $|z - \frac{1}{2}| = \frac{1}{2}$ is mapped onto the line $u = 1$. Likewise, the circle $|z + \frac{1}{2}| = \frac{1}{2}$ is mapped onto the line $u = 0$. The transferred boundary conditions are shown in the figure and $U(u, v) = u$ is the solution. The solution to the Dirichlet problem in Figure 20.38 is



$$u = U(T(z)) = \operatorname{Re}\left(\frac{1}{2}\frac{z+1}{z}\right) = \frac{1}{2} + \frac{1}{2}\frac{x}{x^2+y^2}\,.$$

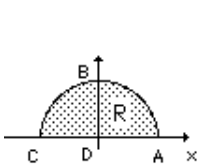
The level curves $u = c$ are the circles with centers on the x-axis which pass through the origin. The level curve $u = \frac{1}{2}$, however, is the vertical line $x = 0$.

19. The linear fractional transformation that sends 1, i, $-i$ to 0, 1, -1 satisfies the cross-ratio equation

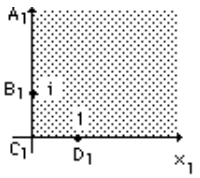
$$\frac{(w-0)(2)}{(w+1)(1)} = \frac{(z-1)(2i)}{(z+i)(i-1)}\,.$$

Solving for w, $w = i\,\dfrac{1-z}{1+z} = T(z)$. Since $T(0) = i$, the image of the disk $|z| \le 1$ is the upper half-plane $v \ge 0$.

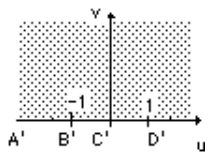
20. The linear fractional transformation that sends 1, i, -1 to ∞, i, 0 is $T(z) = \dfrac{1+z}{1-z}$ and so $f(z) = \left(\dfrac{1+z}{1-z}\right)^2$ maps 1, i, -1 to ∞, -1, 0. The upper semi-circle is mapped by T to the positive imaginary axis, and the real interval $[-1, 1]$ is mapped to the positive real axis. Since $T\left(\dfrac{i}{2}\right) = \dfrac{3}{5} + \dfrac{4}{5}\,i$, the image of R under T is the first quadrant. $w = z_1^2$ doubles the size of the opening so that the image under f is the upper half-plane $v \ge 0$. See the figures below.



(1)



(2)



(3)

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21. $T_2(T_1(z)) = \dfrac{a_2T_1(z)+b_2}{c_2T_1(z)+d_2} = \dfrac{a_2\dfrac{a_1z+b_1}{c_1z+d_1}+b_2}{c_2\dfrac{a_1z+b_1}{c_1z+d_1}+d_2} = \dfrac{a_1a_2z+a_2b_1+b_2c_1z+b_2d_1}{a_1c_2z+b_1c_2+c_1d_2z+d_1d_2} = \dfrac{(a_1a_2+c_1b_2)z+(b_1a_2+d_1b_2)}{(a_1c_2+c_1d_2)z+(b_1c_2+d_1d_2)}$

22. $|w-w_1| = \lambda|w-w_2| \Longrightarrow (u-u_1)^2+(v-v_1)^2 = \lambda^2[(u-u_2)^2+(v-v_2)^2]$
The latter equation may be put in the form

$$Au^2+Bv^2+Cu+Dv+F=0$$

where $A = B = 1-\lambda^2$. If $\lambda = 1$, the line $Cu+Dv+F=0$ results. If $\lambda > 0$ and $\lambda \neq 1$, then the equation defines a circle.

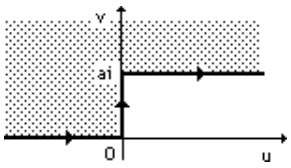
Exercises 20.4

1. $\arg f'(t) = -\frac{1}{2}\operatorname{Arg}(t-1) = \begin{cases} -\pi/2, & t<1 \\ 0, & t>1 \end{cases}$. Since $f(1)=0$, the image is the first quadrant.

2. $\arg f'(t) = -\dfrac{1}{3}\operatorname{Arg}(t+1) = \begin{cases} -\pi/3, & t<-1 \\ 0, & t>-1 \end{cases}$. In (2), $\alpha_1 = 2\pi/3$ and since $f(-1)=0$, the image is the wedge $0 \le \operatorname{Arg} w \le 2\pi/3$.

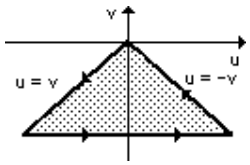
3. $\arg f'(t) = -\dfrac{1}{2}\operatorname{Arg}(t+1)+\dfrac{1}{2}\operatorname{Arg}(t-1) = \begin{cases} 0, & t<-1 \\ \pi/2, & -1<t<1 \\ 0, & t>1 \end{cases}$

and $\alpha_1 = \pi/2$ and $\alpha_2 = 3\pi/2$. Since $f(-1)=0$, the image of the upper half-plane is the region shown in the figure.



4. $\arg f'(t) = -\dfrac{1}{2}\operatorname{Arg}(t+1)-\dfrac{3}{4}\operatorname{Arg}(t-1) = \begin{cases} -5\pi/4, & t<-1 \\ -3\pi/4, & -1<t<1 \\ 0, & t>1 \end{cases}$

and $\alpha_1 = \pi/2$ and $\alpha_2 = \pi/4$. Since $f(-1)=0$, the image of the upper half-plane is the region shown in the figure.



5. Since $\alpha_1 = \alpha_2 = \alpha_3 = \pi/2$, $\alpha_i/\pi - 1 = -1/2$ and so $f'(z) = A(z+1)^{-1/2}z^{-1/2}(z-1)^{-1/2}$ for some constant A.

6. Since $\alpha_1 = \pi/3$ and $\alpha_2 = \pi/2$, $\alpha_1/\pi - 1 = -2/3$ and $\alpha_2/\pi - 1 = -1/2$ and so $f'(z) = A(z+1)^{-2/3}z^{-1/2}$ for some constant A.

7. Since $\alpha_1 = \alpha_2 = 2\pi/3$, $\alpha_i/\pi - 1 = -1/3$ and so $f'(z) = A(z+1)^{-1/3}z^{-1/3}$ for some constant A.

8. Since $\alpha_1 = 3\pi/2$ and $\alpha_2 = \pi/4$, $\alpha_1/\pi - 1 = 1/2$ and $\alpha_2/\pi - 1 = -3/4$. Therefore $f'(z) = A(z+1)^{1/2}z^{-3/4}$ for some constant A.

9. Since $\alpha_1 = \alpha_2 = \pi/2$, $f'(z) = A(z+1)^{-1/2}(z-1)^{-1/2} = A/(z^2-1)^{1/2}$. Therefore $f(z) = A\cosh^{-1}z + B$. But $f(-1) = \pi i$ and $f(1) = 0$. Since $\cosh^{-1}1 = 0$, $B = 0$. Since $\cosh^{-1}(-1) = \pi i$, $\pi i = A(\pi i)$ and so $A = 1$. Hence $f(z) = \cosh^{-1}z$.

10. Since $\alpha_1 = 3\pi/2 = \alpha_2$, $f'(z) = A(z+1)^{1/2}(z-1)^{1/2} = A(z^2-1)^{1/2}$. Therefore

$$f(z) = A\left[\frac{z(z^2-1)^{1/2}}{2} - \frac{1}{2}\operatorname{Ln}(z+(z^2-1)^{1/2}\right] + B.$$

but $f(-1) = -ai$ and $f(1) = ai$. It follows that

$$ai = f(1) = B, \qquad -ai = f(-1) = A\left(-\frac{\pi i}{2}\right) + B$$

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and so $B = ai$ and $A = 4a/\pi$. Therefore

$$f(z) = \frac{4a}{\pi}\left[\frac{z(z^2-1)^{1/2}}{2} - \frac{1}{2}\text{Ln}(z+(z^2-1)^{1/2})\right] + ai = \frac{4a}{\pi}\left[\frac{z(z^2-1)^{1/2}}{2} - \frac{1}{2}\cosh^{-1} z\right] + ai.$$

11. $f'(z) = A(z+1)^{\alpha_1/\pi-1}z^{\alpha_2/\pi-1}(z-1)^{\alpha_3/\pi-1}$ from (3). Since $f(-1) = \pi i$, $\alpha_1 \to \pi$ as $w_1 \to \infty$ in the horizontal direction. Likewise $\alpha_2 \to 0$ and $\alpha_3 \to \pi$. This suggests we examine $f'(z) = Az^{-1} = A/z$. Therefore $f(z) = A\text{Ln}\, z + B$. But $f(-1) = \pi i$ and $f(1) = 0$. It follows that $A = 1$ and $B = 0$ so that $f(z) = \text{Ln}\, z$. We verified in Example 1, Section 20.1 that $f(z) = \text{Ln}\, z$ maps the upper half-plane $y \geq 0$ to the horizontal strip $0 \leq v \leq \pi$.

12. From (3), $f'(z) = Az^{-3/4}(z-1)^{\alpha_2/\pi-1}$. But $\alpha_2 \to \pi$ as $\theta \to 0$. This suggests that we examine $f'(z) = Az^{-3/4}$. Therefore $f(z) = A_1 z^{1/4} + B_1$. But $f(0) = 0$ and $f(1) = 1$ so that $B_1 = 0$ and $A_1 = 1$. Hence $f(z) = z^{1/4}$ and we recognize that this power function maps the upper half-plane onto the wedge $0 \leq \text{Arg}\, w \leq \pi/4$.

13. From (3), $f'(z) = A(z+1)^{\alpha_1/\pi-1}z^{\alpha_2/\pi-1}(z-1)^{\alpha_3/\pi-1}$. But as $u_1 \to 0$, $\alpha_1 \to \pi/2$, $\alpha_2 \to 2\pi$, and $\alpha_3 \to \pi/2$. This suggests that we examine

$$f'(z) = A(z+1)^{-1/2}z(z-1)^{-1/2} = A\,\frac{z}{(z^2-1)^{1/2}}\,.$$

Therefore $f(z) = A(z^2-1)^{1/2} + B$. But $f(-1) = f(1) = 0$ and $f(0) = ai$. This implies that $B = 0$ and $ai = Ai$ or $A = a$. Therefore $f(z) = a(z^2-1)^{1/2}$. By expressing $f(z)$ as the composite of $z_1 = z^2$, $z_2 = z_1 - 1$, $z_3 = z_2^{1/2}$ and $w = az_3$ we can show that the image of the upper half-plane is R'.

14. If $w(t) = u(t) + iv(t)$, then $w'(t) = u'(t) + iv'(t)$ and so

$$\tan(\arg w'(t)) = \frac{v'(t)}{u'(t)} = \frac{dv}{du}\,.$$

If $\arg w'(t)$ is constant, then $dv/du = m$ or $v = mu + b$ for some constants m and b.

Exercises 20.5

1. Using (3) with $x_0 = -1$, $x_1 = 0$, $x_2 = 1$ and $u_1 = -1$ and $u_1 = 1$,

$$u = -\frac{1}{\pi}\text{Arg}\left(\frac{z}{z+1}\right) + \frac{1}{\pi}\text{Arg}\left(\frac{z-1}{z}\right).$$

2. Using (3) with $x_0 = -2$, $x_1 = 0$, $x_2 = 1$ and $u_1 = 5$ and $u_2 = 1$,

$$u = \frac{5}{\pi}\text{Arg}\left(\frac{z}{z+2}\right) + \frac{1}{\pi}\text{Arg}\left(\frac{z-1}{z}\right).$$

3. The harmonic function

$$u_1 = \frac{5}{\pi}[\pi - \text{Arg}(z-1)] = \begin{cases} 5, & x > 1 \\ 0, & x < 1 \end{cases}, \quad \text{and} \quad u_2 = -\frac{1}{\pi}\text{Arg}\left(\frac{z+1}{z+2}\right) + \frac{1}{\pi}\text{Arg}\left(\frac{z}{z+1}\right)$$

from (3) satisfies all boundary conditions except that $u_2 = 0$ for $x > 1$. Therefore $u = u_1 + u_2$ is the solution to the given Dirichlet problem.

4. The harmonic function

$$u_1 = \frac{1}{\pi}\text{Arg}(z+2) = \begin{cases} 1, & x < -2 \\ 0, & x > -2 \end{cases}, \quad \text{and} \quad u_2 = -\frac{1}{\pi}\text{Arg}\left(\frac{z+1}{z+2}\right) + \frac{1}{\pi}\text{Arg}\left(\frac{z+1}{z-1}\right)$$

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satisfies all boundary conditions except that $u_2 = 0$ for $x < -2$. Therefore $u = u_1 + u_2$ is the solution to the given Dirichlet problem.

5. By Theorem 20.5,

$$u(x, y) = \frac{y}{\pi} \int_0^1 \frac{t^2}{(x-t)^2 + y^2}\, dt.$$

If we let $s = x - t$, this integral can be expressed in terms of the natural log and inverse tangent. Using *Maple* we obtain

$$u = \frac{y}{\pi}\left(\frac{y^2 - x^2}{y}\left[\tan^{-1}\left(\frac{x-1}{y}\right) - \tan^{-1}\left(\frac{x}{y}\right)\right] + x \ln \frac{(x-1)^2 + y^2}{x^2 + y^2} + 1\right).$$

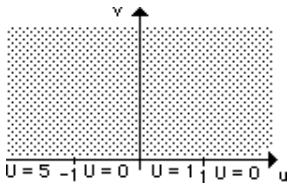
6. From Theorem 20.5,

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\cos t}{(x-t)^2 + y^2}\, dt = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\cos(x-s)}{s^2 + y^2}\, ds$$

letting $s = x - t$. But $\cos(x - s) = \cos x \cos x + \sin x \sin s$. It follows that

$$u(x, y) = \frac{y \cos x}{\pi} \int_{-\infty}^{\infty} \frac{\cos s}{s^2 + y^2}\, ds + \frac{y \sin x}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s^2 + y^2}\, ds = \frac{y \cos x}{\pi}\left(\frac{\pi e^{-y}}{y}\right) = e^{-y} \cos x, \quad y > 0.$$

7. The mapping $f(z) = z^2$ maps R onto the upper half-plane R'. The corresponding boundary value problem in R' is shown in the figure. From (3),



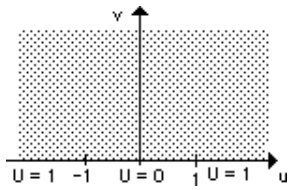
$$U = \frac{5}{\pi}\text{Arg}(w+1) + \frac{1}{\pi}\text{Arg}\left(\frac{w-1}{w}\right)$$

is the solution in R'. Therefore

$$u = U(f(z)) = \frac{5}{\pi}\text{Arg}(z^2+1) + \frac{1}{\pi}\text{Arg}\left(\frac{z^2-1}{z^2}\right)$$

is the solution to the original Dirichlet problem in R.

8. Using H-4 with $a = 3$, $f(z) = \cos(\pi z/3)$ maps R onto the upper half-plane R'. The corresponding Dirichlet problem in R' is shown in the figure. From (3),



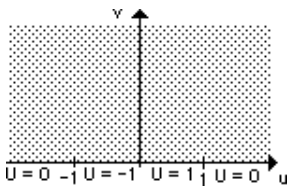
$$U = \frac{1}{\pi}[\pi - \text{Arg}(w-1)] + \frac{1}{\pi}\text{Arg}(w+1)$$

is the solution in R'. Therefore

$$u = U(f(z)) = 1 + \frac{1}{\pi}\left[\text{Arg}(\cos(\pi z/3) + 1) - \text{Arg}(\cos(\pi z/3) - 1)\right]$$

is the solution to the original Dirichlet problem in R.

9. Using H-1, $f(z) = i\,\dfrac{1-z}{1+z}$ maps R onto the upper half-plane R'. The corresponding Dirichlet problem in R' is shown in the figure. From (3),



$$U = -\frac{1}{\pi}\text{Arg}\left(\frac{w}{w+1}\right) + \frac{1}{\pi}\text{Arg}\left(\frac{w-1}{w}\right) = \frac{1}{\pi}\left[\text{Arg}\left(\frac{w-1}{w}\right) - \text{Arg}\left(\frac{w}{w+1}\right)\right].$$

The harmonic function $u = U(f(z))$ may be simplified to

$$u = \frac{1}{\pi}\left[\text{Arg}\left(\frac{(1-i)z - (1+i)}{1-z}\right) - \text{Arg}\left(\frac{1-z}{-(1+i)z + 1 - i}\right)\right]$$

and is the solution to the original Dirichlet problem in R.

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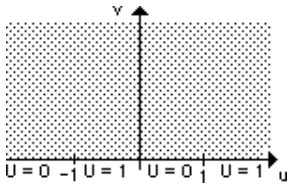
10. Using H-5, $f(z) = \left(\dfrac{1+z}{1-z}\right)^2$ maps R onto the upper half-plane R'. The corresponding Dirichlet problem in R' is shown in the figure. From (3),



$$U = \frac{1}{\pi}\text{Arg}\left(\frac{w}{w+1}\right) + \frac{1}{\pi}[\pi - \text{Arg}(w-1)] = 1 - \frac{1}{\pi}\text{Arg}(w-1) + \frac{1}{w}\text{Arg}\left(\frac{w}{w+1}\right).$$

The harmonic function $u = U(f(z))$ may be simplified to

$$u = 1 - \frac{1}{\pi}\text{Arg}\left(\frac{4z}{(1-z)^2}\right) + \frac{1}{\pi}\text{Arg}\left(\frac{(1+z)^2}{2(1+z^2)}\right).$$

11. From Theorem 20.6, $u(x,y) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \frac{t^2}{\pi^2}\,\frac{1-|z|^2}{|e^{it}-z|^2}\,dt$. Therefore,

$$u(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{t^2}{\pi^2}\,dt = \frac{1}{3}.$$

To estimate $u(0.5,0)$ and $u(-0.5,0)$ we must use a numerical integration method. With the aid of Simpson's Rule, $u(0.5,0) = 0.1516$ and $u(-0.5,0) = 0.5693$.

12. From Theorem 20.6, $u(x,y) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^{-|t|}\,\frac{1-|z|^2}{|e^{it}-z|^2}\,dt$. Therefore,

$$u(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-|t|}dt = \frac{1}{\pi}\int_0^{\pi} e^{-t}\,dt = \frac{1}{\pi}(1-e^{-\pi}).$$

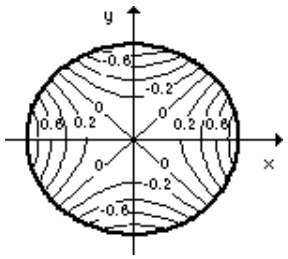
With the aid of Simpson's Rule, $u(0.5,0) = 0.5128$ and $u(-0.5,0) = 0.1623$.

13. $u(0.0) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} u(e^{it})\,dt$. The latter integral is just the average value of $u(e^{it})$ for $-\pi \le t \le \pi$.

14. For $u(e^{i\theta}) = \cos 2\theta$, the Fourier series solution (6) reduces to

$$u(r,\theta) = r^2\cos 2\theta = \text{Re}(z^2) \quad \text{or} \quad u(x,y) = x^2 - y^2.$$

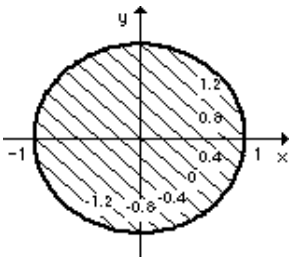
The corresponding system of level curves is shown in the figure.



15. For $u(e^{i\theta}) = \sin\theta + \cos\theta$, the Fourier series solution (6) reduces to

$$u(r,\theta) = r\sin\theta + r\cos\theta \quad \text{or} \quad u(x,y) = y + x.$$

The corresponding system of level curves is shown in the figure.



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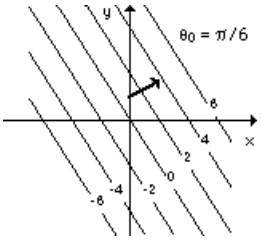
Exercises 20.6

1. $g(z) = \cos\theta_0 - i\sin\theta_0 = e^{-i\theta_0}$ is analytic everywhere and so div$\mathbf{F} = 0$ and curl $\mathbf{F} = \mathbf{0}$ by Theorem 20.7. A complex potential is $G(z) = e^{-i\theta_0} z$ and

$$\phi(x, y) = \text{Re}(G(z)) = x\cos\theta_0 + y\sin\theta_0.$$

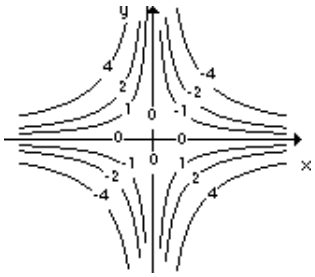
The equipotential lines (corresponding to $\theta_0 = \pi/6$) are shown in the figure.



2. $g(z) = -y + xi = i(x + iy) = iz$ is analytic everywhere and so div $\mathbf{F} = 0$ and curl $\mathbf{F} = \mathbf{0}$ by Theorem 20.7. A complex potential is $G(z) = \frac{i}{2} z^2$ and

$$\phi(x, y) = \text{Re}\left(\frac{i}{2} z^2\right) = -xy.$$

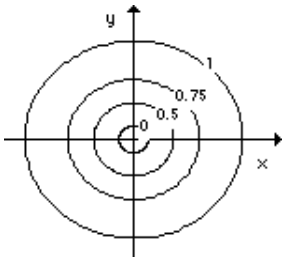
The equipotential lines $-xy = c$ are shown in the figure and are hyperbolas.



3. $g(z) = \dfrac{x}{x^2+y^2} - \dfrac{y}{x^2+y^2}\, i = \dfrac{1}{z}$ is analytic for $z \neq 0$ and so div $\mathbf{F} = 0$ and curl $\mathbf{F} = \mathbf{0}$ by Theorem 20.7. A complex potential is $G(z) = \text{Ln}z$ and

$$\phi(x, y) = \text{Re}(G(z)) = \frac{1}{2}\log_e(x^2 + y^2).$$

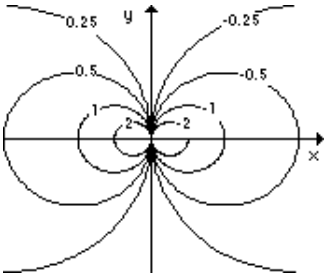
The equipotential lines $\phi(x, y) = c$ are circles $x^2 + y^2 = e^{2c}$ and are shown in the figure.



4. $g(z) = \dfrac{x^2 - y^2 - 2xyi}{(x^2+y^2)^2} = \dfrac{1}{z^2}$ is analytic for $z \neq 0$ and so div $\mathbf{F} = 0$ and curl $\mathbf{F} = \mathbf{0}$ by Theorem 20.7. A complex potential is $G(z) = -\dfrac{1}{z}$ and

$$\phi(x, y) = \text{Re}\left(-\frac{1}{z}\right) = -\frac{x}{x^2+y^2}\,.$$

The equipotential lines $-\dfrac{x}{x^2+y^2} = c$ can be writen as $\left(x + \dfrac{1}{2c}\right)^2 + y^2 = \left(\dfrac{1}{2c}\right)^2$ for $c \neq 0$. See the figure.



5. The mapping $f(z) = z^4$ maps the wedge $0 \leq \text{Arg}z \leq \pi/4$ to the upper half-plane R' and $U = \dfrac{1}{\pi}\text{Arg}w$ is the solution to the corresponding Dirichlet problem in R'. Therefore

$$\phi(x, y) = U(z^4) = \frac{1}{\pi}\text{Arg}z^4 = \frac{4}{\pi}\text{Arg}z$$

is the potential in the wedge. A complex potential is $G(z) = \dfrac{4}{\pi}\text{Ln}z$ and, since $\phi(r, \theta) = \dfrac{4}{\pi}\,\theta$, the equipotential lines are the rays $\theta = \dfrac{\pi}{4}\,c$. Finally

$$\mathbf{F} = \overline{G'(z)} = \frac{4}{\pi}\,\frac{1}{\bar{z}} = \frac{4}{\pi}\left(\frac{x}{x^2+y^2}\,,\, \frac{y}{x^2+y^2}\right).$$

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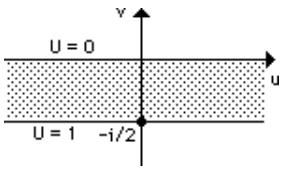
6. The function $f(z) = \dfrac{1}{z}$ maps the original region R to the strip $-\frac{1}{2} \le v \le 0$ (see Example 2, Section 20.1). The boundary conditions transfer as shown in the figure. $U = -2v$ is the solution in the horizontal strip and so



$$\phi(x, y) = -2\text{Im}\left(\frac{1}{z}\right) = \frac{2y}{x^2 + y^2}$$

is the potential in the original region R. The equipotential lines $\dfrac{2y}{x^2 + y^2} = c$ may be written as $x^2 + \left(y + \dfrac{1}{c}\right)^2 = \left(\dfrac{1}{c}\right)^2$ for $c \neq 0$ and are circles. If $c = 0$, we obtain the line $y = 0$. Note that $\phi(x, y) = \text{Re}\left(\dfrac{2i}{z}\right)$ and so $G(z) = \dfrac{2i}{z}$ is a complex potential. The corresponding vector field is

$$\mathbf{F} = \overline{G'(z)} = \frac{2i}{\bar{z}^2} = \left(\frac{-4xy}{(x^2 + y^2)^2}, \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}\right).$$

7. Using H-5, $w = \left(\dfrac{1 - z}{1 + z}\right)^2 = \left(\dfrac{z - 1}{z + 1}\right)^2$ maps R onto the upper half-plane R' and $U = \dfrac{1}{\pi}\text{Arg}w$ is the solution to the corresponding Dirichlet problem in R'. Therefore

$$\mu = \frac{1}{\pi}\text{Arg}\left(\frac{z - 1}{z + 1}\right)^2.$$

In R' the equipotential lines are rays $\theta = \theta_0$. The inverse transformation is $z = S(T(w))$ where $T(w) = w^{1/2}$ and $S(w) = \dfrac{1 + w}{1 - w}$. T maps the ray $\theta = \theta_0$ to the ray $\theta = \theta_0/2$ and S maps $\theta = \theta_0/2$ to an arc of a circle since $S(0) = 1$ and $S(\infty) = -1$ and S is a linear fractional transformation.

8. Using C-1 with $b = 2$ and $c = 4$, we have $a = \dfrac{3 + \sqrt{5}}{2}$ and $r_0 = \dfrac{7 - 3\sqrt{5}}{2}$, so $T(z) = \dfrac{z - a}{az - 1}$ maps R onto the annular region R' defined by $r_0 \le |w| \le 1$. The solution to the corresponding Dirichlet problem in R' is $U = \dfrac{\log_e |w|}{\log_e r_0}$ and so

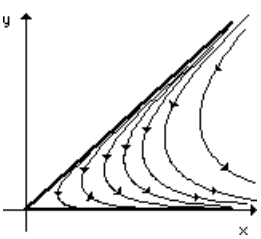
$$\phi = \frac{1}{\log_e r_0} \log_e \left|\frac{z - a}{az - 1}\right|.$$

The level curves of U are circles $|w| = r$ where $r_0 < r < 1$, and the equipotential lines $\phi(x, y) = c$ are the images of these circles under the inverse transformation $T^{-1}(w) = \dfrac{-w + a}{-aw + 1}$. T^{-1} has a pole at $w = \dfrac{1}{a} = \dfrac{2}{3 + \sqrt{5}}$ (≈ 0.38) and $r_0 < \dfrac{1}{a} < 1$. All circles $|w| = r$ are mapped onto circles in the z-plane with the exception of $|w| = \dfrac{1}{a}$ which is mapped onto a line.

9. **(a)** $\psi(x, y) = \text{Im}(z^4) = 4xy(x^2 - y^2)$ and so $\psi(x, y) = 0$ when $y = x$ and $y = 0$.

(b) $\mathbf{V} = \overline{G'(z)} = \overline{4z^3} = 4(x^3 - 3xy^2, y^3 - 3x^2y)$

(c) In polar coordinates $r^4 \sin 4\theta = c$ or $r = (c \csc 4\theta)^{1/4}$, for $0 < \theta < \pi/4$, are the streamlines. See the figure.



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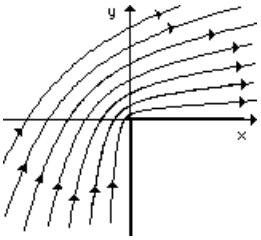
10. **(a)** Since $G(re^{i\theta}) = r^{2/3}e^{i2\theta/3}$, $\psi(r,\theta) = \text{Im}(G(re^{i\theta})) = r^{2/3}\sin\dfrac{2\theta}{3}$. Note that $\psi = 0$ on the boundary where $\theta = 0$ and $\theta = 3\pi/2$.

(b) $\mathbf{V} = \overline{G'(z)} = \overline{\frac{2}{3}z^{-1/3}}$. Therefore, letting $z = re^{i\theta}$, $\mathbf{V} = \dfrac{2}{3}r^{-1/3}(\cos(\theta/3), \sin(\theta/3))$ for $0 < \theta < 3\pi/2$.

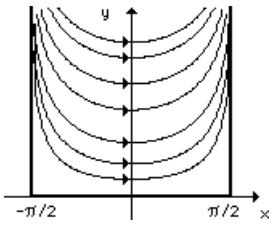
(c) $r^{2/3}\sin(2\theta/3) = c$ implies that $r = [c\csc(2\theta/3)]^{2/3}$ for $0 < \theta < 3\pi/2$. The streamlines are shown in the figure.



11. **(a)** $\psi(x,y) = \text{Im}(\sin z) = \cos x \sinh y$ and $\psi(x,y) = 0$ when $x = \pm\pi/2$ or when $y = 0$.

(b) $\mathbf{V} = \overline{G'(z)} = \overline{\cos z} = (\cos x \cosh y, \sin x \sinh y)$.

(c) $\cos x \sinh y = c \implies y = \sinh^{-1}(c\sec x)$ and the streamlines are shown in the figure.



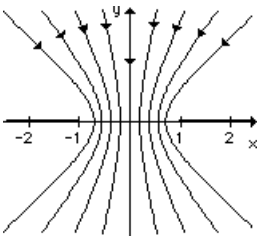
12. **(a)** The image of R under $w = i\sin^{-1} z$ is the horizontal strip (see E-6) $-\pi/2 \le v \le \pi/2$ and

$$\psi(x,y) = \text{Im}(i\sin^{-1} z) = \begin{cases} \pi/2, & x \ge 1 \\ -\pi/2, & x \le -1 \end{cases}.$$

Each piece of boundary is therefore a streamline.

(b) $\mathbf{V} = \overline{G'(z)} = \overline{\dfrac{i}{(1-z^2)^{1/2}}} = \dfrac{-i}{(1-\bar{z}^2)^{1/2}}$

(c) The streamlines are the images of the lines $v = b$, $-\pi/2 < b < \pi/2$ under $z = -i\sin w$ and are therefore hyperbolas. See Example 2, Section 20.2, and the figure. Note that at $z = 0$, $v = -i$ and the flow is downward.



13. **(a)** If $z = re^{i\theta}$, $G(re^{i\theta}) = r^2e^{2i\theta} + \dfrac{1}{r^2}e^{-2i\theta}$ and so

$$\psi(r,\theta) = \text{Im}(G(re^{i\theta})) = \left(r^2 - \frac{1}{r^2}\right)\sin 2\theta.$$

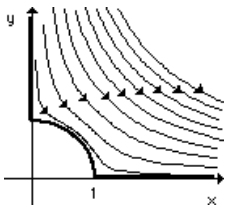
Note that $\psi = 0$ when $r = 1$ or when either $\theta = 0$ or $\theta = \pi/2$. Therefore $\psi = 0$ or the boundary of R.

(b) $\mathbf{V} = \overline{G'(z)} = (\overline{2z - 2z^{-3}})$ and so in polar coordinates

$$\mathbf{V} = 2re^{-i\theta} - \frac{2}{r^3}e^{3i\theta} = 2(r\cos\theta - r^{-3}\cos 3\theta, -r\sin\theta - r^{-3}\sin 3\theta).$$

(c) In rectangular coordinates, the streamlines are

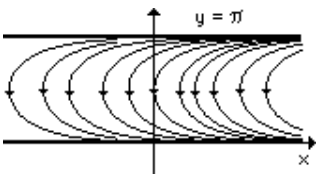
$$\psi(x,y) = 2xy\left[1 - \frac{1}{(x^2+y^2)^2}\right] = c.$$



14. **(a)** $\psi(x,y) = \text{Im}(e^z) = e^x\sin y$ and so $\psi = 0$ when $y = 0$ or π. Therefore $\psi = 0$ on the boundary of R.

(b) $\mathbf{V} = \overline{G'(z)} = \overline{e^z} = (e^x\cos y - e^x\sin y)$

(c) The streamlines are $e^x\sin y = c$ and so $x = \log_e(c\csc y)$ for $0 < y < \pi$. See the figure.



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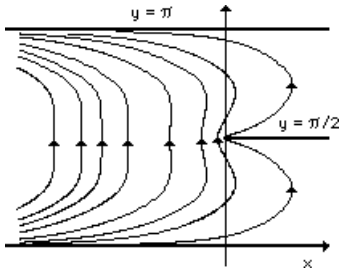
15. **(a)** For $f(z) = \pi i - \frac{1}{2}[\text{Ln}(z+1) + \text{Ln}(z-1)]$

$$f(t) = \pi i - \frac{1}{2}[\log_e|t+1| + \log_e|t-1| + i\text{Arg}(t+1) + i\text{Arg}(t-1)]$$

and so $\text{Im}(f(t)) = \begin{cases} 0, & t < -1 \\ \pi/2, & -1 < t < 1 \\ \pi, & t > 1 \end{cases}$. Hence $\text{Im}(G(z)) = \psi(x,y) = 0$ on the boundary of R.

(b) $x = -\frac{1}{2}[\log_e|t+1+ic| + \log_e|t-1+ic|], \quad y = \pi - \frac{1}{2}[\text{Arg}(t+1+ic) + \text{Arg}(t-1+ic)]$ for $c > 0$

(c)



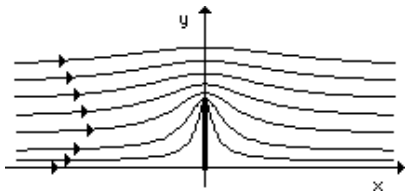
16. **(a)** For $f(z) = (z^2-1)^{1/2}$,

$$f(t) = |t^2-1|^{1/2}\cos\left(\frac{1}{2}\text{Arg}(t^2-1)\right) + i|t^2-1|^{1/2}\sin\left(\frac{1}{2}\text{Arg}(t^2-1)\right)$$

and so $f(t) = \begin{cases} |t^2-1|^{1/2}, & |t| > 1 \\ i|t^2-1|^{1/2}, & |t| < 1 \end{cases}$. Hence $\text{Im}(G(z)) = 0$ on the boundary of R.

(b) $x = |(t+ic)^2-1|^{1/2}\cos\left(\frac{1}{2}\text{Arg}((t+ic)^2-1)\right), \quad y = |(t+ic)^2-1|^{1/2}\sin\left(\frac{1}{2}\text{Arg}((t+ic)^2-1)\right)$ for $c > 0$

(c)



17. **(a)** For $f(z) = \frac{1}{\pi}[(z^2-1)^{1/2} + \cosh^{-1}z]$,

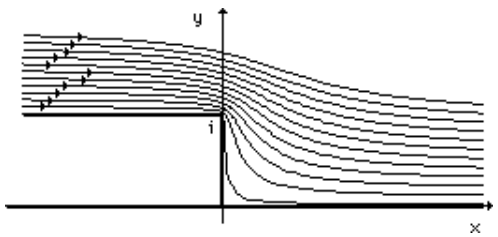
$$f(t) = \frac{1}{\pi}\left[(t^2-1)^{1/2} + \cosh^{-1}t\right) = \frac{1}{\pi}\left((t^2-1)^{1/2} + \text{Ln}(t + (t^2-1)^{1/2})\right]$$

and so $\text{Im}(f(t)) = \begin{cases} 1, & t < -1 \\ 0, & t > 1 \end{cases}$ and $\text{Re}(f(t)) = 0$, for $-1 < t < 1$. Hence $\text{Im}(G(z)) = \psi(x,y) = 0$ on the boundary of R.

(b) $x = \text{Re}\left(\frac{1}{\pi}\left[\left((t+ic)^2-1\right)^{1/2} + \cosh^{-1}(t+ic)\right]\right)$,

$y = \text{Im}\left(\frac{1}{\pi}\left[\left((t+ic)^2-1\right)^{1/2} + \cosh^{-1}(t+ic)\right]\right)$ for $c > 0$

(c)



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18. **(a)** For $f(z) = 2(z+1)^{1/2} + \text{Ln}\left(\dfrac{(z+1)^{1/2}-1}{(z+1)^{1/2}+1}\right)$,

$$f(t) = 2(t+1)^{1/2} + \text{Ln}\frac{(t+1)^{1/2}-1}{(t+1)^{1/2}+1}\,.$$

If we write $(t+1)^{1/2} = |t+1|^{1/2}e^{(i/2)\text{Arg}(t+1)}$, we may conclude that

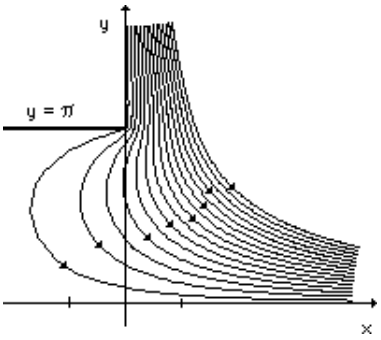
$$\text{Im}(f(t)) = \begin{cases} 0, & t > 0 \\ \pi, & -1 < t < 0 \end{cases} \quad \text{and} \quad \text{Re}(f(t)) = 0 \text{ for } t < -1.$$

Therefore $\text{Im}(G(z)) = \psi(x,y) = 0$ on the boundary of R.

(b) $x = \text{Re}\left[2(t+ic+1)^{1/2} + \text{Ln}\dfrac{(t+ic+1)^{1/2}-1}{(t+ic+1)^{1/2}+1}\right]$

$y = \text{Im}\left[2(t+ic+1)^{1/2} + \text{Ln}\dfrac{(t+ic+1)^{1/2}-1}{(t+ic+1)^{1/2}+1}\right]$ for $c > 0$

(c)



19. In Example 5, $\mathbf{V} = (2x, -2y)$ and so the only stagnation point occurs at $z = 0$. In Example 6, $\mathbf{V} = 1 - 1/\bar{z}^2$ and so if $\mathbf{V} = \mathbf{0}$, $\bar{z}^2 = 1$. Therefore $z = 1$, -1 are the only stagnation points.

20. **(a)** $\psi(x,y) = \text{Im}(G(z)) = k\text{Arg}(z - x_1)$ and so if $\psi(x,y) = c$, $\text{Arg}(z - x_1)$ is constant. This implies that the streamlines are rays with vertex at $z = x_1$.

(b) $\mathbf{V} = \overline{G'(z)} = \overline{\dfrac{k}{z - x_1}} = \dfrac{k}{\overline{z - x_1}} = \dfrac{k}{|z - x_1|^2}(z - x_1)$.

The direction of the flow is determined by the sign of k, and if $k < 0$ the flow is directed towards $z = x_1$.

21. $f(z) = z^2$ maps the first quadrant onto the upper half-plane and $f(\xi_0) = f(1) = 1$. Therefore $G(z) = \text{Ln}(z^2 - 1)$ is the complex potential, and so

$$\psi(x,y) = \text{Arg}(z^2 - 1) = \tan^{-1}\left(\frac{2xy}{x^2 - y^2 - 1}\right)$$

is the streamline function where $\tan^{-1}$ is chosen to be between 0 and π. If $\psi(x,y) = c$, then $x^2 + Bxy - y^2 - 1 = 0$ where $B = -2\cot c$. Each hyperbola in the family passes through $(1,0)$ and the boundary of the first quadrant satisfies $\psi(x,y) = 0$.

22. **(a)** From E-5, $f(z) = e^z$ maps the horizontal strip $0 < y < \pi$ onto the upper half-plane and $f(\xi_0) = f(0) = 1$. Therefore $G(z) = k\text{Ln}(e^z - 1)$ is a complex potential. To construct a sink at $\xi_0 = 0$, we must have $k < 0$.

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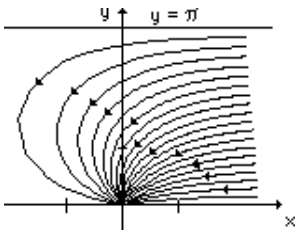
(b) $\psi(x,y) = \text{Im}(k\text{Ln}(e^z - 1)) = k\text{Arg}(e^z - 1) = k\tan^{-1}\left(\dfrac{e^x \sin y}{e^x \cos y - 1}\right)$



where $\tan^{-1}$ is chosen to be between 0 and π. If we set $k = -1$, then the streamlines $\psi(x,y) = c$, $-\pi < c < 0$, satisfy $e^x[B\cos y - \sin y] = B$ where $B = -\tan c$, and so

$$x = \log_e\left[\frac{B}{B\cos y - \sin y}\right].$$

Note that B will vary over all real values. The streamlines are also the images of rays through $w = 1$ under the inverse transformation $z = \text{Ln}w$. See the figure.

23. $\psi = \text{Im}(G(z)) = k\text{Arg}(z-1) - k\text{Arg}(z+1) = k\text{Arg}\left(\dfrac{z-1}{z+1}\right)$.

(See Figure 20.55 in the text). In rectangular coordinates

$$\psi(x,y) = k\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right)$$

where $\tan^{-1}$ is chosen to be between 0 and π. Level curves $\psi(x,y) = c$ can be put in the form

$$x^2 + y^2 - 2By = 1 \quad \text{or} \quad x^2 + (y-B)^2 = 1 + B^2.$$

Each member of the family passes through $(1,0)$ and $(-1,0)$.

24. (a) $\mathbf{V} = \dfrac{a+ib}{\bar{z}} = \left(\dfrac{ax-by}{x^2+y^2}, \dfrac{bx+ay}{x^2+y^2}\right)$ and since $(x'(t), y'(t)) = \mathbf{V}$, the path of the particle satisfies

$$\frac{dx}{dt} = \frac{ax-by}{x^2+y^2}, \qquad \frac{dy}{dt} = \frac{bx+ay}{x^2+y^2}.$$

(b) Switching to polar coordinates,

$$\frac{dr}{dt} = \frac{1}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{1}{r}\left(\frac{ax^2-bxy}{r^2} + \frac{bxy+ay^2}{r^2}\right) = \frac{a}{r}$$

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\frac{dx}{dt} + x\frac{dy}{dt}\right) = \frac{1}{r^2}\left(\frac{-axy+by^2}{r^2} + \frac{bx^2+axy}{r^2}\right) = \frac{b}{r^2}.$$

Therefore $\dfrac{dr}{d\theta} = \dfrac{a}{b}r$ and so $r = ce^{a\theta/b}$.

(c) $\dfrac{dr}{dt} = \dfrac{a}{r} < 0$ if and only if $a < 0$, and in this case r is decreasing and the curve spirals inward. $\dfrac{d\theta}{dt} = \dfrac{b}{r^2} < 0$ if and only if $b < 0$, and in this case θ is decreasing and the curve is traversed clockwise.

Chapter 20 Review Exercises

1. $f(z) = x^2 - y^2 + 2xyi$ and so the hyperbola $xy = 2$ is mapped onto the line $v = 4$.

2. $-i = e^{-i\pi/2}$ and so $f(z) = -iz$ is a rotation through $-90°$.

3. The wedge $0 \le \text{Arg}\, w \le 2\pi/3$. See figure 20.6 in the text.

4. $f'(z) = \sinh z = 0$ for $z = \pm\pi i$, and so $f(z) = \cosh z$ is conformal except at $z = \pm n\pi i$.

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5. True, by Theorem 20.2. $\operatorname{Arg} w$ is harmonic and is the upper half-plane $v \geq 0$.

6. A line, since $|z-1| = 1$ passes through the pole at $z = 2$.

7. 0, 1, ∞.

8. True. $\alpha_1 = \alpha_2 = \alpha_3 = \pi/2$

9. False. $\overline{g(z)} = P - iQ$ is analytic. See Theorem 20.7.

10. $iG(z) = -\psi(x,y) + i\phi(x,y)$ is analytic in R and $\operatorname{Im}(G(z)) = \phi(x,y)$. True

11. If $z = re^{i\theta}$ and $0 < \theta < \pi/2$, $w = \operatorname{Ln} z = \log_e r + i\theta$. Therefore $v = \theta$ and so $0 < v < \pi/2$. The image of the first quadrant is the strip $0 < v < \pi/2$ in the w-plane. Rays $\theta = \theta_0$ are mapped onto horizontal lines $v = \theta_0$.

12. First use $z_1 = z^2$ to map the first quadrant onto the upper half-plane $y_1 \geq 0$, and segment AB to the negative real axis. We then use $w = \frac{1}{\pi}(z_1 + \operatorname{Ln} z_1 + 1)$ to map this half-plane onto the target region R'. The composite transformation is

$$w = \frac{1}{\pi}[z^2 + \operatorname{Ln}(z^2) + 1]$$

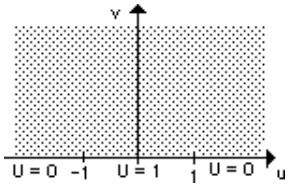
and the image of AB is the ray extending to the left from $w = i$ along the line $v = 1$.

13. From H-4, $z_1 = \cos \pi z$ maps R onto the upper half-plane $y_1 \geq 0$, and AB onto the real segment $(-\infty, -1]$. Using C-4, $w = \dfrac{i - z_1}{1 + z_1}$ maps this upper half-plane onto the target region R'. The composite transformation is

$$w = \frac{i - \cos \pi z}{i + \cos \pi z}$$

and the image of AB is the circular arc lying in quadrants II and III.

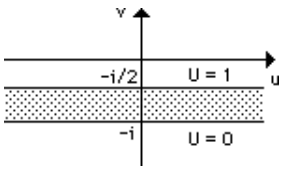
14. The power function $w = z^4$ maps R onto the upper half-plane $v \geq 0$; the transferred boundary conditions are shown in the figure. From (2) of Section 20.5, $U = \dfrac{1}{\pi}\operatorname{Arg}\left(\dfrac{w-1}{w+1}\right)$ and so



$$u = \frac{1}{\pi}\operatorname{Arg}\left(\frac{z^4 - 1}{z^4 + 1}\right)$$

is the solution to the original Dirichlet problem in R.

15. The inversion $w = 1/z$ maps R onto the horizontal strip $-1 \leq v \leq -1/2$ and the transferred boundary conditions are shown in the figure. The solution in the strip is $U = 2v + 2$ and so



$$u = U\left(\frac{1}{z}\right) = 2\operatorname{Im}\left(\frac{1}{z}\right) + 2 = \frac{-2y}{x^2 + y^2} + 2$$

is the solution to the original Dirichlet problem in R.

16. Using the cross ratio formula,

$$\frac{(w - i)(-i + 1)}{(w + 1)(-2i)} = \frac{(z - 1) \cdot 1}{1 \cdot (-2)}$$

we see that $w = -\dfrac{z - i}{z + i}$ maps $1, -1, \infty$ to $i, -i, -1$.

17. **(a)** We note that as $u_1 \to +\infty$, $\alpha_1 \to 0$, $\alpha_2 \to 2\pi$, and $\alpha_3 \to 0$.

(b) Since $f'(z) = A(z+1)^{\mathscr{L}_{1/\pi - 1}} z^{\alpha_2/\pi - 1}(z-1)^{\alpha_3/\pi - 1}$,

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this suggests that we examine

$$f'(z) = A(z+1)^{-1}z(z-1)^{-1} = A\,\frac{z}{z^2-1}\,.$$

We may write

$$f'(z) = \frac{A}{2}\left(\frac{1}{z+1} + \frac{1}{z-1}\right)$$

and so

$$f(z) = \frac{1}{2}\,A[\text{Ln}(z+1) + \text{Ln}(z-1)] + B.$$

(c) For t real, we can write

$$f(t) = \frac{1}{2}\,A[\log_e|t+1| + \log_e|t-1| + i\text{Arg}(t+1) + i\text{Arg}(t-1)] + B.$$

Since $f(0) = \dfrac{\pi i}{2}$, $\dfrac{\pi i}{2} = \dfrac{\pi i}{2}\,A + B$. For t real,

$$f(t) = \begin{cases} \frac{1}{2}A[\ln|t^2-1| + 2\pi i] + B, & t < -1 \\ \frac{1}{2}A\ln|t^2-1| + B, & t > 1 \end{cases}.$$

If A is real and we require that $\text{Im}(f(t)) = 0$ for $t < -1$, then $0 = A\pi i + \text{Im}(B)$. If $\text{Im}(f(t)) = \pi$ for $t > 1$, then $\pi = \text{Im}(B)$. All three equations are satisfied by letting $B = \pi i$ and $A = -1$. Therefore

$$f(z) = \pi i - \frac{1}{2}[\text{Ln}(z+1) + \text{Ln}(z-1)].$$

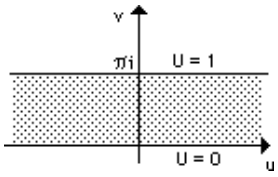
18. (a) From Theorem 20.5,

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\sin t}{(x-t)^2+y^2}\,dt = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\sin(x-s)}{s^2+y^2}]\,ds \quad \text{(letting } s = x-t\text{)}.$$

But $\sin(x-s) = \sin x\cos s - \cos x\sin s$. We now proceed as in the solution to Problem 6, Section 20.5 to show that $u(x,y) = e^{-y}\sin x$.

(b) For $u(e^{i\theta}) = \sin\theta$, the Fourier Series solution (6) in Section 20.5 reduces to $u(r,\theta) = r\sin\theta$.

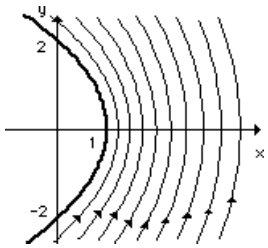
19. If $f(w) = w + e^w + 1$, $G(z) = f^{-1}(z)$ maps R onto the strip $0 \le v \le \pi$ and the transferred boundary conditions are shown in the figure. The solution for the strip is $U = v/\pi$ and so the solution in R is



$$\phi(x,y) = U(G(z)) = \frac{1}{\pi}\text{Im}(G(z)) = \frac{1}{\pi}\psi(x,y).$$

Therefore the equipotential lines $\phi(x,y) = c$ are the streamlines $\psi(x,y) = c\pi$ of the flow in Figure 20.72.

20. $G(re^{i\theta}) = -r^{1/2}\sin(\theta/2) + i[r^{1/2}\cos(\theta/2) - 1]$ and so $\psi(r,\theta) = \sqrt{r}\,\cos(\theta/2) = 1$. If $\psi(r,\theta) = 0$, $r\cos^2(\theta/2) = 1$ or $r\cos\theta + r = 2$ (using $\cos^2(\theta/2) = (1+\cos\theta)/2$.) In rectangular coordinates, $x + \sqrt{x^2+y^2} = 2$ or $y^2 = 4 - 4x$. Therefore the boundary of R is a stramline. To sketch the streamlines, note that $\psi(r,\theta) = c$ implies that $r = (c+1)^2\sec^2(\theta/2)$. See the figure.



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